ANNALES
POLONICI MATHEMATICI
110.2 (2014)

Clifford analysis approach to a self-conjugate Cauchy type integral on Ahlfors regular surfaces

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Abstract. In this note, based on a natural isomorphism between the spaces of differential forms and Clifford algebra-valued multi-vector functions, the Cauchy type integral for self-conjugate differential forms in $\mathbb{R}^n$ is considered.

1. Introduction. As is well known, a k-vector in $\mathbb{R}^n$ can be interpreted as a directed k-dimensional volume. Such entities were first considered by H. Grassmann in the second half of the 19th century. He thus created an algebraic structure which is now commonly known as the exterior algebra. At about the same time, Sir William Hamilton invented his quaternion algebra which enabled him to represent rotations in three-dimensional space. In his 1878 paper, W. K. Clifford united both systems into a single geometric algebra named after him (see [14]).

Clifford analysis offers a function theory associated with the Dirac operator which is a higher dimensional generalization of classical complex analysis in $\mathbb{R}^2$ (identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way) to Euclidean space $\mathbb{R}^n$ ($n \geq 3$).

On the other hand the theory of differential forms provides a generalization in $\mathbb{R}^n$ of holomorphic functions of one complex variable.

Although Clifford analysis seems to be truly appropriate to study differential forms by using Clifford algebras in a beautiful way, this has been mentioned so far only in a few papers. For an overview of the main operator identities and properties of these objects in the Clifford analysis context we refer to [15] and the references quoted there.

In [11] (see also [12]), the authors compare the language of differential forms and that of Clifford algebra-valued multi-vector functions (multi-vector fields) and show that the spaces of smooth differential forms on the

2010 *Mathematics Subject Classification*: Primary 30G35; Secondary 58A10.
Key words and phrases: Clifford analysis, multi-vector fields, differential forms.

DOI: 10.4064/ap110-2-1

one hand, and smooth multi-vector functions on the other are isomorphic in a natural way. Moreover the action of the operator $d - d^*$, where d and d^* are the differential and co-differential operators respectively, on the space of smooth k-forms is identified with the action (on the right) of the Dirac operator D, which plays the role of the Cauchy–Riemann operator on the space of smooth k-vector fields. On the other hand, the action of the operator $d + d^*$ is identified with the action (on the left) of the Dirac operator D.

As is well-known (see [10]), in Clifford analysis there is a well-defined Cauchy type integral for domains with moderately smooth boundary, but the same cannot be said about the theory of differential forms, where an appropriate Cauchy kernel is missing.

In the last decade there was a strong increase of interest in studies of boundary properties of Clifford/Cauchy type integrals on rougher domains (see for instance [1, 8, 9]).

In [2–7] generalizations of basic properties of the Cauchy type integral for k-vector fields using Clifford analysis are proved.

The main goal of this paper is to show how a Cauchy type integral could be directly defined also for so-called self-conjugate non-homogeneous differential forms. Here the full use of the isomorphism between the smooth differential forms on the one hand, and the smooth multi-vector functions on the other is the key point.

The analog of the Cauchy type integral for the theory of self-conjugate differential forms to be introduced in Section 5 is inspired by an integral representation of such differential forms obtained by A. Cialdea [13]. It is crucial to note that the proof of the representation formula there is based on componentwise equations derived for the self-conjugacy condition and using the Poisson kernel for forms, but the same formula can be obtained directly using tools of Clifford analysis. See Theorem 5.3(iii) below.

2. Harmonic differential forms. In this section we follow the notations and conventions used in [11].

Denoting by $\Lambda^k\mathbb{R}^n$ $(0 \le k \le n)$ the space of alternating real-valued differential forms of degree k (briefly k-forms), the well known Grassmann algebra over $\mathbb{R}^n$ is the associative algebra

$$\Lambda\mathbb{R}^n := \bigoplus_{k=0}^{n} \Lambda^k\mathbb{R}^n$$

endowed with the exterior multiplication $\wedge$.

We recall that a k-form in an open domain Ω of $\mathbb{R}^n$ is a map

$$U_k : \Omega \to \Lambda^k\mathbb{R}^n, \quad x \mapsto \sum_{|A|=k} U_{k,A}(x)dx^A,$$

where for each $A \subset \{1, \dots, n\}$, $U_{k,A}$ is a real-valued function in Ω and the set

$$\{dx^A : |A| := \mathrm{card}(A) = k\}$$

is a basis for $\Lambda^k \mathbb{R}^n$.

We denote by $C^1(\Omega, \Lambda^k \mathbb{R}^n)$ the space of smooth k-forms in Ω. A k-form U_k of class $C^1(\Omega, \Lambda^k \mathbb{R}^n)$ is said to be *harmonic* in Ω if it satisfies in Ω the Hodge/de Rham system

$$\begin{cases} dU_k = 0, \\ d^* U_k = 0. \end{cases} \tag{2.1}$$

These definitions can be directly extended to non-homogeneous differential forms: $C^1(\Omega, \Lambda \mathbb{R}^n)$ will denote $\sum_{k=0}^n C^1(\Omega, \Lambda^k \mathbb{R}^n)$.

If $U = \sum_{k=0}^n U_k \in C^1(\Omega, \Lambda \mathbb{R}^n)$ where $U_k \in C^1(\Omega, \Lambda^k \mathbb{R}^n)$ is a k-form, we consider the action of the exterior derivative d and the co-derivative d^* as

$$dU = \sum_{k=0}^n dU_k \quad \text{and} \quad d^* U = \sum_{k=0}^n d^* U_k.$$

Following [13] a non-homogeneous differential form $U \in C^1(\Omega, \Lambda \mathbb{R}^n)$ is said to be *self-conjugate* if

$$dU = d^* U \tag{2.2}$$

in Ω, i.e.,

$$d^* U_1 = 0; \quad dU_{k-1} = d^* U_{k+1} \quad (k = 1, \dots n-1); \quad dU_{n-1} = 0.$$

3. Clifford algebras and multi-vectors. The real Clifford algebra associated with $\mathbb{R}^n$ endowed with the Euclidean metric is the minimal enlargement of $\mathbb{R}^n$ to a real linear associative algebra $\mathbb{R}_{0,n}$ with identity, and such that $x^2 = -|x|^2$ for any $x \in \mathbb{R}^n$.

It follows that if $\{e_j\}_{j=1}^n$ is the standard basis of $\mathbb{R}^n$, then we must have $e_i e_j + e_j e_i = -2\delta_{ij}$. Every element $a \in \mathbb{R}_{0,n}$ is of the form $a = \sum_{A \subseteq N} a_A e_A$, $N = \{1, \dots, n\}$, $a_A \in \mathbb{R}$, where $e_\emptyset := e_0 = 1$, $e_{\{j\}} = e_j$, and $e_A = e_{\alpha_1} \cdots e_{\alpha_k}$ for $A = \{\alpha_1, \dots, \alpha_k\}$ with $\alpha_j \in \{1, \dots, n\}$ and $\alpha_1 < \dots < \alpha_k$, or still as $a = \sum_{k=0}^n [a]_k$, where $[a]_k = \sum_{|A|=k} a_A e_A$ is a so-called k-vector ($k = 0, 1, \dots, n$).

If we denote the space of k-vectors by $\mathbb{R}_{0,n}^k$, then $\mathbb{R}_{0,n} = \sum_{k=0}^n \oplus \mathbb{R}_{0,n}^k$, leading to the identification of $\mathbb{R}^n$ with $\mathbb{R}_{0,n}^1$.

For a 1-vector x and a k-vector Y_k, their product xY_k splits into a $(k-1)$-vector and a $(k+1)$-vector,

$$xY_k = [xY_k]_{k-1} + [xY_k]_{k+1},$$

where

$$[xY_k]_{k-1} = \tfrac{1}{2}(xY_k - (-1)^k Y_k x) \quad \text{and} \quad [xY_k]_{k+1} = \tfrac{1}{2}(xY_k + (-1)^k Y_k x).$$

The inner and outer products between x and Y_k are then defined by

$$x \bullet Y_k := [xY_k]_{k-1} \quad \text{and} \quad x \wedge Y_k := [xY_k]_{k+1}. \tag{3.1}$$

Notice also that

$$[xY_k]_{k-1} = (-1)^{k+1}[Y_k x]_{k-1}, \qquad [xY_k]_{k+1} = (-1)^k [Y_k x]_{k+1}, \tag{3.2}$$

For a deeper discussion of properties of inner and outer products between multi-vectors, we refer the reader to [17].

Conjugation in $\mathbb{R}_{0,n}$ is defined by $\overline{a} := \sum_A a_A \overline{e_A}$, where

$$\overline{e_A} = (-1)^k e_{i_k} \cdots e_{i_2} e_{i_1} \qquad \text{if } e_A = e_{i_1} e_{i_2} \cdots e_{i_k}.$$

In particular for a 1-vector x we have $\overline{x} = -x$.

4. Clifford analysis and harmonic multi-vector fields. Let $\Omega \subset \mathbb{R}^n$ be open and let F be an $\mathbb{R}_{0,n}$-valued function in Ω, say

$$F(x) = \sum_A F_A(x) e_A, \qquad x \in \Omega,$$

with all F_A being real-valued.

$C^1(\Omega, \mathbb{R}_{0,n})$ denotes the space of continuously differentiable $\mathbb{R}_{0,n}$-valued functions in Ω.

We say that F is *right* (resp. *left*) *monogenic* in Ω if $FD = 0$ (resp. $DF = 0$) in Ω, where D denote the Dirac operator in $\mathbb{R}^n$:

$$D = \sum_{j=1}^{n} e_j \partial_{x_j}.$$

An important example of a function which is both right and left monogenic is the fundamental solution of the Dirac operator, given by

$$E(x) = \frac{1}{A_n} \frac{\overline{x}}{|x|^n}, \qquad x \in \mathbb{R}^n \setminus \{0\},$$

where A_n denotes the surface area of the unit sphere in $\mathbb{R}^n$.

The function $E(x-y)$ plays the same role in Clifford analysis as the Cauchy kernel does in complex analysis. For this reason it is also called the Cauchy kernel in $\mathbb{R}^n$.

Let $0 < k \le n$ be fixed. Then the space of C^1-functions from Ω into $\mathbb{R}^k_{0,n}$, called smooth k-vector fields, is denoted by $C^1(\Omega, \mathbb{R}^k_{0,n})$.

Notice that for $F_k \in C^1(\Omega, \mathbb{R}^k_{0,n})$ a straightforward calculation leads to $\overline{DF_k} = \overline{F_k}\,\overline{D}$ with $\overline{D} = -D$ and $\overline{F_k} = (-1)^{k(k+1)/2} F_k$. It follows that for an element in $C^1(\Omega, \mathbb{R}^k_{0,n})$ the notions of left and right monogenicity coincide.

Consequently, we will call $F_k \in C^1(\Omega, \mathbb{R}^k_{0,n})$ *harmonic* in Ω if either $DF_k = 0$ or $F_k D = 0$ in Ω.

A natural isomorphism Θ between the real linear spaces $C^1(\Omega, \mathbb{R}^k_{0,n})$ and $C^1(\Omega, \Lambda^k\mathbb{R}^n)$ may be defined in the following way:

For $F_k = \sum_{|A|=r}(F_k)_A e_A \in C^1(\Omega, \mathbb{R}^k_{0,n})$, put $\Theta F_k = U_k \leftrightarrow U_k = \sum_{|A|=r}(U_k)_A dx^A$, where $(F_k)_A = (U_k)_A$ for each A. This can be found in [11].

Finally notice that through this isomorphism, for $F_k \in C^1(\Omega, \mathbb{R}^k_{0,n})$ and $U_k = \Theta F_k \in C^1(\Omega, \Lambda^k\mathbb{R}^n)$ $(0 < k < n-1)$, we have $D \wedge F_k \leftrightarrow dU_k$ and $D \bullet F_k \leftrightarrow d^*U_k$.

The following equivalence holds:

$$DF_k = 0 \;\Leftrightarrow\; \begin{cases} dU_k = 0, \\ d^*U_k = 0. \end{cases} \tag{4.1}$$

The relation (4.1) implies that F_k being harmonic in Ω is equivalent to saying that $U_k = \Theta F_k$ is a harmonic k-form in Ω.

More generally, Θ may be extended by linearity to $C^1(\Omega, \mathbb{R}_{0,n})$, thus leading to a linear isomorphism $\Theta : C^1(\Omega, \mathbb{R}_{0,n}) \to C^1(\Omega, \Lambda\mathbb{R}^n)$. Hence, let $U = \sum_{k=0}^n U_k$ be a non-homogeneous differential form and consider the $\mathbb{R}_{0,n}$-valued function $F = \sum_{k=0}^n F_k$, where $\Theta F_k = U_k$. Then it may be easily verified that

$$FD = 0 \;\Leftrightarrow\; (d - d^*)U = 0. \tag{4.2}$$

The relation (4.2) obviously indicates that the theory of right monogenic functions and the theory of self-conjugate differential forms in an open domain $\Omega \subset \mathbb{R}^n$ are equivalent. For a more elaborate description of the interplay between multi-vector functions and differential forms, we refer the reader to [11].

5. Boundary properties of self-conjugate differential forms. Let $\mathbf{K} \subset \mathbb{R}^n$ be a compact set and $C^{0,\alpha}(\mathbf{K}, \mathcal{A})$, where $\mathcal{A}$ is either $\mathbb{R}_{0,n}$ or $\Lambda\mathbb{R}^n$, denote respectively the class of $\mathbb{R}_{0,n}$-valued functions or the class of non-homogeneous differential forms on $\mathbf{K}$ satisfying the Hölder condition with exponent $0 < \alpha \le 1$. The latter means that all the components $(F_k)_A$ or $(U_k)_A$ have the cited property as real valued-functions.

Let Ω be a bounded oriented connected open subset of $\mathbb{R}^n$ whose boundary is a compact topological surface Γ with $\mathcal{H}^{n-1}(\Gamma) < \infty$, where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

We say that Γ is *Ahlfors regular* if there exist $c_1, c_2 \in (0, \infty)$ such that for each $0 < r \le \operatorname{diam} \Gamma$ and $z \in \Gamma$,

$$c_1 r^{n-1} \le \mathcal{H}^{n-1}(\{\xi \in \Gamma : |\xi - z| < r\}) \le c_2 r^{n-1}.$$

Let us introduce the temporary notations $\Omega = \Omega_+$, $\Omega_- = \mathbb{R}^n \setminus \overline{\Omega}_+$.

For any $\mathbb{R}_{0,n}$-valued continuous function F the (right) Clifford/Cauchy type integral and its singular version are given by the formulas

$$\mathcal{C}[F](z) := \int_{\Gamma} F(\xi)\nu(\xi)E(\xi - z)\, d\mathcal{H}^{n-1}(\xi), \qquad z \notin \Gamma,$$

$$\mathcal{S}[F](z) :=$$

$$2 \lim_{\delta \to 0} \int_{\Gamma \setminus \{\xi \in \Gamma : |\xi - z| < \delta\}} (F(\xi) - F(z))\nu(\xi)E(\xi - z)\, d\mathcal{H}^{n-1}(\xi) + F(z), \qquad z \in \Gamma,$$

where $\nu(\xi)$ is the outward unit normal vector on Γ introduced by Federer. For a thorough treatment and for references to the extensive literature on the subject one may refer to the book [16].

We shall now recall useful properties of the Clifford/Cauchy type integral over Ahlfors regular surfaces. For details concerning the proof, we refer to [1, 8, 9].

Theorem 5.1. *Let* $F \in C^{0,\alpha}(\Gamma, \mathbb{R}_{0,n})$, $0 < \alpha < 1$. *Then*

(i) $\mathcal{C}F \in C^{0,\alpha}(\Omega_{\pm} \cup \Gamma, \mathbb{R}_{0,n})$ *with* $\mathcal{C}F(\infty) = 0$.

(ii) $\mathcal{C}F$ *is right monogenic in* $\mathbb{R}^n \setminus \Gamma$.

(iii) (Cauchy's integral formula) *If* F *is right monogenic in* Ω, *then*

$$F(x) = (\mathcal{C}F)(x), \qquad x \in \Omega.$$

(iv) (Plemelj–Sokhotzki formula) *For all* $z \in \Gamma$,

$$(\mathcal{C}^{\pm}F)(z) = \lim_{\Omega_{\pm} \ni x \to z} (\mathcal{C}F)(x) = \tfrac{1}{2}(\mathcal{S}F(z) \pm F(z)).$$

(v) *In order for* F *to be the boundary value of a function* $F^{\pm}$ *which is right monogenic in* $\Omega_{\pm}$ *respectively, it is necessary and sufficient that*

$$F(z) = \pm\mathcal{S}F(z), \qquad \forall z \in \Gamma.$$

(vi) $\mathcal{S}$ *is bounded on* $C^{0,\alpha}(\Gamma, \mathbb{R}_{0,n})$. *Moreover* $\mathcal{S}^2 = \mathcal{I}$, *where* $\mathcal{I}$ *is the identity operator.*

The reason for the following definition of self-conjugate Cauchy type integrals is clear; it reflects the profound relation between self-conjugate differential forms and Clifford analysis.

Definition 5.2. Let $U = \sum_{k=0}^{n} U_k$ be a non-homogeneous differential form and put $F = \sum_{k=0}^{n} F_k$, where $\Theta F_k = U_k$. Then

(i) The Cauchy type integral for the theory of self-conjugate forms is given by

$$\mathsf{C}[U](z) := \Theta\mathcal{C}[F](z), \qquad z \notin \Gamma,$$

(ii) In the same way, we define the singular Cauchy type integral by

$$\mathsf{S}[U](z) := \Theta\mathcal{S}[F](z), \qquad z \in \Gamma.$$

The basic properties of $\mathsf{C}[U]$ and $\mathsf{S}[U]$ are established in our next theorem.

THEOREM 5.3. *Let* $U \in C^{0,\alpha}(\Gamma, \Lambda\mathbb{R}^n))$, $0 < \alpha < 1$. *Then*

(i) $\mathsf{C}[U] \in C^{0,\alpha}(\Omega_\pm \cup \Gamma, \Lambda\mathbb{R}^n))$ *with* $\mathsf{C}[U](\infty) = 0$.
(ii) $\mathsf{C}[U]$ *is self-conjugate in* $\mathbb{R}^n \setminus \Gamma$.
(iii) (Cauchy's integral formula) *If* U *is self-conjugate in* Ω, *then*

$$U(x) = \mathsf{C}[U](x), \; x \in \Omega.$$

(iv) (Plemelj–Sokhotzki formula) *For all* $z \in \Gamma$,

$$\mathsf{C}^{\pm}[U](z) = \lim_{\Omega_\pm \ni x \to z} \mathsf{C}[U](x) = \tfrac{1}{2}(\mathsf{S}[U](z) \pm U(z)).$$

(v) *In order for* U *to be the boundary value of a differential form* $U^{\pm}$ *which is self-conjugate in* $\Omega_\pm$ *respectively, it is necessary and sufficient that*

$$U(z) = \pm \mathsf{S}U(z), \qquad \forall z \in \Gamma.$$

(vi) S *is bounded on* $C^{0,\alpha}(\Gamma, \Lambda\mathbb{R}^n)$. *Moreover* $\mathsf{S}^2 = \mathcal{I}$, *where* $\mathcal{I}$ *is the identity operator.*

Main ideas of the proof. The results do not need separate proofs, they are straightforward adaptation of the corresponding ones in Theorem 5.1 if we use Definition 5.2 by fully exploiting the isomorphism Θ. For instance, it is easily seen that $\mathsf{C}[U]$ is a self-conjugate differential form in $\mathbb{R}^n \setminus \Gamma$, since $\mathcal{C}F$ is right monogenic in $\mathbb{R}^n \setminus \Gamma$.

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Received 5.1.2013
and in final form 30.3.2013 (2994)