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Asymptotic behavior of the sectional curvature of the Bergman metric for annuli

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Abstract. We extend and simplify results of [Din 2010] where the asymptotic behavior of the holomorphic sectional curvature of the Bergman metric in annuli is studied. Similarly to [Din 2010] the description enables us to construct an infinitely connected planar domain (in our paper it is a Zalcman type domain) for which the supremum of the holomorphic sectional curvature is two, whereas its infimum is equal to $-\infty$.

For a domain $D \subset \mathbb{C}^n$, $j = 0, 1, \ldots$, $z \in D$, $X \in \mathbb{C}^n$ define

$$J_D^{(j)}(z;X) := \sup\{|f^{(j)}(z)(X)|^2 : f \in L_h^2(D),\ f(z) = 0, \ldots, f^{(j-1)}(z) = 0,\ \|f\|_{L^2(D)} \le 1\}.$$

Note that the functions above are the squares of the operator norms of continuous operators defined on a closed subspace of $L_h^2(D)$.

Let us restrict ourselves to the case when D is bounded. Note that $J_D^{(0)}(z;X)$ is independent of $X \neq 0$ and is equal to the Bergman kernel $K_D(z,z)$. Moreover, we may express the Bergman metric as $\beta_D^2(z;X) = J_D^{(1)}(z;X)/J_D^{(0)}(z;X)$, $X \neq 0$. And finally the sectional curvature is given by the formula

$$R_D(z;X) = 2 - \frac{J_D^{(0)}(z;X) J_D^{(2)}(z;X)}{J_D^{(1)}(z;X)^2}, \qquad X \neq 0.$$

Below we list a number of simple properties of the above functions.

The transformation formula for a biholomorphic mapping $F : D_1 \to D_2$ is

$$J_{D_1}^{(j)}(z;X) = |\det F'(z)|^2 J_{D_2}^{(j)}(F(z); F'(z)X),$$

from which we get, among other things, the biholomorphic invariance of the sectional curvature: $R_{D_1}(z;X) = R_{D_2}(F(z); F'(z)X)$.

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If $D_1 \subset D_2$ then $J_{D_1}^{(j)} \geq J_{D_2}^{(j)}$.

We shall also need the continuity property of the functions just introduced with respect to increasing families of domains.

Proposition 1.

(1) *Let D be a bounded domain in $\mathbb{C}^n$. Let $D = \bigcup_{\nu=1}^{\infty} D_\nu$ where $D_\nu \subset D_{\nu+1}$, D_ν is a domain in $\mathbb{C}^n$. Then for any j the sequence $(J_{D_\nu}^{(j)})_\nu$ is increasing and convergent locally uniformly on $D\times\mathbb{C}^n$ to $J_D^{(j)}$. In particular, the sequence (β_{D_ν}) (respectively, $(R_{D_\nu})_\nu$) is locally uniformly convergent to β_D (respectively, R_D) on $D \times (\mathbb{C}^n \setminus \{0\})$.*

(2) *Let D be a bounded domain in $\mathbb{C}^n$. Assume that $D = \bigcup_{\nu=1}^{\infty} G_\nu$ where G_ν is a domain in $\mathbb{C}^n$. Assume additionally that for any compact set $K \subset D$ there is a ν_0 such that $K \subset G_\nu$ for any $\nu \geq \nu_0$. Then the sequence $(J_{G_\nu}^{(j)})_{\nu=1}^{\infty}$ is locally uniformly conergent to $J_D^{(j)}$. In particular, the sequence (β_{G_ν}) (respectively, (R_{G_ν})) is locally uniformly convergent to β_D (respectively, R_D) on $D \times (\mathbb{C}^n \setminus \{0\})$.*

For a domain $D \subset \mathbb{C}$, $z \in D$ we put $J_D^{(j)}(z) := J_D^{(j)}(z;1)$, $\beta_D(z) := \beta_D(z;1)$, $R_D(z) := R_D(z;1)$. Recall that $J_D^{(j)} = J_{D\setminus A}^{(j)}$ on $D \setminus A$ where A is any closed polar set in D such that $D \setminus A$ is connected.

Denote $P(\lambda_0, r, R) := \{\lambda \in \mathbb{C} : r < |\lambda - \lambda_0| < R\}$, $0 \leq r < R \leq \infty$, $\lambda_0 \in \mathbb{C}$. We also put $P(r,R) := P(0,r,R)$.

We are going to prove the following result.

Theorem 2. *Let $r \in (0,1)$, $\alpha \in (0,1)$. Then*

$$r^{2\alpha} J_{P(r,1)}^{(0)}(r^\alpha) \sim \frac{1}{-\log r}, \qquad r^{4\alpha} J_{P(r;1)}^{(1)}(r^\alpha) \sim \frac{2r^{2\alpha} + 2r^{2(1-\alpha)}}{1-r^2},$$

$$r^{6\alpha} J_{P(r;1)}^{(2)}(r^\alpha) = \frac{A(r)}{B(r)},$$

where

$$A(r) \sim \frac{r^2}{(1-r^2)^2}(-2^4) + \frac{r^{6(1-\alpha)}}{(1-r^2)(1-r^4)}(A) + \frac{r^{6\alpha}}{(1-r^2)(1-r^4)}(-2^5),$$

$$B(r) \sim \frac{2r^{2\alpha} + 2r^{2(1-\alpha)}}{1-r^2},$$

for some $A < -100$. The symbol $\varphi(r) \sim \psi(r)$ means that for any sufficiently small $\varepsilon > 0$ one has $\varphi(r) - \psi(r) = \psi(r) o(r^\varepsilon)$.

In particular,

$$\lim_{r\to 0^+} R_{P(r;1)}(r^\alpha) = \begin{cases} -\infty & \text{for } \alpha \in (1/3, 2/3), \\ 2 & \text{for } \alpha \in (0, 1/3] \cup [2/3, 1). \end{cases}$$

The above theorem gives a generalization of a result from [Din 2010] (where the cases $\alpha = 1/2$, $\alpha = 0.3$ and $\alpha = 0.7$ have been handled). Additionally, we present in Remark 4 the precise asymptotic behavior of $R_{P(r;1)}(r^\alpha)$ as $r \to 0^+$.

As in [Din 2010], we can make use of Theorem 2 to construct an infinitely connected planar bounded domain with the supremum of the sectional curvature equal to 2 and its infimum equal to $-\infty$. The domain constructed by us is a Zalcman-type domain (unlike that in [Din 2010]) and the proof of the above fact does not use, in contrast to [Din 2010], any sophisticated means.

Recall that the example from [Din 2010] (and certainly also the one in Corollary 3) may be seen as the final one in presenting examples where the supremum of the sectional curvature may be 2 (see [Chen-Lee 2009]) or its infimum may be equal to $-\infty$ (see [Her 2007])—the example has both properties simultaneously.

COROLLARY 3. *Let $\theta \in (0,1)$. Then there is a strictly increasing sequence $(n_k)_k$ of positive integers such that $\bar{\triangle}(\theta^{n_k}, \theta^{2n_k}) \cap \bar{\triangle}(\theta^{n_l}, \theta^{2n_l}) = \emptyset$ for $k \neq l$, $\bar{\triangle}(\theta^{n_k}, \theta^{2n_k}) \subset \frac{1}{2}\mathbb{D}$ and*

$$\sup\{R_D(z) : z \in D\} = 2, \qquad \inf\{R_D(z) : z \in D\} = -\infty,$$

where $D = \frac{1}{2}\mathbb{D} \setminus \left(\bigcup_{k=1}^\infty \bar{\triangle}(\theta^{n_k}, \theta^{2n_k}) \cup \{0\}\right)$ and $\bar{\triangle}(a, r) := \{z \in \mathbb{C} : |z - a| \leq r\}$.

Proof of Theorem 2. We start with the analysis of some more general situation. For $0 < r < R$ denote $\alpha_n^{r,R} := \|\lambda^n\|^2_{P(r,R)}$, $n \in \mathbb{Z}$.

Note that

$$\frac{1}{2\pi}\,\alpha_n^{r,R} = \begin{cases} \dfrac{R^{2(n+1)} - r^{2(n+1)}}{2(n+1)}, & n \neq -1, \\ \log R - \log r, & n = -1. \end{cases}$$

For $f \in L_h^2(P(r,R))$, $f(\lambda) = \sum_{n\in\mathbb{Z}} a_n \lambda^n$ we have the following identity:

$$\|f\|^2_{P(r,R)} = \sum_{n\in\mathbb{Z}} |a_n|^2 \alpha_n^{r,R}.$$

Assume now that $r < 1 < R$. Notice that

$$\begin{aligned} |f(1)|^2 &= \Big|\sum_{n\in\mathbb{Z}} a_n\Big|^2 = \Big|\sum_{n\in\mathbb{Z}} a_n \sqrt{\alpha_n^{r,R}}\,\frac{1}{\sqrt{\alpha_n^{r,R}}}\Big|^2 \\ &\leq \sum_{n\in\mathbb{Z}} |a_n|^2 \alpha_n^{r,R} \sum_{n\in\mathbb{Z}} \frac{1}{\alpha_n^{r,R}} = \|f\|^2_{P(r,R)} \sum_{n\in\mathbb{Z}} \frac{1}{\alpha_n^{r,R}}. \end{aligned}$$

Therefore, $J^{(0)}_{P(r,R)}(1) \leq \sum_{n\in\mathbb{Z}} 1/\alpha_n^{r,R}$. In fact, we have equality above—it is sufficient to take $f \in L_h^2(P(r,R))$ with $a_n = 1/\alpha_n^{r,R}$.

Our next aim is to give a formula for $J^{(1)}_{P(r,R)}(1)$ (which together with the previous one and general properties of the Bergman metric gives a formula for the Bergman metric of an arbitrary annulus at any point—see Remark 4).

We prove the equality

$$J^{(1)}_{P(r,R)}(1)=\sum_{n\in\mathbb{Z}}\frac{(n-\beta)^2}{\alpha_n^{r,R}} \tag{1}$$

for a suitably chosen $\beta\in\mathbb{R}$ (to be given precisely later).

Let us start with $f\in L^2_h(P_{r,R})$ of the form $f(\lambda)=\sum_{n\in\mathbb{Z}}a_n\lambda^n$ such that $\sum_{n\in\mathbb{Z}}a_n=f(1)=0$. For such an f the following estimate holds:

$$\begin{aligned}|f'(1)|^2&=\Big|\sum_{n\in\mathbb{Z}}na_n\Big|^2=\Big|\sum_{n\in\mathbb{Z}}(n-\beta)a_n\Big|^2=\Big|\sum_{n\in\mathbb{Z}}\frac{n-\beta}{\sqrt{\alpha_n^{r,R}}}\,a_n\sqrt{\alpha_n^{r,R}}\Big|^2\\&\le\sum_{n\in\mathbb{Z}}\frac{(n-\beta)^2}{\alpha_n^{r,R}}\sum_{n\in\mathbb{Z}}|a_n|^2\alpha_n^{r,R}=\sum_{n\in\mathbb{Z}}\frac{(n-\beta)^2}{\alpha_n^{r,R}}\,\|f\|^2_{P(r,R)}.\end{aligned}$$

This gives the inequality "$\le$" (with arbitrary β). Now we take f with $a_n=n-\beta/\alpha_n^{r,R}$, where β is such that $\sum_{n\in\mathbb{Z}}a_n=f(1)=0$. If such a β could be found we would get the equality in (1). But this means that we need to find a β such that $\sum_{n\in\mathbb{Z}}(n-\beta)/\alpha_n^{r,R}=0$, which is satisfied exactly if

$$\beta=\frac{\sum_{n\in\mathbb{Z}}\frac{n}{\alpha_n^{r,R}}}{\sum_{n\in\mathbb{Z}}\frac{1}{\alpha_n^{r,R}}}.$$

Consequently, with this β we get the equality

$$\begin{aligned}J^{(1)}_{P(r,R)}(1)&=\sum_{n\in\mathbb{Z}}\frac{(n-\beta)^2}{\alpha_n^{r,R}}=\sum_{n\in\mathbb{Z}}\frac{n^2-\beta n}{\alpha_n^{r,R}}+\beta\sum_{n\in\mathbb{Z}}\frac{\beta-n}{\alpha_n^{r,R}}\\&=\sum_{n\in\mathbb{Z}}\frac{n^2-n\beta}{\alpha_n^{r,R}}=\frac{\varphi_{r,R}(2)\varphi_{r,R}(0)-\varphi_{r,R}(1)^2}{\varphi_{r,R}(0)},\end{aligned}$$

where $\varphi_{r,R}(j):=\sum_{n\in\mathbb{Z}}n^j/\alpha_n^{r,R}$.

Let us now go on to the case of the annulus $P(r,1)$ where $0<r<1$. Our aim is to get the asymptotic behavior of the curvature of $P(r,1)$ at r^α (for a fixed $\alpha\in(0,1)$) as $r\to0^+$. First recall that

$$J^{(j)}_{P(r,1)}(r^\alpha)=r^{-2(j+1)\alpha}J^{(j)}_{P(r^{1-\alpha},r^{-\alpha})}(1).$$

For simplicity we shall write $\alpha_n=\alpha_n^{r^{1-\alpha},r^{-\alpha}}$ and $J^{(j)}(1)=J^{(j)}_{P(r^{1-\alpha},r^{-\alpha})}(1)$.

Then we get the following formulas:

$$\frac{\alpha_n}{2\pi} = \begin{cases} \dfrac{1 - r^{2(n+1)}}{2(n+1)r^{2(n+1)\alpha}}, & n \neq -1, \\ -\log r, & n = -1. \end{cases}$$

From now on we forget about the constant 2π. Skipping the factor $1/(2\pi)$ is justified by the formula for $2 - R(r, \alpha)$ in Remark 4.

Note that for $n \geq 0$ the following formula holds:

$$\alpha_{-n-2} = \frac{1 - r^{2(n+1)}}{2(n+1)r^{2(n+1)(1-\alpha)}}.$$

Let us define (for $j = 0, 1, \ldots$)

$$\begin{aligned} \varphi(j) &:= \sum_{n\in\mathbb{Z}} \frac{n^j}{\alpha_n} \\ &= \frac{(-1)^j}{-\log r} + \sum_{n=0}^{\infty} \frac{2(n+1)}{1 - r^{2(n+1)}} (n^j r^{2(n+1)\alpha} + (-1)^j (n+2)^j r^{2(n+1)(1-\alpha)}) \\ &=: \frac{(-1)^j}{-\log r} + \psi(j). \end{aligned}$$

Then we have

$$J^{(0)}(1) = \varphi(0), \qquad J^{(1)}(1) = \frac{\varphi(2)\varphi(0) - \varphi(1)^2}{\varphi(0)}.$$

Note that the above formulas depend on r and α.

Our next aim is to find a formula for $J^{(2)}(1)$. We proceed as above.

Let us start with $f \in \mathcal{O}(P(r^{1-\alpha}, r^{-\alpha}))$ with $f(\lambda) = \sum_{n\in\mathbb{Z}} a_n \lambda^n$ such that $\sum_{n\in\mathbb{Z}} a_n = f(1) = 0$ and $\sum_{n\in\mathbb{Z}} n a_n = f'(1) = 0$. Then

$$\begin{aligned} |f''(1)|^2 &= \Big|\sum_{n\in\mathbb{Z}} n(n-1)a_n\Big|^2 = \Big|\sum_{n\in\mathbb{Z}} (n^2 - \beta n - \gamma)a_n\Big|^2 \\ &= \Big|\sum_{n\in\mathbb{Z}} \frac{n^2 - \beta n - \gamma}{\sqrt{\alpha_n}} a_n \sqrt{\alpha_n}\Big|^2 \leq \sum_{n\in\mathbb{Z}} \frac{(n^2 - \beta n - \gamma)^2}{\alpha_n} \sum_{n\in\mathbb{Z}} |a_n|^2 \alpha_n. \end{aligned}$$

As before if we find β, γ such that for $a_n = (n^2 - \beta n - \gamma)/\alpha_n$ the equalities $\sum_{n\in\mathbb{Z}} n a_n = \sum_{n\in\mathbb{Z}} a_n = 0$ hold then we shall have the equality

$$J^{(2)}(1) = \sum_{n\in\mathbb{Z}} \frac{(n^2 - \beta n - \gamma)^2}{\alpha_n} = \sum_{n\in\mathbb{Z}} \frac{n^2(n^2 - \beta n - \gamma)}{\alpha_n}.$$

The above properties are satisfied iff for some $\beta, \gamma \in \mathbb{R}$,

$$\sum_{n\in\mathbb{Z}} \frac{n^2 - \beta n - \gamma}{\alpha_n} = 0, \qquad \sum_{n\in\mathbb{Z}} n\, \frac{n^2 - \beta n - \gamma}{\alpha_n} = 0.$$

The above is equivalent to the following system:

$$\begin{cases} \beta \sum_{n\in\mathbb{Z}} \dfrac{n}{\alpha_n} + \gamma \sum_{n\in\mathbb{Z}} \dfrac{1}{\alpha_n} = \sum_{n\in\mathbb{Z}} \dfrac{n^2}{\alpha_n}, \\ \beta \sum_{n\in\mathbb{Z}} \dfrac{n^2}{\alpha_n} + \gamma \sum_{n\in\mathbb{Z}} \dfrac{n}{\alpha_n} = \sum_{n\in\mathbb{Z}} \dfrac{n^3}{\alpha_n}. \end{cases}$$

Since $\big(\sum_{n\in\mathbb{Z}} \frac{n}{\alpha_n}\big)^2 - \sum_{n\in\mathbb{Z}} \frac{n^2}{\alpha_n} \sum_{n\in\mathbb{Z}} \frac{1}{\alpha_n} < 0$, the above system has the unique solution

$$\beta = \frac{\sum_{n\in\mathbb{Z}} \frac{n^2}{\alpha_n} \sum_{n\in\mathbb{Z}} \frac{n}{\alpha_n} - \sum_{n\in\mathbb{Z}} \frac{n^3}{\alpha_n} \sum_{n\in\mathbb{Z}} \frac{1}{\alpha_n}}{\big(\sum_{n\in\mathbb{Z}} \frac{n}{\alpha_n}\big)^2 - \sum_{n\in\mathbb{Z}} \frac{n^2}{\alpha_n} \sum_{n\in\mathbb{Z}} \frac{1}{\alpha_n}} = \frac{\varphi(2)\varphi(1) - \varphi(3)\varphi(0)}{\varphi(1)^2 - \varphi(2)\varphi(0)},$$

$$\gamma = \frac{\sum_{n\in\mathbb{Z}} \frac{n}{\alpha_n} \sum_{n\in\mathbb{Z}} \frac{n^3}{\alpha_n} - \big(\sum_{n\in\mathbb{Z}} \frac{n^2}{\alpha_n}\big)^2}{\big(\sum_{n\in\mathbb{Z}} \frac{n}{\alpha_n}\big)^2 - \sum_{n\in\mathbb{Z}} \frac{n^2}{\alpha_n} \sum_{n\in\mathbb{Z}} \frac{1}{\alpha_n}} = \frac{\varphi(1)\varphi(3) - \varphi(2)^2}{\varphi(1)^2 - \varphi(2)\varphi(0)}.$$

Therefore, we may write

$$\begin{aligned} J^{(2)}(1) &= \varphi(4) - \beta\varphi(3) - \gamma\varphi(2) \\ &= \frac{\varphi(4)\varphi(1)^2 - \varphi(4)\varphi(2)\varphi(0) - 2\varphi(3)\varphi(2)\varphi(1) + \varphi(3)^2\varphi(0) + \varphi(2)^3}{\varphi(1)^2 - \varphi(2)\varphi(0)}. \end{aligned}$$

So let us fix $\alpha \in (0,1)$. Then for any $\varepsilon > 0$ small enough,

$$\varphi(0) = \frac{1}{-\log r} + \frac{2r^{2\alpha}}{1-r^2} + \frac{2r^{2(1-\alpha)}}{1-r^2} + o(r^{2\alpha+\varepsilon}) + o(r^{2(1-\alpha)+\varepsilon}).$$

The asymptotic behavior of $\varphi(1)^2 - \varphi(2)\varphi(0)$ is the following. The coefficient of the expression of highest order (i.e. of $\frac{1}{(-\log r)^2}$) vanishes and the coefficient of $\frac{1}{-\log r}$ is

$$-(\psi(2) + \psi(0) + 2\psi(1)) = -\sum_{n=0}^{\infty} \frac{2(n+1)^3}{1-r^{2(n+1)}} (r^{2(n+1)\alpha} + r^{2(n+1)(1-\alpha)}).$$

The remaining expressions are $\psi(1)^2 - \psi(2)\psi(0)$. Therefore, one can easily verify that the asymptotic behavior is the following: for any $\varepsilon > 0$ small enough,

$$\varphi(1)^2 - \varphi(2)\varphi(0) = \frac{1}{-\log r}\left(\frac{2r^{2\alpha}}{1-r^2} + \frac{2r^{2(1-\alpha)}}{1-r^2}\right) + o(r^{2\alpha+\varepsilon}) + o(r^{2(1-\alpha)+\varepsilon}).$$

We still have to consider the asymptotic behavior of the expression $\varphi(4)\varphi(1)^2 - \varphi(4)\varphi(2)\varphi(0) - 2\varphi(3)\varphi(2)\varphi(1) + \varphi(3)^2\varphi(0) + \varphi(2)^3$.

First note that the coefficients of the expressions $\frac{1}{(-\log r)^j}$, $j = 2, 3$, vanish. On the other hand, the coefficient of $\frac{1}{-\log r}$ is

$$\psi(1)^2 - 2\psi(1)\psi(4) - \psi(4)\psi(2) - \psi(4)\psi(0) - \psi(2)\psi(0)$$
$$- 2(-\psi(2)\psi(1) + \psi(3)\psi(1) - \psi(2)\psi(3)) + \psi^2(3) - 2\psi(0)\psi(3) + 3\psi(2)^2.$$

One can calculate that for any $\varepsilon > 0$ small enough the last expression equals

$$\frac{r^2}{(1-r^2)^2}(-2^4) + \frac{r^{6(1-\alpha)}}{(1-r^2)(1-r^4)}(A)$$
$$+ \frac{r^{6\alpha}}{(1-r^2)(1-r^4)}(-2^5) + o(r^2) + o(r^{6(1-\alpha)+\varepsilon}) + o(r^{6\alpha+\varepsilon})$$

for some $A < -100$.

Combining all the above results we easily get the desired asymptotic behavior as claimed in the theorem. ∎

Remark 4. Recall the formula for the curvature

$$R_{P(r,1)}(r^\alpha) = 2 - R(r,\alpha) := 2 - \frac{J^{(0)}(1)J^{(2)}(1)}{(J^{(1)}(1))^2}.$$

Then the result of Theorem 2 gives, in particular, the asymptotic behavior of the expression $R(r,\alpha)$ (and consequently the asymptotic behavior of the holomorphic curvature) as $r \to 0^+$, which is as follows:

$$\begin{cases} \dfrac{1}{-\log r} & \text{for } \alpha \in (0, 1/3], \\ \dfrac{1}{r^{6\alpha-2}(-\log r)} & \text{for } \alpha \in (1/3, 1/2], \\ \dfrac{1}{r^{6(1-\alpha)-2}(-\log r)} & \text{for } \alpha \in (1/2, 2/3), \\ \dfrac{1}{-\log r} & \text{for } \alpha \in [2/3, 1). \end{cases}$$

Remark 5. Note that in the proof of Theorem 2 we have obtained a formula for the Bergman kernel and metric in the annulus (compare [Her 1983], [Jar-Pfl 1993]) and a relatively simple expression for the sectional curvature of the annulus.

Proof of Corollary 3. We construct inductively sequences (n_k), (x_k), (y_k) and (r_k) such that $\theta^{n_1} + \theta^{2n_1} < x_1, y_1 < 1/2$ and for any $k = 1, 2, \ldots$ the following properties hold: $\theta^{n_{k+1}} + \theta^{2n_{k+1}} < x_{k+1}, y_{k+1} < \theta^{n_k} - \theta^{2n_k}$, $\theta^{n_{k+1}} + \theta^{2n_{k+1}} < r_{k+1} < \theta^{n_k} - \theta^{2n_k}$ and for any compact $L \subset \bar{\triangle}(0, r_{k+1})$ for which $\Omega = \frac{1}{2}\mathbb{D} \setminus (\bigcup_{j=1}^k \bar{\triangle}(\theta^{n_j}, \theta^{2n_j}) \cup L)$ is connected the inequalities $R_\Omega(x_j) > 2 - 1/j$, $R_\Omega(y_j) < -j$ hold for any $j = 1, \ldots, k$.

Then we put $D := \frac{1}{2}\mathbb{D} \setminus (\bigcup_{j=1}^\infty \bar{\triangle}(\theta^{n_j}, \theta^{2n_j}) \cup \{0\})$. From the properties stated above we will obtain the inequalities $R_D(x_k) > 2 - 1/k$, $R_D(y_k) < -k$, which finishes the proof.

We turn to the construction of the above sequences. We put $r_1 := 1/4$. The possibility of choosing n_1, x_1, y_1 as desired follows from Theorem 2 together with the biholomorphic invariance of the sectional curvature (we have to choose n_1 sufficiently large). The possibility of choosing r_2 follows from Proposition 1. Now assume the system as above has been chosen for $j = 1, \dots, k$ (with the choice of n_j, x_j, y_j, $j = 1, \dots, k$, and r_j, $j = 1, \dots, k+1$).

First note that choosing $n_{k+1} > n_k$ so that $\theta^{n_{k+1}} + \theta^{2n_{k+1}} < r_{k+1}$ we achieve that the recursively defined set $D_{k+1} = \frac{1}{2}\mathbb{D} \setminus (\bigcup_{j=1}^{k+1} \bar{\triangle}(\theta^{n_j}, \theta^{2n_j}))$ satisfies $R_{D_{k+1}}(x_j) > 2 - 1/j$, $R_{D_{k+1}}(y_j) < -j$, $j = 1, \dots, k$. Moreover, notice that after we choose n_{k+1} and x_{k+1}, y_{k+1} with $\theta^{n_{k+1}} + \theta^{2n_{k+1}} < x_{k+1}, y_{k+1} < \theta^{n_k} - \theta^{2n_k}$ and $R_{D_{k+1}}(x_{k+1}) > 2 - 1/(k+1)$, $R_{D_{k+1}}(y_{k+1}) < -(k+1)$ we easily get the existence of the desired r_{k+2} from Proposition 1. Therefore, what we need is to choose $n_{k+1} \gg n_k$ and properly select x_{k+1}, y_{k+1}. We define x_{k+1}, y_{k+1} to be equal to $\theta^{n_{k+1}} + \theta^{\alpha 2n_{k+1}}$, where $\alpha = 1/4$ in the case of x_{k+1} and $\alpha = 1/2$ in the case of y_{k+1}.

Note that for $r < r^\alpha < s < a$,

$$J^{(j)}_{P(r,s)}(r^\alpha) = J^{(j)}_{P(r/s,1)}\left(\left(\frac{r}{s}\right)^\beta\right) s^{-2(j+1)}, \quad \text{where } \beta = \frac{\alpha \log r - \log s}{\log r - \log s}.$$

If $n_{k+1} \gg n_k$ then $s = r_{k+1} - \theta^{n_{k+1}}$ is very close to r_{k+1}, and $\beta \approx \alpha$. Then we obtain

$$\begin{aligned}
R_{D_{k+1}}(x_{k+1}) &= 2 - \frac{J^{(0)}_{D_{k+1}}(x_{k+1}) J^{(2)}_{D_{k+1}}(x_{k+1})}{(J^{(1)}_{D_{k+1}}(x_{k+1}))^2} \\
&\ge 2 - \frac{J^{(0)}_{P(\theta^{n_{k+1}}, \theta^{2n_{k+1}}, s)}(x_{k+1}) J^{(2)}_{P(\theta^{n_{k+1}}, \theta^{2n_{k+1}}, s)}(x_{k+1})}{(J^{(1)}_{P(\theta^{n_{k+1}}, \theta^{2n_{k+1}}, 1)}(x_{k+1}))^2} \\
&= 2 - \frac{J^{(0)}_{P(\theta^{n_{k+1}}, \theta^{2n_{k+1}}, s)}(\theta^{\alpha 2n_{k+1}}) J^{(2)}_{P(\theta^{n_{k+1}}, \theta^{2n_{k+1}}, s)}(\theta^{\alpha 2n_{k+1}})}{(J^{(1)}_{P(\theta^{n_{k+1}}, \theta^{2n_{k+1}}, 1)}(\theta^{\alpha 2n_{k+1}}))^2}.
\end{aligned}$$

Substituting in the last formula $\alpha = 1/4$ we find that, in view of Theorem 2, if $n_{k+1} \gg n_k$ then the last expression is greater than $2 - 1/(k+1)$. Analogously we get the desired estimate for $R_{D_{k+1}}(y_{k+1})$ (but in this case we substitute $\alpha = 1/2$). ■

Remark 6. It would be interesting to find a precise description of Zalcman-type domains having the property as stated in Corollary 3. Note that such a description (complete or at least partial) has been given for the boundary behavior of the Bergman kernel, Bergman metric or Bergman completeness (see [Juc 2004], [Pfl-Zwo 2003], [Zwo 2002]).

The construction presented in Corollary 3 is similar to the one presented in [Jar-Pfl-Zwo 2000] where the first example of a fat bounded planar domain which is not Bergman exhaustive has been given.

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References

[Chen-Lee 2009] B. Y. Chen and H. J. Lee, *Bergman kernel and complex singularity exponent*, Sci. China 52 (2009), 2590–2603.

[Din 2010] Ż. Dinew, *An example for the holomorphic sectional curvature of the Bergman metric*, Ann. Polon. Math. 98 (2010), 147–167.

[Her 1983] G. Herbort, *Über die Geodätischen der Bergmanmetrik*, Schriftenreihe Math. Inst. Univ. Münster (2) 46 (1983).

[Her 2007] —, *An example of a pseudoconvex domain whose holomorphic sectional curvature of the Bergman metric is unbounded*, Ann. Polon. Math. 92 (2007), 29–39.

[Jar-Pfl 1993] M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, de Gruyter, 1993.

[Jar-Pfl-Zwo 2000] M. Jarnicki, P. Pflug and W. Zwonek, *On Bergman completeness of non-hyperconvex domains*, Univ. Iagel. Acta Math. 38 (2000), 169–184.

[Juc 2004] P. Jucha, *Bergman completeness of Zalcman type domains*, Studia Math. 163 (2004), 71–83.

[Pfl-Zwo 2003] P. Pflug and W. Zwonek, *Logarithmic capacity and Bergman functions*, Arch. Math. (Basel) 80 (2003), 536–552.

[Zwo 2002] W. Zwonek, *Wiener's type criterion for Bergman exhaustiveness*, Bull. Polish Acad. Sci. Math. 50 (2002), 297–311.

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