ANNALES
POLONICI MATHEMATICI
98.3 (2010)

Uniform attractors for nonautonomous parabolic equations involving weighted p-Laplacian operators

by Cung The Anh and Nguyen Van Quang (Hanoi)

Abstract. We consider the first initial boundary value problem for nonautonomous quasilinear degenerate parabolic equations involving weighted p-Laplacian operators, in which the nonlinearity satisfies the polynomial condition of arbitrary order and the external force is normal. Using the asymptotic *a priori* estimate method, we prove the existence of uniform attractors for this problem. The results, in particular, improve some recent ones for nonautonomous p-Laplacian equations.

1. Introduction. Nonautonomous equations appear in many applications in the natural sciences, so they are of great importance and interest. The long-time behavior of solutions of such equations have been studied extensively in the last years.

In the book [14], Haraux considered some special classes of such systems and systematically studied the notion of a uniform attractor. As is known, a general method for considering the existence of the uniform attractor for nonautonomous equations was introduced by Chepyzhov and Vishik in [8]. This method has been used to study the existence of the uniform attractor for many equations arising in mathematical physics (see e.g. [8, 9, 10]). However, it is unsatisfactory that the method of Chepyzhov and Vishik can only be used to deal with the problems with translation compact symbols, while in applications symbols of many problems are not translation compact. Recently, Lu et al. [18] gave a necessary and sufficient condition for the existence of the uniform attractor for nonautonomous systems, which can be applied for more general symbols.

In this paper we study the following nonautonomous quasilinear degenerate parabolic equation with variable, nonnegative coefficients, defined on an arbitrary domain (bounded or unbounded) $\Omega \subset \mathbb{R}^N$, $N \geq 2$:

2010 *Mathematics Subject Classification*: 35B41, 35K65, 35D05.
Key words and phrases: nonautonomous quasilinear parabolic equation, weighted p-Laplacian operator, global solution, uniform attractor, compactness method, monotonicity method, asymptotic *a priori* estimate method.

DOI: 10.4064/ap98-3-5

$$(1.1)\qquad \begin{cases} u_t - \operatorname{div}(\rho|\nabla u|^{p-2}\nabla u) + f(x,u) = g(t), & x \in \Omega,\ t > 0,\\ u|_{t=\tau} = u_\tau(x), & x \in \Omega,\\ u|_{\partial\Omega} = 0, \end{cases}$$

where $p \geq 2$, $u_\tau \in L^2(\Omega)$ is given, and the diffusion coefficient ρ, the nonlinearity f, and the external force g satisfy some conditions specified later.

The degeneracy of problem (1.1) is considered in the sense that the measurable, nonnegative diffusion coefficient $\rho(x)$ is allowed to have at most a finite number of (essential) zeroes at some points. More precisely, we assume that the function $\rho : \Omega \to \mathbb{R}$ satisfies the following assumptions:

(H1) when the domain Ω is bounded,

($\mathcal{H}_\alpha$) $\rho \in L^1_{\mathrm{loc}}(\Omega)$ and for some $\alpha \in (0,p)$, $\liminf_{x\to z} |x-z|^{-\alpha}\rho(x) > 0$ for every $z \in \overline{\Omega}$,

and when the domain Ω is unbounded,

($\mathcal{H}^\infty_{\alpha,\beta}$) ρ satisfies condition ($\mathcal{H}_\alpha$) and $\liminf_{|x|\to\infty} |x|^{-\beta}\rho(x) > 0$ for some $\beta > p + (N/2)(p-2)$.

The physical motivation of the assumption ($\mathcal{H}_\alpha$) is related to the modelling of reaction-diffusion processes in composite materials, occupying a bounded domain Ω, which at some points behave as *perfect insulators*. Following [12, p. 79], when at some points the medium is perfectly insulating, it is natural to assume that $\rho(x)$ vanishes at those points. On the other hand, when condition ($\mathcal{H}^\infty_{\alpha,\beta}$) is satisfied, it is easy to see that the diffusion coefficient has to be unbounded. Physically, this situation corresponds to a nonhomogeneous medium, occupying the unbounded domain Ω, which behaves as a perfect conductor in $\Omega \setminus B_R(0)$ (see [12, p. 79]), and as a perfect insulator at a finite number of points in $B_R(0)$. Note that in various diffusion processes, the equation involves a diffusion $\rho(x) \sim |x|^\alpha$, $\alpha \in (0,p)$, in the case of a bounded domain, and $\rho(x) \sim |x|^\alpha + |x|^\beta$, $\alpha \in (0,p)$, $\beta > p$, in the case of an unbounded domain.

In the case of a bounded domain and $\rho(x)$ satisfying condition ($\mathcal{H}_\alpha$), problem (1.1) contains some important classes of parabolic equations, such as the semilinear heat equations (when $\rho = 1$, $p = 2$), semilinear degenerate parabolic equations (when $p = 2$), the p-Laplacian equations (when $\rho = 1$, $p \neq 2$), etc. The long-time behavior of solutions to p-Laplacian equations has been studied by many authors in the last years (see e.g. [6, 7, 11, 22, 23]). In the autonomous degenerate case, that is, the case of g independent of time t, the existence and long-time behavior of solutions to problem (1.1) when $p = 2$ have been studied in [15, 16] and recently in [2, 1]; the quasilinear case $1 < p \neq 2$ is investigated in [3].

In this paper we continue the study of the long-time behavior of solutions to problem (1.1) by allowing the external force g to depend on t. To study problem (1.1) we assume that

(H2) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|f(x,u)| \le C_1|u|^{q-1} + h_1(x), \tag{1.2}$$

$$uf(x,u) \ge C_2|u|^q - h_2(x), \tag{1.3}$$

$$F(x,u) = \int_0^u f(x,\xi)\,d\xi \ge C_3|u|^q + h_3(x), \tag{1.4}$$

$$f'_u(x,u) \ge -l \quad \text{for some } l > 0, \tag{1.5}$$

where $q \ge 2$; $C_1, C_2, C_3 > 0$; $h_1 \in L^{q'}(\Omega)$ and $h_2, h_3 \in L^1(\Omega) \cap L^\infty(\Omega)$ are nonnegative functions.

(H3) The external force $g \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$ satisfies

$$g \in L^2_{\text{n}}(\mathbb{R}; L^2(\Omega)), \quad g' \in L^2_{\text{b}}(\mathbb{R}; L^2(\Omega)), \tag{1.6}$$

where $L^2_{\text{n}}(\mathbb{R}; L^2(\Omega))$ and $L^2_{\text{b}}(\mathbb{R}; L^2(\Omega))$ are the sets of translation normal functions and translation bounded functions (see Section 2.2 for their definitions).

(H4) $[1, p^*_\alpha) \cap \mathcal{I}[p', q'] \neq \emptyset$, where α is given in $(\mathcal{H}_\alpha)$, if Ω is a bounded domain; and $(p^*_\beta, p] \cap \mathcal{I}[p', q'] \neq \emptyset$, where α, β are given in $(\mathcal{H}^\infty_{\alpha,\beta})$, if Ω is an unbounded domain.

Here p' denotes the conjugate exponent of p, i.e., $1/p + 1/p' = 1$; $p^*_\gamma := pN/(N - p + \gamma)$, for $\gamma \in \mathbb{R}^+$; and $\mathcal{I}[p,q] := \{tp + (1-t)q \mid 0 \le t \le 1\}$.

In order to study problem (1.1) we introduce the natural energy space $\mathcal{D}^{1,p}_0(\Omega, \rho)$ defined as the closure of $C^\infty_0(\Omega)$ in the norm

$$\|u\|_{\mathcal{D}^{1,p}_0(\Omega,\rho)} := \Big(\int_\Omega \rho(x)|\nabla u|^p\,dx\Big)^{1/p},$$

and prove some compactness results (see Section 2.1 for more details). Using the compactness and monotonicity methods [17], we prove the global existence of a weak solution to problem (1.1). The main aim of this paper is to study the existence of an $(L^2(\Omega), \mathcal{D}^{1,p}_0(\Omega,\rho) \cap L^q(\Omega))$-uniform attractor for a family of processes associated to problem (1.1).

Let us describe the methods used in the paper (we refer to Section 2.3 for definitions of related concepts). First, we use *a priori* estimates to show the existence of an $(L^2(\Omega), \mathcal{D}^{1,p}_0(\Omega,\rho) \cap L^q(\Omega))$-uniformly absorbing set for the family of processes. By the compactness of the embedding $\mathcal{D}^{1,p}_0(\Omega,\rho) \hookrightarrow L^2(\Omega)$, the family of processes is $(L^2(\Omega), L^2(\Omega))$-uniformly asymptotically compact. This immediately implies the existence of an $(L^2(\Omega), L^2(\Omega))$-uniform attractor. When proving the existence of an $(L^2(\Omega), L^q(\Omega))$-uniform attractor and an $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega,\rho)\cap L^q(\Omega))$-uniform attractor, to overcome the difficulty caused by the lack of embbeding results, we use the asymptotic *a priori* estimate method initiated in [19] for autonomous equations and developed in [18] for nonautonomous equations. This method has been applied successfully for some classes of partial differential equations (see e.g. [18–21, 23, 24]). One of the main new features in our paper is that the existence of uniform attractors is proved for a class of quasilinear degenerate parabolic equations in an arbitrary (bounded or unbounded) domain. It is worth noticing that, when $\rho = 1$, we recover/improve the recent results in [7, 20] for nonautonomous heat equations and nonautonomous p-Laplacian equations in bounded domains.

The content of the paper is as follows. In Section 2, we prove some compactness results and recall some concepts and results on uniform attractors which we will use. In Sections 3 and 4, we focus on the case of an unbounded domain and the diffusion coefficient ρ satisfying condition $(\mathcal{H}^{\infty}_{\alpha,\beta})$ because it is more complicated. Section 3 is devoted to the proof of the existence and uniqueness of a global weak solution to problem (1.1) by using the compactness and monotonicity methods. In Section 4, using the asymptotic *a priori* estimate method, we prove the existence of uniform attractors in various spaces. In the last section, we give some remarks on similar results for a bounded domain and ρ satisfying condition $(\mathcal{H}_\alpha)$.

2. Preliminary results

2.1. Function spaces and operators. In order to study problem (1.1), we introduce the weighted Sobolev space $\mathcal{D}_0^{1,p}(\Omega,\rho)$ defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{\mathcal{D}_0^{1,p}(\Omega,\rho)} := \Big(\int_\Omega \rho(x)|\nabla u|^p\,dx\Big)^{1/p},$$

and denote by $\mathcal{D}^{-1,p'}(\Omega,\rho)$ the dual space of $\mathcal{D}_0^{1,p}(\Omega,\rho)$.

We now give some compactness results, which are generalizations of the results in the case $p = 2$ of Caldiroli and Musina [5]. The first result comes from [3, Proposition 2.1].

Proposition 2.1. *Assume that Ω is a bounded domain in $\mathbb{R}^N$, $N \ge 2$, and ρ satisfies the hypothesis $(\mathcal{H}_\alpha)$. Then there is a compact embedding $\mathcal{D}_0^{1,p}(\Omega,\rho) \subset L^r(\Omega)$ whenever $1 \le r < p^*_\alpha$.*

We now prove embedding results in the case of an unbounded domain.

Proposition 2.2. *Assume that Ω is an unbounded domain in $\mathbb{R}^N$, $N \ge 2$, and ρ satisfies the hypothesis $(\mathcal{H}^{\infty}_{\alpha,\beta})$. Then there is a compact embedding $\mathcal{D}_0^{1,p}(\Omega,\rho) \subset L^r(\Omega)$ for every $r \in (p^*_\beta, p^*_\alpha)$, $r \ge 1$.*

Proof. Let $\{u_m\}$ be a sequence in $\mathcal{D}_0^{1,p}(\Omega,\rho)$ such that $u_m \rightharpoonup 0$ in $\mathcal{D}_0^{1,p}(\Omega,\rho)$. For any fixed $r \in (p_\beta^*, p_\alpha^*), r \geq 1$, we have to prove that $u_m \to 0$ in $L^r(\Omega)$. For $R > 0$, write B_R for the ball centered at 0 with radius R. Using Proposition 2.1, we see that $\mathcal{D}_0^{1,p}(B_R,\rho) \hookrightarrow L^r(B_R)$ compactly. Then $\|u_m\|_{L^r(B_R)} \to 0$ as $m \to \infty$. Indeed, assume that, on the contrary, there exist $\eta > 0$ and a subsequence of u_m, still denoted by u_m, such that

$$\int_{\Omega\setminus B_R} |u_m|^r \geq \eta \quad \text{for all } R > 0. \tag{2.1}$$

Choose a function $\varphi \in C^\infty(\mathbb{R}^N)$ such that

- $0 \leq \varphi \leq 1$,
- $\varphi = 0$ in B_R and $\varphi = 1$ in $\Omega \setminus B_{2R}$.

Now putting $\hat{u}_m = \varphi u_m$, we have

$$\int_\Omega \rho(x)|\nabla \hat{u}_m|^p \leq C\Big(\int_\Omega \rho(x)|\nabla u_m|^p + \int_\Omega \rho(x)|u_m|^p|\nabla\varphi|^p\Big). \tag{2.2}$$

One can rewrite the last integral as

$$\int_\Omega \rho(x)|u_m|^p|\nabla\varphi|^p = \int_{\Omega\cap(B_{2R}\setminus B_R)} \rho(x)|u_m|^p|\nabla\varphi|^p. \tag{2.3}$$

Using Proposition 2.1 again for the bounded domain $\Omega \cap (B_{2R} \setminus B_R)$, we find that $u_m \to 0$ a.e. in $\Omega \cap (B_{2R} \setminus B_R)$ and hence

$$\int_\Omega \rho(x)|u_m|^p|\nabla\varphi|^p = o(1) \quad \text{as } m \to \infty. \tag{2.4}$$

Taking $\gamma \in (0,p)$ such that $p_\gamma^* = r$ and using the Caffarelli–Kohn–Nirenberg inequality [4], we have

$$\begin{aligned}\Big(\int_{\Omega\setminus B_{2R}} |u_m|^{p_\gamma^*}\Big)^{p/p_\gamma^*} &= \Big(\int_{\Omega\setminus B_{2R}} |\hat{u}_m|^{p_\gamma^*}\Big)^{p/p_\gamma^*} \leq \Big(\int_\Omega |\hat{u}_m|^{p_\gamma^*}\Big)^{p/p_\gamma^*} \\ &\leq \int_\Omega |x|^\gamma|\nabla\hat{u}_m|^p.\end{aligned} \tag{2.5}$$

Since ρ satisfies $(\mathcal{H}_{\alpha,\beta}^\infty)$, one can see that

$$\rho(x) \geq \delta|x|^{\beta-\gamma}|x|^\gamma \geq \delta R^{\beta-\gamma}|x|^\gamma$$

for some $\delta > 0$ and all $x \in \Omega \setminus B_R$ with R large enough. Combining this with (2.5), we get

$$\Big(\int_{\Omega\setminus B_{2R}} |u_m|^{p_\gamma^*}\Big)^{p/p_\gamma^*} \leq CR^{\beta-\gamma}\int_\Omega \rho(x)|\nabla u_m|^p + o(1). \tag{2.6}$$

Taking (2.1) into account, we see that (2.6) leads to a contradiction when R is chosen large enough. The proof is complete. ■

Remark 2.1. From the above proof, we see that the conclusion of Proposition 2.2 is still valid for all $\beta > p$. The condition $\beta > p+(N/2)(p-2)$ in assumption $(\mathcal{H}^{\infty}_{\alpha,\beta})$ ensures that $\mathcal{D}^{1,p}_0(\Omega,\rho) \hookrightarrow L^2(\Omega)$ compactly, which is necessary later. Put

$$L_{p,\rho}u := -\mathrm{div}(\rho|\nabla u|^{p-2}\nabla u).$$

The following proposition, whose proof is straightforward, gives some important properties of the operator $L_{p,\rho}$.

Proposition 2.3. *The operator $L_{p,\rho}$ maps $\mathcal{D}^{1,p}_0(\Omega,\rho)$ into its dual $\mathcal{D}^{-1,p'}(\Omega,\rho)$. Moreover,*

(1) *$L_{p,\rho}$ is hemicontinuous, i.e., for all $u,v,w \in \mathcal{D}^{1,p}_0(\Omega,\rho)$, the map $\lambda \mapsto \langle L_{p,\rho}(u+\lambda v), w\rangle$ is continuous from $\mathbb{R}$ to $\mathbb{R}$.*
(2) *$L_{p,\rho}$ is strongly monotone when $p \geq 2$, i.e., there exists $\delta > 0$ such that*

$$\langle L_{p,\rho}u - L_{p,\rho}v, u-v\rangle \geq \delta\|u-v\|^p_{\mathcal{D}^{1,p}_0(\Omega,\rho)} \quad \textit{for all } u,v \in \mathcal{D}^{1,p}_0(\Omega,\rho).$$

2.2. The translation normal functions

Definition 2.1. Assume that $\mathcal{E}$ is a reflexive Banach space.

(1) A function $\varphi \in L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{E})$ is said to be *translation bounded* if

$$\|\varphi\|^2_{L^2_b} = \|\varphi\|_{L^2_b(\mathbb{R};\mathcal{E})} = \sup_{t\in\mathbb{R}} \int_t^{t+1} \|\varphi\|^2_{\mathcal{E}}\,ds < \infty.$$

(2) A function $\varphi \in L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{E})$ is said to be *translation normal* if for any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+\eta} \|\varphi\|^2_{\mathcal{E}}\,ds < \varepsilon.$$

(3) A function $\varphi \in L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{E})$ is said to be *translation compact* if the closure of $\{\varphi(\cdot+h) \mid h \in \mathbb{R}\}$ is compact in $L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{E})$.

Denote by $L^2_{\mathrm{b}}(\mathbb{R};\mathcal{E}), L^2_{\mathrm{n}}(\mathbb{R};\mathcal{E})$ and $L^2_{\mathrm{c}}(\mathbb{R};\mathcal{E})$ the sets of all translation bounded, translation normal and translation compact functions in $L^2_{\mathrm{loc}}(\mathbb{R};\mathcal{E})$, respectively. It is well-known (see [18]) that

$$L^2_{\mathrm{c}}(\mathbb{R};\mathcal{E}) \subset L^2_{\mathrm{n}}(\mathbb{R};\mathcal{E}) \subset L^2_{\mathrm{b}}(\mathbb{R};\mathcal{E}).$$

Let $\mathcal{H}_{\mathrm{w}}(g)$ be the closure of the set $\{g(\cdot+h) \mid h \in \mathbb{R}\}$ in $L^2_{\mathrm{b}}(\mathbb{R};L^2(\Omega))$ in the weak topology. The following results are proved in [18].

Lemma 2.4 ([18, Proposition 3.1]).

(1) *For all $\sigma \in \mathcal{H}_{\mathrm{w}}(g)$, $\|\sigma\|^2_{L^2_{\mathrm{b}}} \leq \|g\|^2_{L^2_{\mathrm{b}}}$.*
(2) *The translation group $\{T(h)\}$ is weakly continuous on $\mathcal{H}_{\mathrm{w}}(g)$.*

(3) $T(h)\mathcal{H}_{\rm w}(g) = \mathcal{H}_{\rm w}(g)$ *for* $h \geq 0$.
(4) $\mathcal{H}_{\rm w}(g)$ *is weakly compact.*

Lemma 2.5 ([18, Lemma 3.1]). *If* $g \in L^2_{\rm n}(\mathbb{R};\mathcal{E})$ *then for any* $\tau \in \mathbb{R}$,

$$\lim_{\gamma\to+\infty} \sup_{t\geq\tau} \int_\tau^t e^{-\gamma(t-\tau)} \|\varphi\|^2_{\mathcal{E}}\, ds = 0 \quad \textit{for all } \varphi \in \mathcal{H}_{\rm w}(g).$$

2.3. Uniform attractors. Let Σ be a parameter set and X, Y two Banach spaces with $Y \subset X$ continuously. $\{U_\sigma(t,\tau) \mid t \geq \tau,\ \tau \in \mathbb{R}\}$, $\sigma \in \Sigma$, is said to be a *family of processes* in X, if for each $\sigma \in \Sigma$, $\{U_\sigma(t,\tau)\}$ is a *process*, that is, a two-parameter family of mappings from X to X satisfying

$$U_\sigma(t,s)U_\sigma(s,\tau) = U_\sigma(t,\tau), \quad \forall t \geq s \geq \tau,\ \tau \in \mathbb{R},$$
$$U_\sigma(\tau,\tau) = \mathrm{Id}, \quad \text{the identity operator},\ \tau \in \mathbb{R},$$

where Σ is called the *symbol space*, $\sigma \in \Sigma$ is the *symbol*. Denote by $\mathcal{B}(X)$ the set of all bounded subsets of X.

Definition 2.2. A set $B_0 \in \mathcal{B}(Y)$ is said to be (X,Y)-*uniformly absorbing* for the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ if for any $\tau \in \mathbb{R}$ and every $B \in \mathcal{B}(X)$, there exists $t_0 = t_0(\tau, B) \geq \tau$ such that $\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)B \subset B_0$ for all $t \geq t_0$. A set $P \subset Y$ is said to be (X,Y)-*uniformly attracting* if, for any fixed $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(X)$, $\lim_{t\to+\infty} \sup_{\sigma\in\Sigma} \mathrm{dist}_Y(U_\sigma(t,\tau)B, P) = 0$.

Definition 2.3. A closed set $\mathcal{A}_\Sigma \subset Y$ is said to be an (X,Y)-*uniform attractor* for the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ if it is an (X,Y)-uniformly attracting set, and it is contained in any closed (X,Y)-uniformly attracting set for the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$.

Theorem 2.6 ([7, Theorem 3.9]). *Let* $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ *be a family of processes acting on* X *such that:*

(1) $U_\sigma(t+h,\tau+h) = U_{T(h)\sigma}(t,\tau)$, *where* $\{T(h) \mid h \geq 0\}$ *is a family of operators acting on* Σ *and satisfying* $T(h)\Sigma = \Sigma$ *for all* $h \in \mathbb{R}^+$;
(2) Σ *is a weakly compact set and* $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ *is* $(X\times\Sigma, Y)$-*weakly continuous, i.e., for any fixed* $t \geq \tau$, $\tau \in \mathbb{R}$, *the mapping* $(u,\tau) \mapsto U_\sigma(t,\tau)u$ *is weakly continuous from* $X \times \Sigma$ *to* Y;
(3) $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ *is* (X,Y)-*uniformly asymptotically compact, i.e., it possesses a compact* (X,Y)-*uniformly attracting set.*

Then the family $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$ *possesses an* (X,Y)-*uniform attractor* $\mathcal{A}_\Sigma$, *which is compact in* Y *and attracts all bounded subsets of* X *in the topology of* Y. *Moreover*

$$\mathcal{A}_\Sigma = \omega_{\tau,\Sigma}(B_0) = \bigcap_{t\geq\tau} \overline{\bigcup_{\sigma\in\Sigma} \bigcup_{s\geq t} U_\sigma(s,\tau)B_0},$$

where B_0 *is any bounded* (X,Y)-*uniformly absorbing set of* $\{U_\sigma(t,\tau)\}_{\sigma\in\Sigma}$.

3. Existence of weak solutions. Recall that in this section and the next section we only consider the case of an unbounded domain and ρ satisfying condition $(\mathcal{H}^{\infty}_{\alpha,\beta})$. Since $\beta > p + (N/2)(p-2)$, we have $\mathcal{D}^{1,p}_0(\Omega,\rho) \hookrightarrow L^2(\Omega)$ compactly thanks to Proposition 2.2. Denote

$$V = L^p(\tau,T;\mathcal{D}^{1,p}_0(\Omega,\rho)) \cap L^q(\tau,T;L^q(\Omega)),$$
$$V^* = L^{p'}(\tau,T;\mathcal{D}^{-1,p'}(\Omega,\rho)) + L^{q'}(\tau,T;L^{q'}(\Omega)),$$

where p' is the conjugate index of p. In what follows, we assume that $u_\tau \in L^2(\Omega)$ is given.

DEFINITION 3.1. A function u is called a *weak solution* of (1.1) on (τ,T) iff

$$u \in V, \quad \frac{\partial u}{\partial t} \in V^*,$$
$$u|_{t=\tau} = u_\tau(x) \quad \text{a.e. in } \Omega,$$

and

$$\int_\tau^T \int_\Omega \left(\frac{\partial u}{\partial t}\varphi + \rho|\nabla u|^{p-2}\nabla u \nabla\varphi + f(x,u)\varphi - g\varphi \right) dx\,dt = 0$$

for all test functions $\varphi \in V$.

It is known (see e.g. [13, Theorem 3, p. 287]) that if $u \in V$ and $du/dt \in V^*$ then $u \in C([\tau,T];L^2(\Omega))$. This makes the initial condition in problem (1.1) meaningful.

THEOREM 3.1. *For any given $\tau, T \in \mathbb{R}$ and $u_\tau \in L^2(\Omega)$, problem* (1.1) *has a unique weak solution on (τ,T).*

Proof. Consider the approximating solution $u_n(t)$ in the form

$$u_n(t) = \sum_{k=1}^n u_{nk}(t)e_k,$$

where $\{e_j\}_{j\in\mathbb{N}}$ is a basis of $\mathcal{D}^{1,p}_0(\Omega,\rho) \cap L^q(\Omega)$ which is orthonormal in $L^2(\Omega)$. We get u_n from solving the problem

$$\left\langle \frac{du_n}{dt}, e_k \right\rangle + \langle L_{p,\rho}u_n, e_k\rangle + \langle f(x,u_n), e_k\rangle = \langle g, e_k\rangle,$$
$$(u_n(\tau), e_k) = (u_\tau, e_k), \quad k = 1,\ldots,n.$$

Using the Peano theorem, we get the local existence of u_n. We have

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + \|u_n\|^p_{\mathcal{D}^{1,p}_0(\Omega,\rho)} + \int_\Omega f(x,u_n)u_n = \int_\Omega g(t)u_n.$$

Since $\beta > p + (N/2)(p-2)$ we have $\mathcal{D}^{1,p}_0(\Omega,\rho) \subset L^2(\Omega)$ continuously and therefore there exists $\lambda > 0$ such that $\|u\|^p_{\mathcal{D}^{1,p}_0(\Omega,\rho)} \ge \lambda\|u\|^p_{L^2(\Omega)} \ge$

$\lambda\|u\|^2_{L^2(\Omega)} - \lambda$. Using hypothesis (1.3) and the Cauchy inequality, we get

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + \|u_n\|^p_{\mathcal{D}_0^{1,p}(\Omega,\rho)} + C_2\|u_n\|^q_{L^q(\Omega)} - \|h_2\|_{L^1(\Omega)} \le \frac{1}{2\lambda}\|g(t)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u_n\|^2_{L^2(\Omega)}.$$

Hence

$$\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + \|u_n\|^p_{\mathcal{D}_0^{1,p}(\Omega,\rho)} + 2C_2\|u_n\|^q_{L^q(\Omega)} \le \frac{1}{\lambda}\|g(t)\|^2_{L^2(\Omega)} + 2\|h_2\|_{L^1(\Omega)} + \lambda. \tag{3.1}$$

We show that the solution u_n can be extended to the interval $[\tau,\infty)$. Indeed, from (3.1), we have

$$\frac{d}{dt}\|u_n\|^2_{L^2(\Omega)} + \lambda\|u_n\|^2_{L^2(\Omega)} \le \frac{1}{\lambda}\|g(t)\|^2_{L^2(\Omega)} + 2\|h_2\|_{L^1(\Omega)} + 2\lambda.$$

Then by the Gronwall inequality, we obtain

$$\begin{aligned}\|u_n(t)\|^2_{L^2(\Omega)} &\le \|u_n(\tau)\|^2_{L^2(\Omega)}e^{-\lambda(t-\tau)} + \frac{1}{\lambda}\int_\tau^t e^{-\lambda(t-s)}\|g(s)\|^2_{L^2(\Omega)}\,ds \\ &\quad + (2\|h_2\|_{L^1(\Omega)} + 2\lambda)(1-e^{-\lambda(t-\tau)}) \\ &\le \|u_n(\tau)\|^2_{L^2(\Omega)}e^{-\lambda(t-\tau)} + \frac{1}{\lambda}(1-e^{-\lambda})^{-1}\|g\|^2_{L^2_{\mathrm{b}}} \\ &\quad + (2\|h_2\|_{L^1(\Omega)} + 2\lambda)(1-e^{-\lambda(t-\tau)}),\end{aligned} \tag{3.2}$$

where we have used the fact that

$$\begin{aligned}&\int_\tau^t e^{-\lambda(t-s)}\|g(s)\|^2_{L^2(\Omega)}\,ds \\ &\quad\le \int_t^{t-1} e^{-\lambda(t-s)}\|g(s)\|^2_{L^2(\Omega)}\,ds + \int_{t-2}^{t-1} e^{-\lambda(t-s)}\|g(s)\|^2_{L^2(\Omega)}\,ds + \cdots \\ &\quad\le \int_{t-1}^{t}\|g(s)\|^2_{L^2(\Omega)}\,ds + e^{-\lambda}\int_{t-2}^{t-1}\|g(s)\|^2_{L^2(\Omega)}\,ds + \cdots \\ &\quad\le (1+e^{-\lambda}+e^{-2\lambda}+\cdots)\|g\|^2_{L^2_{\mathrm{b}}} = (1-e^{-\lambda})^{-1}\|g\|^2_{L^2_{\mathrm{b}}}.\end{aligned}$$

We now establish some *a priori* estimates for u_n. Integrating (3.1) on $[\tau,t]$, $\tau < t \le T$, we have

$$\begin{aligned}&\|u_n(t)\|^2_{L^2(\Omega)} + \int_\tau^t\|u_n\|^p_{\mathcal{D}_0^{1,p}(\Omega,\rho)} + 2C_2\int_\tau^t\|u_n\|^q_{L^q(\Omega)} \\ &\qquad\le \|u_n(\tau)\|^2_{L^2(\Omega)} + \int_\tau^t\|g(s)\|^2_{L^2(\Omega)}\,ds + (2\|h_2\|_{L^1(\Omega)} + \lambda)(t-\tau).\end{aligned}$$

The last inequality implies that

$$\begin{aligned}&\{u_n\} \text{ is bounded in } L^\infty(\tau,T;L^2(\Omega)),\\&\{u_n\} \text{ is bounded in } L^p(\tau,T;\mathcal{D}_0^{1,p}(\Omega,\rho)),\\&\{u_n\} \text{ is bounded in } L^q(\tau,T;L^q(\Omega)).\end{aligned}$$

Using hypothesis (1.2), we get

$$\int_\tau^T \|f(x,u_n)\|^{q'}_{L^{q'}(\Omega)} \le \int_\tau^T\int_\Omega (h_1+C_1|u_n|^{q-1})^{q'} \le \int_\tau^T\int_\Omega C(|h_1|^{q'}+|u_n|^q).$$

It follows that $\{f(x,u_n)\}$ is bounded in $L^{q'}(\tau,T;L^{q'}(\Omega))$. For any $v \in L^p(\tau,T;\mathcal{D}_0^{1,p}(\Omega,\rho))$, we have

$$\begin{aligned}|\langle L_{p,\rho}u_n,v\rangle| &= \Big|\int_\tau^T dt\int_\Omega \rho^{(p-1)/p}|\nabla u_n|^{p-2}\nabla u_n(\rho^{1/p}\nabla v)\Big|\\ &\le \|u_n\|^{p/p'}_{L^p(\tau,T;\mathcal{D}_0^{1,p}(\Omega,\rho))}\|v\|_{L^p(\tau,T;\mathcal{D}_0^{1,p}(\Omega,\rho))},\end{aligned}$$

where we have used the Hölder inequality. Because of the boundedness of $\{u_n\}$ in $L^p(\tau,T;\mathcal{D}_0^{1,p}(\Omega,\rho))$, we infer that $\{L_{p,\rho}u_n\}$ is bounded in $L^{p'}(\tau,T;\mathcal{D}^{-1,p}(\Omega,\rho))$. From the above estimates, we have

$$\begin{aligned}u_n &\rightharpoonup u &&\text{in } L^p(\tau,T;\mathcal{D}_0^{1,p}(\Omega,\rho)),\\ f(x,u_n) &\rightharpoonup \eta &&\text{in } L^{q'}(\tau,T;L^{q'}(\Omega)),\\ L_{p,\rho}u_n &\rightharpoonup \chi &&\text{in } L^{p'}(\tau,T;\mathcal{D}^{-1,p}(\Omega,\rho)),\end{aligned}$$

up to a subsequence. By rewriting the equation as

$$\frac{du_n}{dt} = -L_{p,\rho}u_n - f(x,u_n) + g, \tag{3.3}$$

we see that $\{du_n/dt\}$ is bounded in V^*. By hypothesis (H4) and Proposition 2.2, one can take a number $r \in (p^*_\beta, p] \cap \mathcal{I}[p',q']$ such that

$$\mathcal{D}_0^{1,p}(\Omega,\rho) \subset\subset L^r(\Omega). \tag{3.4}$$

Since $r' \in \mathcal{I}[p,q]$, we have

$$L^p(\Omega)\cap L^q(\Omega) \subset L^{r'}(\Omega),$$

and therefore

$$L^r(\Omega) \subset L^{p'}(\Omega) + L^{q'}(\Omega). \tag{3.5}$$

Using Proposition 2.2 again, we see that

$$\mathcal{D}_0^{1,p}(\Omega,\rho) \subset L^p(\Omega).$$

This and (3.5) imply that

$$L^r(\Omega) \subset W^* := \mathcal{D}^{-1,p'}(\Omega,\rho) + L^{q'}(\Omega).$$

Now with (3.4), we have an evolution triple

$$\mathcal{D}_0^{1,p}(\Omega,\rho) \subset\subset L^p(\Omega) \subset W^*. \tag{3.6}$$

The boundedness of $\{u_n'\}$ in V^* ensures that $\{u_n'\}$ is also bounded in $L^s(\tau,T; W^*)$, where $s=\min\{p',q'\}$. Thanks to the Aubin–Lions Lemma [17, p. 58], $\{u_n\}$ is precompact in $L^p(\tau,T;L^r(\Omega))$ and therefore in $L^r(\tau,T;L^r(\Omega))$ since $r\le p$. Hence we can assume that $u_n\to u$ strongly in $L^r(\tau,T;L^r(\Omega))$, so $u_n\to u$ a.e. in $\Omega\times[\tau,T]$ up to a subsequence. Since $f(x,\cdot)$ is continuous, it follows that $f(x,u_n)\to f(x,u)$ a.e. in $\Omega\times[\tau,T]$. Thanks to Lemma 1.3 in [17, Chapter 1], one has

$$f(x,u_n) \rightharpoonup f(x,u) \quad \text{in } L^{q'}(\tau,T;L^{q'}(\Omega)). \tag{3.7}$$

Thus, from (3.3) we have

$$u' = -\chi - f(x,u) + g \quad \text{in } V^*. \tag{3.8}$$

We now show that $\chi = L_{p,\rho}u$. Since $L_{p,\rho}$ is monotone, we have

$$X_n = \int_\tau^T \langle L_{p,\rho}u_n - L_{p,\rho}u, u_n-u\rangle\,dt \ge 0.$$

Note that $\{u_n(T)\}$ is bounded in $L^2(\Omega)$, so $u_n(T)\rightharpoonup u(T)$ in $L^2(\Omega)$, up to a subsequence. Because

$$\int_\tau^T \langle L_{p,\rho}u_n,u_n\rangle\,dt = -\int_\tau^T dt\int_\Omega (f(x,u_n)u_n - gu_n)\,dx + \frac12\|u_n(\tau)\|^2_{L^2(\Omega)} - \frac12\|u_n(T)\|^2_{L^2(\Omega)}, \tag{3.9}$$

we obtain

$$\begin{aligned}\limsup_{n\to\infty} X_n \le {}& -\int_\tau^T (f(x,u),u)\,dt + \frac12\|u(\tau)\|^2_{L^2(\Omega)} - \frac12\|u(T)\|^2_{L^2(\Omega)}\\ &- \int_\tau^T(\chi,v)\,dt - \int_\tau^T (L_{p,\rho}v,u-v)\,dt + \int_\tau^T (g,u)\,dt,\end{aligned} \tag{3.10}$$

where we have used the facts that $u_n(\tau)\to u_\tau$ in $L^2(\Omega)$, $\|u(T)\|_{L^2(\Omega)}\le \liminf_{n\to\infty}\|u_n(T)\|_{L^2(\Omega)}$. On the other hand, by integrating by parts, from (3.7) we have

$$\int_\tau^T (f,u)\,dt + \frac12\|u(\tau)\|^2_{L^2(\Omega)} - \frac12\|u(T)\|^2_{L^2(\Omega)} + \int_\tau^T (g,u)\,dt = \int_\tau^T (\chi,u)\,dt,$$

and therefore thanks to (3.9) and (3.10) one gets

$$\int_\tau^T (\chi - L_{p,\rho} v, u - v)\, dt \geq 0.$$

We now use the hemicontinuity of the operator $L_{p,\rho}$ to prove that $\chi = L_{p,\rho} u$. Taking $v = u - \lambda w$, where $\lambda > 0$ and $w \in L^p(\tau, T; \mathcal{D}_0^{1,p}(\Omega, \rho))$, we obtain

$$\lambda \int_\tau^T (\chi - L_{p,\rho}(u - \lambda w), w)\, dt \geq 0,$$

hence

$$\int_\tau^T (\chi - L_{p,\rho}(u - \lambda w), w)\, dt \geq 0; \tag{3.11}$$

letting $\lambda \to 0$ in (3.11), we conclude that

$$\int_\tau^T (\chi - L_{p,\rho} u, w) dt \geq 0 \quad \text{for all } w.$$

So $\chi = L_{p,\rho} u$. Thus

$$u' = -L_{p,\rho} u - f(x, u) + g \quad \text{in } V^*.$$

It remains to show that $u(\tau) = u_\tau$. Choosing some $\varphi \in C^1([\tau, T]; \mathcal{D}_0^{1,p}(\Omega, \rho) \cap L^q(\Omega))$ with $\varphi(T) = 0$, observe that $\varphi \in V$, so in the "limiting equation" one can integrate by parts in the t variable to obtain

$$\int_\tau^T -(u, \varphi') + \int_\tau^T \int_\Omega \rho(x) |\nabla u|^{p-2} \nabla u \nabla \varphi + \int_\tau^T \int_\Omega (f(x, u) - g)\varphi = (u(\tau), \varphi(\tau)).$$

Doing the same in the Galerkin approximations yields

$$\int_\tau^T -(u_n, \varphi') + \int_\tau^T \int_\Omega \rho(x) |\nabla u_n|^{p-2} \nabla u_n \nabla \varphi + \int_\tau^T \int_\Omega (f(x, u_n) - g)\varphi = (u_n(\tau), \varphi(\tau)).$$

Taking limits as $n \to \infty$ we conclude that

$$\int_\tau^T -(u, \varphi') + \int_\tau^T \int_\Omega \rho(x) |\nabla u|^{p-2} \nabla u \nabla \varphi + \int_\tau^T \int_\Omega (f(x, u) - g)\varphi = (u_\tau, \varphi(\tau)),$$

since $u_n(\tau) \to u_\tau$. Thus, $u(\tau) = u_\tau$.

We now verify the uniqueness and continuous dependence of the solution. Let u_1, u_2 be two solutions of problem (1.1) with the initial data $u_1(\tau), u_2(\tau)$, respectively. From (1.1), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_1-u_2\|^2_{L^2(\Omega)} &\le \frac{1}{2}\frac{d}{dt}\|u_1-u_2\|^2_{L^2(\Omega)} + \langle L_{p,\rho}u_1 - L_{p,\rho}u_2, u_1-u_2\rangle \\ &= -\langle f(x,u_1)-f(x,u_2), u_1-u_2\rangle \\ &\le l\|u_1-u_2\|^2_{L^2(\Omega)}.\end{aligned}$$

Applying the Gronwall lemma, we obtain

$$\|u_1(t)-u_2(t)\|^2_{L^2(\Omega)} \le e^{2l(t-\tau)}\|u_1(\tau)-u_2(\tau)\|_{L^2(\Omega)}.$$

This implies the uniqueness (if $u_1(\tau)=u_2(\tau)$) and the continuous dependence of the solution. ■

4. Existence of uniform attractors. From Theorem 3.1, we can define a family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_{\mathrm{w}}(g)}$, acting on $L^2(\Omega)$ as $U_\sigma(t,\tau)u_\tau = u(t)$, where $u(t)$ is the unique weak solution of problem (1.1) with the initial condition u_τ and the external force $\sigma(t)$. By the uniqueness of the weak solution, we have

$$U_\sigma(t+h,\tau+h) = U_{T(h)\sigma}(t,\tau), \quad \forall\sigma\in\mathcal{H}_{\mathrm{w}}(g),\ t\ge\tau,\ \tau\in\mathbb{R},\ h\ge 0.$$

Here $\mathcal{H}_{\mathrm{w}}(g)$ denotes the closure of the set $\{g(\cdot+h)\mid h\in\mathbb{R}\}$ in $L^2_{\mathrm{b}}(\mathbb{R};L^2(\Omega))$ in the weak topology. We first prove the following

PROPOSITION 4.1. *The family of processes* $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_{\mathrm{w}}(g)}$ *associated to* (1.1) *is* $(L^2(\Omega)\times\mathcal{H}_{\mathrm{w}}(g), L^2(\Omega))$*-weakly continuous, and* $(L^2(\Omega)\times\mathcal{H}_{\mathrm{w}}(g), \mathcal{D}^{1,p}_0(\Omega,\rho)\cap L^q(\Omega))$*-weakly continuous.*

Proof. Fix $t\ge\tau$, $\tau\in\mathbb{R}$ and assume that $u_{\tau_n}\rightharpoonup u_\tau$ weakly in $L^2(\Omega)$ and $\sigma_n\rightharpoonup\sigma_0$ weakly in $\mathcal{H}_{\mathrm{w}}(g)$. Denote $u_n(t)=U_{\sigma_n}(t,\tau)u_{\tau_n}$. Similarly to the proof of Theorem 3.1, the estimates for u_n are valid for $u_n(t)$. Thus, we can find a subsequence $u_m(t)$ of $u_n(t)$ such that $u_m(t)\rightharpoonup w(t)$ weakly in $L^2(\Omega)$ and in $\mathcal{D}^{1,p}_0(\Omega,\rho)\cap L^q(\Omega)$, where w is the solution of problem (1.1) with initial condition u_τ. Due to the uniqueness of the solution, we deduce that $U_{\sigma_m}(t,\tau)u_{\tau_m}\rightharpoonup U_{\sigma_0}(t,\tau)u_\tau$ weakly in $L^2(\Omega)$ and in $\mathcal{D}^{1,p}_0(\Omega,\rho)\cap L^q(\Omega)$. This holds for all subsequences of $\{U_{\sigma_n}(t,\tau)u_{\tau_n}\}$ and therefore also for $\{U_{\sigma_n}(t,\tau)u_{\tau_n}\}$. ■

PROPOSITION 4.2. *The family of processes* $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_{\mathrm{w}}(g)}$ *associated to problem* (1.1) *has a bounded* $(L^2(\Omega),\mathcal{D}^{1,p}_0(\Omega,\rho)\cap L^q(\Omega))$*-uniformly absorbing set.*

Proof. Suppose that $B\subset L^2(\Omega)$ is bounded, $u_\tau\in B$, $\sigma\in\mathcal{H}_{\mathrm{w}}(g)$ and $u=U_\sigma(t,\tau)u_\tau$. Then the same estimates of u_n are valid for u with $g(t)$ replaced by $\sigma(t)$. Similarly to (3.2) we get

$$\begin{aligned}\|u(t)\|^2_{L^2(\Omega)} &\le \|u(\tau)\|^2_{L^2(\Omega)}e^{-\lambda(t-\tau)} + \frac{1}{\lambda}\int_\tau^t e^{-\lambda(t-s)}\|g(s)\|^2_{L^2(\Omega)}\\ &\quad + 2(\|h_2\|_{L^1(\Omega)}+\lambda)(1-e^{-\lambda(t-\tau)})\\ &\le \|u(\tau)\|^2_{L^2(\Omega)}e^{-\lambda(t-\tau)} + \frac{1}{\lambda}(1-e^{-\lambda})^{-1}\|g\|^2_{L^2_{\mathrm{b}}}\\ &\quad + 2(\|h_2\|_{L^1(\Omega)}+\lambda)(1-e^{-\lambda(t-\tau)}).\end{aligned}$$

The last inequality implies that there exists $T_1 = T_1(B,\tau)$ such that

$$\|u(t)\|^2_{L^2(\Omega)} \le C(\|g\|^2_{L^2_{\mathrm{b}}}) \quad \text{for all } t \ge T_1,\ u_\tau \in B,\ \sigma \in \mathcal{H}_{\mathrm{w}}(g). \tag{4.1}$$

By (3.1) and (4.1),

$$\int_t^{t+1} (\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\rho)} + \|u\|^q_{L^q(\Omega)}) \le C_4 \quad \text{for any } t \ge T_1. \tag{4.2}$$

Combining (1.4) with (4.2), we get

$$\int_t^{t+1} \Big(\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\rho)} + \int_\Omega F(x,u)\Big) \le C_5 \quad \text{for any } t \ge T_1. \tag{4.3}$$

On the other hand, multiplying (1.1) by u_t, we obtain

$$\|u_t\|_2^2 + \frac{d}{dt}\Big(\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\rho)} + \int_\Omega F(x,u)\Big) \le \frac{1}{2}\|\sigma(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_t\|^2_{L^2(\Omega)}, \tag{4.4}$$

thus

$$\frac{d}{dt}\Big(\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\rho)} + \int_\Omega F(x,u)\Big) \le \frac{1}{2}\|\sigma(t)\|^2_{L^2(\Omega)}. \tag{4.5}$$

From (4.3) and (4.5), by virtue of the uniform Gronwall lemma [22, p. 91], we get

$$\|u\|^p_{\mathcal{D}_0^{1,p}(\Omega,\rho)} + \int_\Omega F(x,u) \le C_6 \quad \text{for any } t \ge T_1.$$

Hence, using (1.4), we obtain

$$\|u(t)\|^p_{\mathcal{D}_0^{1,p}(\Omega,\rho)} + \|u(t)\|^q_{L^q(\Omega)} \le C_7 \quad \text{for any } t \ge T_1. \tag{4.6}$$

This last inequality implies the existence of a bounded $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega,\rho)\cap L^q(\Omega))$-uniformly absorbing set B_0. ■

The set B_0 is of course a bounded $(L^2(\Omega), L^2(\Omega))$- and $(L^2(\Omega), L^q(\Omega))$-uniformly absorbing set for the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_{\mathrm{w}}(g)}$. By Theorem 2.6, to prove the existence of a uniform attractor, we only need to verify that $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_{\mathrm{w}}(g)}$ is uniformly asymptotically compact.

4.1. $(L^2(\Omega), L^2(\Omega))$**-uniform attractor.** Because $\mathcal{D}_0^{1,p}(\Omega,\rho) \subset L^2(\Omega)$ compactly, $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_{\rm w}(g)}$ is $(L^2(\Omega), L^2(\Omega))$-uniformly asymptotically compact. Thus, we immediately get the following result.

THEOREM 4.3. *The family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_{\rm w}(g)}$ associated to problem* (1.1) *has an $(L^2(\Omega), L^2(\Omega))$-uniform attractor $\mathcal{A}_2$, which is compact in $L^2(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ in the topology of $L^2(\Omega)$. Moreover,*

$$\mathcal{A}_2 = \omega_{\tau,\mathcal{H}_{\rm w}(g)}(B_0),$$

where B_0 is any $(L^2(\Omega), L^2(\Omega))$-uniformly absorbing set in $L^2(\Omega)$.

4.2. $(L^2(\Omega), L^q(\Omega))$**-uniform attractor.** To prove that the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_{\rm w}(g)}$ is $(L^2(\Omega), L^q(\Omega))$-uniformly asymptotically compact, we use the following result (see [7, Corollary 3.12]).

LEMMA 4.4. *Let $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_{\rm w}(g)}$ be a family of processes on $L^2(\Omega)$ that is $(L^2(\Omega), L^2(\Omega))$-uniformly asymptotically compact. Then this family is $(L^2(\Omega), L^q(\Omega))$-uniformly asymptotically compact, $2 \le q \le \infty$, if*

(1) *$\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_{\rm w}(g)}$ has a bounded $(L^2(\Omega), L^q(\Omega))$-uniformly absorbing set B_0;*
(2) *for any $\varepsilon > 0$, $\tau \in \mathbb{R}$ and any bounded subset $B \subset L^2(\Omega)$, there exist positive constants $T = T(B,\varepsilon,\tau)$ and $M = M(\varepsilon)$ such that*

$$\int_{\Omega(|U_\sigma(t,\tau)u_\tau|\ge M)} |U_\sigma(t,\tau)u_\tau|^q\,dx < \varepsilon \text{ for any } u_\tau \in B,\ t \ge T,\ \sigma \in \mathcal{H}_{\rm w}(g).$$

Here $\Omega(|u| \ge M) := \{x \in \Omega \mid |u(x)| \ge M\}$.

THEOREM 4.5. *The family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_{\rm w}(g)}$ associated to problem* (1.1) *has an $(L^2(\Omega), L^q(\Omega))$-uniform attractor $\mathcal{A}_q$, which is compact in $L^q(\Omega)$ and attracts every bounded subset of $L^2(\Omega)$ in the topology of $L^q(\Omega)$. Moreover,*

$$\mathcal{A}_q = \omega_{\tau,\mathcal{H}_{\rm w}(g)}(B_0),$$

where B_0 is any $(L^2(\Omega), L^q(\Omega))$-uniformly absorbing set.

Proof. By Lemma 4.4 and Theorem 2.6, it is sufficient to show that for any $\varepsilon > 0, \tau \in \mathbb{R}$ and any bounded subset $B \subset L^2(\Omega)$, there exist positive constants $T = T(B,\varepsilon,\tau)$ and $M = M(\varepsilon)$ such that

$$\int_{\Omega(|U_\sigma(t,\tau)u_\tau|\ge M)} |U_\sigma(t,\tau)u_\tau|^q\,dx < \varepsilon \quad \text{for any } u_\tau \in B,\ t \ge T,\ \sigma \in \mathcal{H}_{\rm w}(g).$$

Take $M > 0$ large enough such that $\tilde{C}_2|u|^{q-1} \le f(x,u)$ in $\Omega_M = \{x \in \Omega \mid u(x,t) \ge M\}$ and denote

$$u_M^+ = \begin{cases} u - M, & u \ge M, \\ 0, & u \le M. \end{cases}$$

In Ω_M we see that

$$\sigma(t)(u_M^+)^{q-1} \le \frac{\tilde{C}_2}{2}(u_M^+)^{2q-2} + \frac{1}{2\tilde{C}_2}|\sigma(t)|^2 \le \frac{\tilde{C}_2}{2}(u_M^+)^{q-1}|u|^{q-1} + \frac{1}{2\tilde{C}_2}|\sigma(t)|^2,$$

and

$$\begin{aligned} f(u)(u_M^+)^{q-1} &\ge \tilde{C}_2|u|^{q-1}(u_M^+)^{q-1} \ge \frac{\tilde{C}_2}{2}(u_M^+)^{q-1}|u|^{q-1} + \frac{\tilde{C}_2}{2}|u|^{q-2}(u_M^+)^q \\ &\ge \frac{\tilde{C}_2}{2}(u_M^+)^{q-1}|u|^{q-1} + \frac{\tilde{C}_2 M^{q-2}}{2}(u_M^+)^q. \end{aligned}$$

Multiplying equation (1.1) by $|u_M^+|^{q-1}$ and using the above inequalities, we deduce that

$$\frac{1}{q}\frac{d}{dt}\|u_M^+\|_{L^q(\Omega_M)}^q + \frac{q-1}{2}\int_{\Omega_M}\rho(x)|\nabla u_M^+|^p|u_M^+|^{q-2} + \frac{\tilde{C}_2 M^{q-2}}{2}\int_{\Omega_M}|u_M^+|^q \le \int_{\Omega_M}\frac{1}{2\tilde{C}_2}|\sigma|^2.$$

Therefore

$$\frac{d}{dt}\|u_M^+\|_{L^q(\Omega_M)}^q + \frac{\tilde{C}_2 M^{q-2} q}{2}\int_{\Omega_M}|u_M^+|^q \le \int_{\Omega_M}\frac{q}{2\tilde{C}_2}|\sigma|^2.$$

Letting $k = T_B$, where T_B is chosen such that $\|U_\sigma(t,\tau)u_\tau\|_{L^q(\Omega)}^q \le \rho_q$ for all $t \ge T_B$ we deduce that

$$\begin{aligned} \|u_M^+(t)\|_{L^q(\Omega)}^q &\le \|u_M^+(k)\|_{L^q(\Omega)}^q e^{-\lambda(t-k)} + \int_k^t\Big(e^{-\lambda(t-s)}\frac{q}{2\tilde{C}_2}\int_{\Omega_M}|\sigma|^2\Big) \\ &\le \|u_M^+(k)\|_{L^q(\Omega)}^q e^{-\lambda(t-k)} + \frac{q}{2\tilde{C}_2}\int_k^t e^{-\lambda(t-s)}\|\sigma\|_{L^2(\Omega_M)}^2, \end{aligned} \tag{4.7}$$

where $\lambda = \tilde{C}_2 M^{q-2} q/2$. By Lemma 2.5, we have

$$\frac{q}{2\tilde{C}_2}\int_k^t e^{-\lambda(t-s)}\|\sigma\|_{L^2(\Omega_M)}^2 \le \frac{\varepsilon}{2^{q+2}} \tag{4.8}$$

for $\sigma \in \mathcal{H}_{\mathrm{w}}(g)$, $M \ge M_1$ for some $M_1 > 0$.

Letting $T_1 = \frac{1}{\lambda}\ln\big(\frac{2^{q+3}\rho_q}{\varepsilon}\big) + k$, we get

$$\|u_M^+\|_{L^q(\Omega_M)}^q e^{-\lambda(t-k)} \le \frac{\varepsilon}{2^{q+2}} \quad \text{for all } t > T_1. \tag{4.9}$$

From (4.7)–(4.9), we deduce that

$$\int_{\Omega_M}|u_M^+|^q \le \frac{\varepsilon}{2^{q+1}} \quad \text{for } t > T_1,\ \sigma \in \mathcal{H}_{\mathrm{w}}(g),\ M \ge M_1. \tag{4.10}$$

Repeating the same steps above, just taking $|u_M^-|^{q-2}u_M^-$ instead of $|u_M^+|^{q-1}$, there exist M_2 and T_2 such that

$$\int_{\Omega_M} |u_M^-|^q \leq \frac{\varepsilon}{2^{q+1}} \quad \text{for } t > T_2,\ \sigma \in \mathcal{H}_{\mathrm{w}}(g),\ M \geq M_2, \tag{4.11}$$

where

$$u_M^- = \begin{cases} u + M, & u \leq -M, \\ 0, & u \geq -M. \end{cases}$$

Taking $M_3 = \max(M_1, M_2)$, we obtain

$$\int_{\Omega_{M_3}} |(|u| - M_3)|^q \leq 2\varepsilon \quad \text{for } t > \max(T_1, T_2),\ \sigma \in \mathcal{H}_{\mathrm{w}}(g). \tag{4.12}$$

Therefore

$$\begin{aligned} \int_{\Omega_{2M_3}} |u|^q &= \int_{\Omega_{2M_3}} ((|u| - M_3) + M_3)^q \leq 2^q \int_{\Omega_{2M_3}} (|u| - M_3)^q + \int_{\Omega_{2M_3}} M_3^q \\ &\leq 2^q \int_{\Omega_{2M_3}} (|u| - M_3)^q + \int_{\Omega_{2M_3}} (|u| - M_3)^q \leq 2^{q+1} \frac{\varepsilon}{2^{q+1}} = \varepsilon. \end{aligned}$$

This completes the proof. ∎

REMARK 4.1. In fact, if we are only concerned with the existence of the $(L^2(\Omega), L^2(\Omega))$-uniform attractor and the $(L^2(\Omega), L^q(\Omega))$-uniform attractor for the family of processes $\{U(t,\tau)\}$, then the assumption (H2) can be replaced by a weaker assumption: $f \in C(\mathbb{R})$ satisfies

$$\begin{gathered} C_1|u|^q - C_0 \leq f(u)u \leq C_2|u|^q + C_0, \quad q \geq 2, \\ (f(u) - f(v))(u - v) \geq -C|u - v|^2 \quad \text{for any } u, v \in \mathbb{R}, \end{gathered}$$

and we only need to assume that $p > 1$, which ensures that the operator $L_{p,\sigma}$ is monotone (but not strongly monotone when $1 < p < 2$). However, we need to use the stronger assumptions, namely $f \in C^1(\mathbb{R})$ satisfying (H2) and $p \geq 2$, in the next section to prove the existence of an $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega, \sigma) \cap L^q(\Omega))$-uniform attractor.

4.3. $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega, \rho) \cap L^q(\Omega))$**-uniform attractor.** First, we shall prove the following lemma.

LEMMA 4.6. *For any bounded subset $B \subset L^2(\Omega), \tau \in \mathbb{R}$ and $\sigma \in \mathcal{H}_{\mathrm{w}}(g)$, there exists a positive constant $T = T(B, \tau) \geq \tau$ such that*

$$\left\| \frac{d}{dt}(U_\sigma(t,\tau)u_\tau)\Big|_{t=s} \right\|_{L^2(\Omega)}^2 \leq C \quad \textit{for any } u_\tau \in B,\ s \geq T,\ \sigma \in \mathcal{H}_{\mathrm{w}}(g),$$

where C is independent of B and σ.

Proof. We give some formal calculations; a rigorous proof is done by use of Galerkin approximations. Letting $u = U_\sigma(t,\tau)u_\tau$, then differentiating (1.1) with the external force $\sigma \in \mathcal{H}_{\mathrm{w}}(g)$ in time, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L^2(\Omega)} + \int_\Omega \rho|\nabla u|^{p-2}|\nabla u_t|^2\,dx + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u.\nabla u_t)^2 \\ = -\int_\Omega f'(u)|u_t|^2 + \int_\Omega \sigma'(t)u_t \\ \le l\|u_t\|^2_{L^2(\Omega)} + \frac{1}{2}\|\sigma'(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_t\|^2_{L^2(\Omega)}.\end{aligned}$$

Therefore

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L^2(\Omega)} \le \left(l+\frac{1}{2}\right)\|u_t\|^2_{L^2(\Omega)} + \frac{1}{2}\|\sigma'(t)\|^2_{L^2(\Omega)}. \tag{4.13}$$

Using (4.3) and (4.4), we have

$$\int_t^{t+1} \|u_t\|^2_{L^2(\Omega)} \le C \quad \text{for } t \text{ large enough.} \tag{4.14}$$

From (4.13) and (4.14), by the uniform Gronwall lemma [22, p. 91], we get

$$\int_\Omega |u_t|^2 dx \le C \quad \text{for } t \text{ large enough,}$$

where C is independent of σ. The proof is complete. ■

THEOREM 4.7. *The family of processes* $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_{\mathrm{w}}(g)}$ *associated to problem* (1.1) *has an* $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega,\rho)\cap L^q(\Omega))$*-uniform attractor* $\mathcal{A}$*, which is compact in* $\mathcal{D}_0^{1,p}(\Omega,\rho)\cap L^q(\Omega)$ *and attracts every bounded subset of* $L^2(\Omega)$ *in the topology of* $\mathcal{D}_0^{1,p}(\Omega,\rho)\cap L^q(\Omega)$*. Moreover,*

$$\mathcal{A} = \omega_{\tau,\mathcal{H}_{\mathrm{w}}(g)}(B_0),$$

where B_0 *is any* $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega,\rho)\cap L^q(\Omega))$*-uniformly absorbing set.*

Proof. Let B_0 be an $(L^2(\Omega), \mathcal{D}_0^{1,p}(\Omega,\rho)\cap L^q(\Omega))$-uniformly absorbing set. We only need to show that for any $u_{\tau_n} \in B_0$, $\sigma_n \in \mathcal{H}_{\mathrm{w}}(g)$, $t_n \to \infty$, $\{U_{\sigma_n}(t_n,\tau)u_{\tau_n}\}$ is precompact in $\mathcal{D}_0^{1,p}(\Omega,\rho)\cap L^q(\Omega)$. Thanks to Theorem 4.3, it is sufficient to verify that for any $u_{\tau_n} \in B_0$, $\sigma_n \in \mathcal{H}_{\mathrm{w}}(g)$, $t_n \to \infty$, $\{U_{\sigma_n}(t_n,\tau)u_{\tau_n}\}$ is precompact in $\mathcal{D}_0^{1,p}(\Omega,\rho)$.

By Theorems 3.1 and 4.3, we can assume that $U_{\sigma_n}(t,\tau)u_{\tau_n}$ is a Cauchy sequence in both $L^q(\Omega)$ and $L^2(\Omega)$. We will prove that $U_{\sigma_n}(t_n,\tau)u_{\tau_n}$ is a Cauchy sequence in $\mathcal{D}_0^{1,p}(\Omega,\rho)$. Denote $u_n^{\sigma_n}(t_n) = U_{\sigma_n}(t_n,\tau)u_{\tau_n}$. By the strong monotonicity of the operator $L_{p,\rho}$ when $p \ge 2$, we have

$$
\begin{aligned}
&\delta\|u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)\|^p_{\mathcal{D}_0^{1,p}(\Omega,\rho)}\\
&\quad\le \langle L_{p,\rho}u_n^{\sigma_n}(t_n)-L_{p,\rho}u_m^{\sigma_m}(t_m),u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)\rangle\\
&\quad\le \int_\Omega\left|\frac{d}{dt}u_n^{\sigma_n}(t_n)-\frac{d}{dt}u_m^{\sigma_m}(t_m)\right||u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)|\,dx\\
&\qquad+\int_\Omega|f(x,u_n^{\sigma_n}(t_n))-f(x,u_m^{\sigma_m}(t_m))|\,|u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)|\,dx\\
&\qquad+\int_\Omega|\sigma_n(t_n)-\sigma_m(t_m)|\,|u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)|\,dx\\
&\quad\le\left\|\frac{d}{dt}u_n^{\sigma_n}(t_n)-\frac{d}{dt}\,u_m^{\sigma_m}(t_m)\right\|_{L^2(\Omega)}\|u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)\|_{L^2(\Omega)}\\
&\qquad+C\|\sigma_n(t_n)-\sigma_m(t_m)\|_{L^2(\Omega)}\|u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)\|_{L^2(\Omega)}\\
&\qquad+\|f(x,u_n^{\sigma_n}(t_n))-f(x,u_m^{\sigma_m}(t_m))\|_{L^{q'}(\Omega)}\|u_n^{\sigma_n}(t_n)-u_m^{\sigma_m}(t_m)\|_{L^q(\Omega)}.
\end{aligned}
$$

Thanks to Lemma 4.6 and since $\{f(x,u_n(t_n))\}$ is bounded in $L^{q'}(\Omega)$, the proof is complete. ■

5. Some remarks on the case of a bounded domain. In this section we discuss the case of a bounded domain $\Omega\subset\mathbb{R}^N$, $N\ge 2$, and the weight function $\rho(x)$ satisfying condition $(\mathcal{H}_\alpha)$. Under conditions (H1)–(H4) when Ω is bounded, using Proposition 2.1 and repeating the arguments in the case of an unbounded domain, one can prove that all results in Sections 3 and 4 are still valid in this case.

Note that in the case of a bounded domain and $\rho(x)$ satisfying condition $(\mathcal{H}_\alpha)$, problem (1.1) contains some important classes of parabolic equations, such as semilinear heat equations (when $\rho=\text{const}>0$, $p=2$), semilinear degenerate parabolic equations (when $p=2$), p-Laplacian equations (when $\rho=1$, $p\neq 2$), etc. Thus, in some sense, our results recover/extend some known results on the existence and long-time behavior of solutions to nonautonomous semilinear heat equations and nonautonomous p-Laplacian equations in bounded domains [7, 8, 10, 20].

References

[1] C. T. Anh, N. D. Binh and L. T. Thuy, *On the global attractors for a class of semilinear degenerate parabolic equations*, Ann. Polon. Math. 98 (2010), 71–89.

[2] C. T. Anh and P. Q. Hung, *Global existence and long-time behavior of solutions to a class of degenerate parabolic equations*, ibid. 93 (2008), 217–230.

[3] C. T. Anh and T. D. Ke, *Long-time behavior for quasilinear parabolic equations involving weighted p-Laplacian operators*, Nonlinear Anal. 71 (2009), 4415–4422.

[4] L. Caffarelli, R. Kohn and L. Nirenberg, *First order interpolation inequalities with weights*, Compos. Math. 53 (1984), 259–275.

[5] P. Caldiroli and R. Musina, *On a variational degenerate elliptic problem*, Nonlinear Differential Equations Appl. 7 (2000), 187–199.

[6] A. N. Carvalho, J. W. Cholewa and T. Dlotko, *Global attractors for problems with monotone operators*, Boll. Uni. Mat. Ital. Sez. B (8) 2 (1999), 693–706.

[7] G. X. Chen and C. K. Zhong, *Uniform attractors for nonautonomous p-Laplacian equations*, Nonlinear Anal. 68 (2008), 3349–3363.

[8] V. V. Chepyzhov and M. I. Vishik, *Attractors of nonautonomous dynamical systems and their dimension*, J. Math. Pures Appl. 73 (1994), 279–333.

[9] —, —, *Nonautonomous evolutionary equations with translation compact symbols and their attractors*, C. R. Acad. Sci. Paris Sér. I 321 (1995), 153–158.

[10] —, —, *Attractors for Equations of Mathematical Physics*, Amer. Math. Soc. Colloq. Publ. 49, Amer. Math. Soc., Providence, RI, 2002.

[11] J. W. Cholewa and T. Dlotko, *Global Attractors in Abstract Parabolic Problems*, Cambridge Univ. Press, 2000.

[12] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology, Vol. I: Physical origins and classical methods*, Springer, Berlin, 1985.

[13] L. C. Evans, *Partial Differential Equations*, Grad. Stud. Math. 19, Amer. Math. Soc., 1998.

[14] A. Haraux, *Systèmes Dynamiques Dissipatifs et Applications*, Paris, Masson, 1991.

[15] N. I. Karachalios and N. B. Zographopoulos, *Convergence towards attractors for a degenerate Ginzburg–Landau equation*, Z. Angew. Math. Phys. 56 (2005), 11–30.

[16] —, —, *On the dynamics of a degenerate parabolic equation: Global bifurcation of stationary states and convergence*, Calc. Var. Partial Differential Equations 25 (2006), 361–393.

[17] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[18] S. S. Lu, H. Q. Wu and C. K. Zhong, *Attractors for nonautonomous 2D Navier–Stokes equations with normal external force*, Discrete Contin. Dynam. Systems 23 (2005), 701–719.

[19] Q. F. Ma, S. H. Wang and C. K. Zhong, *Necessary and sufficient conditions for the existence of global attractor for semigroups and applications*, Indiana Univ. Math. J. 51 (2002), 1541–1559.

[20] H. Song and C. K. Zhong, *Attractors of nonautonomous reaction-diffusion equations in* L^p, Nonlinear Anal. 68 (2008), 1890–1897.

[21] C. Y. Sun, S. Y. Wang and C. K. Zhong, *Global attractors for a nonclassical diffusion equation*, Acta Math. Sinica (English Ser.) 23 (2007), 1271–1280.

[22] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, 2nd ed., Springer, 1997.

[23] M. H. Yang, C. Y. Sun and C. K. Zhong, *Global attractors for p-Laplacian equations*, J. Math. Anal. Appl. 327 (2007), 1130–1142.

[24] C. K. Zhong, M. H. Yang and C. Y. Sun, *The existence of global attractors for the norm-to-weak continuous semigoup and application to the nonlinear reaction-diffusion equations*, J. Differential Equations 15 (2006), 367–399.

Cung The Anh (corresponding author), Nguyen Van Quang
Department of Mathematics
Hanoi National University of Education
136 Xuan Thuy, Cau Giay
Hanoi, Vietnam
E-mail: anhctmath@hnue.edu.vn
quangnvmath@gmail.com

Received 17.8.2009
and in final form 23.10.2009 (2061)