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Solutions of singular semilinear elliptic equations with critical weighted Hardy–Sobolev exponents

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Abstract. Some solutions are obtained for a class of singular semilinear elliptic equations with critical weighted Hardy–Sobolev exponents by variational methods and some analysis techniques.

1. Introduction and main results. Consider the following semilinear elliptic problem

$$
(1.1)\quad \begin{cases} -\operatorname{div}(|x|^{-2a}\nabla u) - \mu \dfrac{u}{|x|^{2(1+a)}} = \dfrac{|u|^{2^*(a,s)-2}}{|x|^s}u + \dfrac{f(x,u)}{|x|^\sigma}, & x \in \Omega \setminus \{0\}, \\ u = 0, & x \in \partial\Omega, \end{cases}
$$

where Ω is an open bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, $0 \in \Omega$, $0 \leq a < \sqrt{\overline{\mu}}$, $\overline{\mu} \triangleq (N-2)^2/4$, $0 \leq \mu < (\sqrt{\overline{\mu}} - a)^2$, $\frac{2Na}{N-2} \leq s < 2(1+a)$, $0 \leq \sigma < 2(1+a)$, $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, and

$$
2^*(a,s) \triangleq \frac{2(N-s)}{N-2(1+a)}.
$$

Note that $2^*(0,s) = \frac{2(N-s)}{N-2}$ is the Hardy–Sobolev critical exponent and

$$
2^* \triangleq 2^*(0,0) = \frac{2N}{N-2}
$$

is the Sobolev critical exponent. In the case $\mu = 0$, problem (1.1) is related to the well known Caffarelli–Kohn–Nirenberg inequalities (see [CKN])

$$
(1.2)\qquad \Big(\int_{\mathbb{R}^N} |x|^{-s}|u|^{2^*(a,s)}\,dx \Big)^{\frac{2}{2^*(a,s)}} \leq C_{a,s} \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\,dx,
$$

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for all $u \in C_0^\infty(\mathbb{R}^N)$, where $-\infty < a < \sqrt{\bar{\mu}}$ and $\frac{2Na}{N-2} \le s \le 2(1+a)$. For sharp constants and extremal functions, see [CW]. If $s = 2(1+a)$ and $2^*(a,s) = 2$ in (1.2), we get the following weighted Hardy inequality (see [CW, CC]):

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2(1+a)}}\,dx \le \frac{1}{(\sqrt{\bar{\mu}}-a)^2} \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2\,dx \tag{1.3}$$

for all $u \in C_0^\infty(\mathbb{R}^N)$. If $a = 0$, (1.3) becomes the well known Hardy inequality

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,dx \le \frac{1}{\bar{\mu}} \int_{\mathbb{R}^N} |\nabla u|^2\,dx \quad \text{for all } u \in C_0^\infty(\mathbb{R}^N).$$

For $\mu \in [0, (\sqrt{\bar{\mu}}-a)^2)$, we use $H_a = H_0^1(\Omega, |x|^{-2a})$ to denote the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \Big(\int_\Omega \Big(|x|^{-2a}|\nabla u|^2 - \mu\frac{u^2}{|x|^{2(1+a)}}\Big)\,dx\Big)^{1/2},$$

which is equivalent to the usual norm of $H_0^1(\Omega, |x|^{-2a})$ due to (1.3), and

$$A = A_{a,s,\mu}(\Omega) \triangleq \inf_{u \in H_a\setminus\{0\}} \frac{\|u\|^2}{\big(\int_\Omega \frac{|u|^{2^*(a,s)}}{|x|^s}\,dx\big)^{\frac{2}{2^*(a,s)}}} \tag{1.4}$$

is the best Hardy–Sobolev constant, which is independent of Ω (see [KLP]).

Problem (1.1) in the case $a = s = 0$ and $\sigma = 0$ has been studied by some authors (see [CH, CW, GP, T]), and some interesting results were obtained. In particular, if $\mu = 0$, the problem has been widely studied since Brezis and Nirenberg (see [ABC, BN, J]); some other authors paid much attention to the singular problem with Hardy–Sobolev critical exponents (the case $a = 0$, $s \ne 0$, $\sigma = 0$) (see [DT, GK, GY, KP1, KP2]). But there are few results dealing with the case $a \ne 0$, $s \ne 0$, $\sigma \ne 0$ and the general form $f(x,t)$. In [HWT], the authors only studied the case $\sigma = 0$ for the general form $f(x,t)$ under suitable conditions; in [K], the authors only studied the special case $a = 0$ and $f(x,t) = \lambda|t|^{q-1}t$ with suitable q. In this paper, we use a variational method to deal with problem (1.1) and generalize the results in [HWT].

Due to the lack of compactness of the embedding $H_a \hookrightarrow L^{2^*}(\Omega)$ (see [GY]), we cannot use the standard variational argument directly. The corresponding energy functional fails to satisfy the classical Palais–Smale ((PS) for short) condition in H_a. However, a local (PS) condition can be established in a suitable range. Then the existence result is obtained via constructing a minimax level within this range and using the Mountain Pass Lemma due to A. Ambrosetti and P. H. Rabinowitz (see [Ra]).

Here are the main results of this paper:

THEOREM 1.1. *Suppose that* $0 \le a < \sqrt{\mu}$, $0 \le \mu < (\sqrt{\mu} - a)^2$, $\frac{2Na}{N-2} \le s < 2(1+a)$, $0 \le \sigma < 2(1+a)$ *and*

- (f_1) $f \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R})$, *and* $f(x,t)/t \to 0$ $(t \to 0^+)$, $f(x,t)/t^{r-1} \to 0$ $(t \to \infty)$ *uniformly for* $x \in \overline{\Omega}$, *where* $r = 2^*$ *if* $0 \le \sigma < \frac{2Na}{N-2}$, *and* $r = 2^*(a,\sigma)$ *if* $\frac{2Na}{N-2} \le \sigma < 2(1+a)$;
- (f_2) *there exists a constant* $\rho > 2$ *such that* $0 < \rho F(x,t) \le f(x,t)t$ *for all* $x \in \overline{\Omega}$ *and* $t \in \mathbb{R}^+ \setminus \{0\}$, *where* $F(x,t)$ *is the primitive function of* $f(x,t)$ *defined by* $F(x,t) = \int_0^t f(x,s)\,ds$.

Assume that

$$\rho > \max\left\{2, \frac{N-\sigma}{\gamma}, \frac{N-\sigma-2\beta}{\sqrt{\mu}-a}\right\}, \tag{1.5}$$

where $\beta = \sqrt{(\sqrt{\mu}-a)^2 - \mu}$ *and* $\gamma = \sqrt{\mu} - a + \beta$. *Then problem* (1.1) *has a positive weak solution.*

COROLLARY 1.2. *Suppose that* $N \ge 4(1+a)$, $0 \le a < \sqrt{\mu}$, $0 \le \mu < (\sqrt{\mu} - a)^2 - (1+a)^2$, $\frac{2Na}{N-2} \le s < 2(1+a)$ *and* $0 \le \sigma < 2(1+a)$. *Assume that* (f_1) *and* (f_2) *hold. Then problem* (1.1) *has a positive solution.*

THEOREM 1.3. *Suppose that* $0 \le a < \sqrt{\mu}$, $0 \le \mu < (\sqrt{\mu} - a)^2$, $\frac{2Na}{N-2} \le s < 2(1+a)$, $0 \le \sigma < 2(1+a)$ *and*

- (f_3) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, *and* $f(x,t)/t \to 0$ $(|t| \to 0)$, $f(x,t)/t^{r-1} \to 0$ $(|t| \to \infty)$ *uniformly for* $x \in \overline{\Omega}$;
- (f_4) *there exists a constant* $\rho > 2$ *such that* $0 < \rho F(x,t) \le f(x,t)t$ *for all* $x \in \overline{\Omega}$ *and* $t \in \mathbb{R} \setminus \{0\}$.

Assume that (1.5) *holds. Then problem* (1.1) *has at least two distinct nontrivial solutions.*

COROLLARY 1.4. *Suppose that* $N \ge 4(1+a)$, $0 \le a < \sqrt{\mu}$, $0 \le \mu \le (\sqrt{\mu} - a)^2 - (1+a)^2$, $\frac{2Na}{N-2} \le s < 2(1+a)$ *and* $0 \le \sigma < 2(1+a)$. *Assume that* (f_3) *and* (f_4) *hold. Then problem* (1.1) *has at least two distinct nontrivial solutions.*

REMARK 1.5. Our theorems generalize the results in [HWT] where the authors only studied the case $\sigma = 0$ with general $f(x,t)$. Moreover, Theorem 1.1 also generalizes [K, Theorem 1.1] where the author only considered the special situation that $a = 0$ and $f(x,t) = \lambda |t|^{q-1} t$ with suitable q.

2. Proofs. In order to study the existence of positive solutions for problem (1.1) we shall first consider the existence of nontrivial solutions to the problem

(2.1)
$$\begin{cases} -\operatorname{div}(|x|^{-2a}\nabla u) - \mu\dfrac{u}{|x|^{2(1+a)}} = \dfrac{(u^+)^{2^*(a,s)-1}}{|x|^s} + \dfrac{f(x,u^+)}{|x|^\sigma}, & x \in \Omega\setminus\{0\},\\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $u^+ = \max\{u,0\}$. The energy functional corresponding to problem (2.1) is given by

$$I(u) = \frac{1}{2}\int_\Omega \left(|x|^{-2a}|\nabla u|^2 - \mu\frac{u^2}{|x|^{2(1+a)}}\right)dx - \frac{1}{2^*(a,s)}\int_\Omega \frac{(u^+)^{2^*(a,s)}}{|x|^s}\,dx - \int_\Omega \frac{F(x,u^+)}{|x|^\sigma}\,dx, \qquad u \in H_a.$$

By the weighted Hardy–Sobolev inequality (1.3) and (f_1), $I \in C^1(H_a,\mathbb{R})$. Now it is well known that there exists a one-to-one correspondence between the weak solutions of problem (2.1) and the critical points of I on H_a. More precisely, we say that $u \in H_a$ is a weak solution of (2.1) if for any $v \in H_a$,

$$\begin{aligned}\langle I'(u), v\rangle &= \int_\Omega \left(|x|^{-2a}\nabla u\nabla v - \mu\frac{uv}{|x|^{2(1+a)}}\right)dx\\ &\quad - \int_\Omega \frac{(u^+)^{2^*(a,s)-1}}{|x|^s}v\,dx - \int_\Omega \frac{f(x,u^+)v}{|x|^\sigma}\,dx\\ &= 0.\end{aligned}$$

Let $\{u_n\}$ be a sequence in H_a and $c \in \mathbb{R}$. Then $\{u_n\}$ is said to be a $(\mathrm{PS})_c$ *sequence* in H_a if $I(u_n) \to c$, $I'(u_n) \to 0$ in $(H_a)^*$ as $n \to \infty$. We say I satisfies the $(\mathrm{PS})_c$ *condition* if any $(\mathrm{PS})_c$ sequence $\{u_n\} \subset H_a$ has a convergent subsequence.

Lemma 2.1. *Suppose that* $0 \le a < \sqrt{\mu}$, $0 \le \mu < (\sqrt{\mu} - a)^2$, $\frac{2Na}{N-2} \le s < 2(1+a)$ *and* $0 \le \sigma < 2(1+a)$. *Assume* (f_1) *and* (f_2) *hold. Suppose* $c \in \left(0, \frac{2^*(a,s)-2}{2\cdot 2^*(a,s)}A^{\frac{2^*(a,s)}{2^*(a,s)-2}}\right)$. *Then I satisfies the* $(\mathrm{PS})_c$ *condition.*

Proof. Suppose that $\{u_n\}$ is a $(\mathrm{PS})_c$ sequence in H_a. By (f_2), we have

$$\begin{aligned} c + 1 + o(1)\|u_n\| &\ge I(u_n) - \frac{1}{\theta}\langle I'(u_n), u_n\rangle\\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 + \left(\frac{1}{\theta} - \frac{1}{2^*(a,s)}\right)\int_\Omega \frac{(u_n^+)^{2^*(a,s)}}{|x|^s}\,dx\\ &\quad - \int_\Omega \frac{F(x,u_n^+) - \frac{1}{\theta}f(x,u_n^+)u_n}{|x|^\sigma}\,dx\\ &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2, \end{aligned}$$

where $\theta = \min\{\rho, 2^*(a,s)\}$. Hence we conclude $\{u_n\}$ is a bounded sequence in H_a, $\|u_n\| \le C_0 < \infty$. Taking a subsequence if necessary, we can get

$$\begin{cases} u_n \to u & \text{weakly in } H_a, \\ u_n \to u & \text{in } L^\gamma(\Omega),\ 1 < \gamma < 2^*, \\ u_n \to u & \text{a.e. in } \Omega, \end{cases}$$

as $n \to \infty$. It follows from (f_1) that there exists $\delta_1 > 0$ such that

$$|f(x,t)| < t \quad \text{for all } t \in [0,\delta_1] \text{ and } x \in \overline{\Omega},$$

and for any $\varepsilon > 0$, there is $\delta_2 > \delta_1$ such that

$$|f(x,t)| < \varepsilon t^{r-1} \quad \text{for all } t > \delta_2 \text{ and } x \in \overline{\Omega}.$$

Moreover, there exists $M > 0$ such that

$$|f(x,t)| \le M \quad \text{for all } x \in \overline{\Omega} \text{ and } t \in [\delta_1, \delta_2].$$

Hence, we deduce that

$$|f(x,t)| \le t + \varepsilon t^{r-1} + M \le \varepsilon t^{r-1} + (1 + M\delta_1^{-1})t$$

for all $t > 0$ and all $x \in \overline{\Omega}$. Then, for any $\varepsilon > 0$, there exists $a(\varepsilon) > 0$ such that

$$|f(x,t)t| \le \varepsilon |t|^r + a(\varepsilon)|t|^2 \quad \text{for all } x \in \overline{\Omega} \text{ and } t > 0.$$

By the weighted Hardy–Sobolev inequality (1.3), there exists a constant $C > 0$ such that

$$\int_\Omega \frac{|u|^2}{|x|^\sigma}\,dx = \int_\Omega \frac{|u|^2}{|x|^{2(1+a)}} |x|^{2(1+a)-\sigma}\,dx \le C\|u\|^2 \tag{2.2}$$

for all $u \in H_a$. Therefore, there exists a constant $\delta > 0$ such that

$$\int_E \frac{|u|^2}{|x|^\sigma}\,dx < \frac{\varepsilon}{a(\varepsilon)}$$

for any subset $E \subseteq \Omega$ with $\operatorname{meas}(E) < \delta$, where $\operatorname{meas}(\cdot)$ denotes the usual Lebesgue measure in $\mathbb{R}^N$.

In addition, there also exist constants $C_2 > C_1 > 0$ such that

$$\int_\Omega \frac{|u|^{2^*}}{|x|^\sigma}\,dx \le \int_\Omega \frac{|u|^{2^*}}{|x|^{\frac{2Na}{N-2}}} |x|^{\frac{2Na}{N-2}-\sigma}\,dx \le C_1\|u\|^{2^*}$$

for $0 \le \sigma < \frac{2Na}{N-2}$ and all $u \in H_a$; and

$$\int_\Omega \frac{|u|^{2^*(a,\sigma)}}{|x|^\sigma}\,dx \le C_2\|u\|^{2^*(a,\sigma)}$$

for $\frac{2Na}{N-2} \le \sigma < 2(1+a)$ and all $u \in H_a$. So we get

$$\int_\Omega \frac{|u|^r}{|x|^\sigma}\,dx \le C_2\|u\|^r$$

for $0 \le \sigma < 2(1+a)$ and all $u \in H_a$. Now, set $\delta_0 = \min\{\delta, \varepsilon/a(\varepsilon)\}$; when $E \subset \Omega$ with $\mathrm{meas}(E) < \delta_0$, we get

$$\begin{aligned}\left|\int_E \frac{f(x,u_n^+)u_n}{|x|^\sigma}\,dx\right| &\le \int_E \frac{|f(x,u_n^+)u_n|}{|x|^\sigma}\,dx \le a(\varepsilon)\int_E \frac{|u_n^2|}{|x|^\sigma}\,dx + \varepsilon\int_E \frac{|u_n|^r}{|x|^\sigma}\,dx\\ &\le \varepsilon + \varepsilon C_2 C_0^r.\end{aligned}$$

Hence $\left\{\int_\Omega \frac{f(x,u_n^+)u_n}{|x|^\sigma}\,dx : n \in \mathbb{N}\right\}$ is equi-absolutely-continuous. According to the Vitali convergence theorem (see [Ru]), we deduce that

$$\int_\Omega \frac{f(x,u_n^+)u_n}{|x|^\sigma}\,dx \to \int_\Omega \frac{f(x,u^+)u}{|x|^\sigma}\,dx \tag{2.3}$$

as $n \to \infty$. Similarly, we have

$$\int_\Omega \frac{F(x,u_n^+)}{|x|^\sigma}\,dx \to \int_\Omega \frac{F(x,u^+)}{|x|^\sigma}\,dx \tag{2.4}$$

as $n \to \infty$. Let $v_n = u_n - u$. Since $I'(u_n) \to 0$ in $(H_a)^*$, we obtain

$$\|u_n\|^2 - \int_\Omega \frac{(u_n^+)^{2^*(a,s)}}{|x|^s}\,dx - \int_\Omega \frac{f(x,u_n^+)u_n}{|x|^\sigma}\,dx = o(1).$$

From the Brezis–Lieb lemma (see [HL]), we have

$$\begin{cases}\displaystyle\int_\Omega \frac{|u_n|^2}{|x|^{2(1+a)}}\,dx - \int_\Omega \frac{|u_n-u|^2}{|x|^{2(1+a)}}\,dx \to \int_\Omega \frac{|u|^2}{|x|^{2(1+a)}}\,dx,\\ \displaystyle\int_\Omega \frac{|u_n|^{2^*(a,s)}}{|x|^s}\,dx - \int_\Omega \frac{|u_n-u|^{2^*(a,s)}}{|x|^s}\,dx \to \int_\Omega \frac{|u|^{2^*(a,s)}}{|x|^s}\,dx,\\ \displaystyle\int_\Omega \frac{|\nabla u_n|^2}{|x|^{2a}}\,dx - \int_\Omega \frac{|\nabla u_n-\nabla u|^2}{|x|^{2a}}\,dx \to \int_\Omega \frac{|\nabla u|^2}{|x|^{2a}}\,dx,\\ \displaystyle\int_\Omega \frac{|u_n|^{2^*(a,s)-2}}{|x|^s}u_n v\,dx \to \int_\Omega \frac{|u|^{2^*(a,s)-2}}{|x|^s}uv\,dx, \quad v \in H_a,\end{cases} \tag{2.5}$$

as $n \to \infty$. By (2.3) and (2.5), we get

$$\begin{aligned}O(1) = \|v_n\|^2 + \|u\|^2 &- \int_\Omega \frac{(v_n^+)^{2^*(a,s)}}{|x|^s}\,dx - \int_\Omega \frac{(u^+)^{2^*(a,s)}}{|x|^s}\,dx\\ &- \int_\Omega \frac{f(x,u^+)u}{|x|^\sigma}\,dx\end{aligned} \tag{2.6}$$

and

$$(2.7)\qquad \lim_{n\to\infty}\langle I'(u_n),u\rangle = \langle I'(u),u\rangle = \|u\|^2 - \int_\Omega \frac{(u^+)^{2^*(a,s)}}{|x|^s}\,dx - \int_\Omega \frac{f(x,u^+)u}{|x|^\sigma}\,dx = 0.$$

It follows from (2.7) that

$$I(u) = I(u) - \frac{1}{2}\langle I'(u),u\rangle = \left(\frac{1}{2}-\frac{1}{2^*(a,s)}\right)\int_\Omega \frac{(u^+)^{2^*(a,s)}}{|x|^s}\,dx + \frac{1}{2}\int_\Omega \frac{f(x,u^+)u}{|x|^\sigma}\,dx - \int_\Omega \frac{F(x,u^+)}{|x|^\sigma}\,dx.$$

From (f_2), we conclude that

$$(2.8)\qquad I(u)\ge 0.$$

Since $I(u_n)\to c$ $(n\to\infty)$, combining (2.4) with (2.5), we obtain

$$\begin{aligned} I(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2^*(a,s)}\int_\Omega \frac{(u_n^+)^{2^*(a,s)}}{|x|^s}\,dx - \int_\Omega \frac{F(x,u_n^+)}{|x|^\sigma}\,dx \\ &= \frac{1}{2}\|v_n\|^2 + \frac{1}{2}\|u\|^2 - \frac{1}{2^*(a,s)}\int_\Omega \frac{(v_n^+)^{2^*(a,s)}}{|x|^s}\,dx \\ &\quad - \frac{1}{2^*(a,s)}\int_\Omega \frac{(u^+)^{2^*(a,s)}}{|x|^s}\,dx - \int_\Omega \frac{F(x,u^+)}{|x|^\sigma}\,dx + o(1) \\ &= I(u) + \frac{1}{2}\|v_n\|^2 - \frac{1}{2^*(a,s)}\int_\Omega \frac{(v_n^+)^{2^*(a,s)}}{|x|^s}\,dx + o(1) \\ &= c + o(1). \end{aligned}$$

Therefore,

$$(2.9)\qquad I(u) + \frac{1}{2}\|v_n\|^2 - \frac{1}{2^*(a,s)}\int_\Omega \frac{(v_n^+)^{2^*(a,s)}}{|x|^s}\,dx = c + o(1).$$

From (2.6) and (2.7), we have

$$\|v_n\|^2 - \int_\Omega \frac{(v_n^+)^{2^*(a,s)}}{|x|^s}\,dx = o(1).$$

Then $\|v_n\|\to 0$ as $n\to\infty$. Indeed, otherwise there exists a subsequence, still denoted by v_n, such that

$$(2.10)\qquad \|v_n\|^2 \to k \quad\text{and}\quad \int_\Omega \frac{(v_n^+)^{2^*(a,s)}}{|x|^s}\,dx \to k \quad \text{as } n\to\infty,$$

where k is a positive constant. By (1.4), we deduce that

$$\|v_n\|^2 \geq A\Big(\int_\Omega \frac{(v_n^+)^{2^*(a,s)}}{|x|^s}\Big)^{\frac{2}{2^*(a,s)}} \qquad \text{for all } n \in \mathbb{N};$$

hence $k \geq Ak^{\frac{2}{2^*(a,s)}}$, i.e., $k \geq A^{\frac{2}{2^*(a,s)-2}}$, which, together with (2.9) and (2.10), shows that

$$I(u) = c - \frac{1}{2}k + \frac{1}{2^*(a,s)}k \leq c - \frac{2^*(a,s)-2}{2\cdot 2^*(a,s)}A^{\frac{2^*(a,s)}{2^*(a,s)-2}} < 0.$$

This contradicts (2.8).

Therefore, $\|v_n\|^2 \to 0$ as $n \to \infty$, which implies that $u_n \to u$ in H_a as $n \to \infty$. From the discussion above, I satisfies the $(\mathrm{PS})_c$ condition. ■

Recently, the authors of [KLP] proved that, for $0 \leq a < \sqrt{\mu}$, $0 \leq \mu < (\sqrt{\mu}-a)^2$, $\frac{2Na}{N-2} \leq s < 2(1+a)$ and $\beta = \sqrt{(\sqrt{\mu}-a)^2-\mu}$, A is attained when $\Omega = \mathbb{R}^N$ by the functions

$$y_\varepsilon(x) = \frac{(2\varepsilon\cdot 2^*(a,s)\beta^2)^{\frac{1}{2^*(a,s)-2}}}{|x|^{\gamma'}(\varepsilon + |x|^{(2^*(a,s)-2)\beta})^{\frac{2}{2^*(a,s)-2}}}$$

for all $\varepsilon > 0$, where $\gamma' = \sqrt{\mu} - a - \beta$. Moreover, the functions $y_\varepsilon(x)$ solve the equation

$$-\operatorname{div}(|x|^{-2a}\nabla u) - \mu\frac{u}{|x|^{2(1+a)}} = \frac{|u|^{2^*(a,s)-2}}{|x|^s}u \quad \text{in } \mathbb{R}^N\setminus\{0\},$$

and satisfy

$$\int_{\mathbb{R}^N}\Big(|x|^{-2a}|\nabla y_\varepsilon(x)|^2 - \mu\frac{y_\varepsilon^2(x)}{|x|^{2(1+a)}}\Big)\,dx = \int_{\mathbb{R}^N}\frac{y_\varepsilon^{2^*(a,s)}(x)}{|x|^s}\,dx = A^{\frac{2^*(a,s)}{2^*(a,s)-2}}.$$

Let

$$C_\varepsilon = (2\varepsilon\cdot 2^*(a,s)\beta^2)^{\frac{1}{2^*(a,s)-2}}, \qquad U_\varepsilon(x) = y_\varepsilon(x)/C_\varepsilon.$$

Choose a cut-off function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi(x) = 1$ for $|x| \leq R$, $\varphi(x) = 0$ for $|x| \geq 2R$ and $0 \leq \varphi(x) \leq 1$, where $B_{2R}(0) \subset \Omega$. Set $u_\varepsilon(x) = \varphi(x)U_\varepsilon(x)$, $v_\varepsilon(x) = u_\varepsilon(x)/(\int_\Omega |u_\varepsilon|^{2^*(a,s)}|x|^{-s}\,dx)^{1/2^*(a,s)}$, so that $\int_\Omega |v_\varepsilon|^{2^*(a,s)}|x|^{-s}\,dx = 1$. Then we can get the following results by the methods used in [GY]:

$$A + C_3\varepsilon^{\frac{2}{2^*(a,s)-2}} \leq \|v_\varepsilon\|^2 \leq A + C_4\varepsilon^{\frac{2}{2^*(a,s)-2}}, \tag{2.11}$$

and

(2.12)
$$\begin{cases} C_5\varepsilon^{\frac{q}{2^*(a,s)-2}} \le \int\limits_\Omega \frac{|v_\varepsilon|^q}{|x|^\sigma}\,dx \le C_6\varepsilon^{\frac{q}{2^*(a,s)-2}}, & 1 \le q < (N-\sigma)/\gamma, \\ C_5\varepsilon^{\frac{q}{2^*(a,s)-2}}|\ln\varepsilon| \le \int\limits_\Omega \frac{|v_\varepsilon|^q}{|x|^\sigma}\,dx \le C_6\varepsilon^{\frac{q}{2^*(a,s)-2}}|\ln\varepsilon|, & q = (N-\sigma)/\gamma, \\ C_5\varepsilon^{\frac{N-\sigma-q(\sqrt{\mu}-a)}{(2^*(a,s)-2)\beta}} \le \int\limits_\Omega \frac{|v_\varepsilon|^q}{|x|^\sigma}\,dx \le C_6\varepsilon^{\frac{N-\sigma-q(\sqrt{\mu}-a)}{(2^*(a,s)-2)\beta}}, & q > (N-\sigma)/\gamma. \end{cases}$$

Moreover, we obtain

$$\int\limits_\Omega \frac{|v_\varepsilon|^r}{|x|^\sigma}\,dx \le C_2(2A)^{r/2} \quad \text{as } \varepsilon \to 0^+. \tag{2.13}$$

In fact, using the Hardy–Sobolev inequality and (2.11), one deduces

$$\int\limits_\Omega \frac{|v_\varepsilon|^r}{|x|^\sigma}\,dx \le C_2\|v_\varepsilon\|^r \le C_2(A + C_4\varepsilon^{\frac{2}{2^*(a,s)-2}})^{r/2} \le C_2(2A)^{r/2} \quad \text{as } \varepsilon \to 0^+.$$

Lemma 2.2. *Suppose that* $0 \le a < \sqrt{\mu}$, $0 \le \mu < (\sqrt{\mu}-a)^2$, $\frac{2Na}{N-2} \le s < 2(1+a)$ *and* $0 \le \sigma < 2(1+a)$. *Assume that* (f$_1$), (f$_2$) *and* (1.5) *hold. Then there exists* $u_0 \in H_a$, $u_0 \not\equiv 0$, *such that*

$$\sup_{t\ge 0} I(tu_0) < \frac{2^*(a,s)-2}{2\cdot 2^*(a,s)} A^{\frac{2^*(a,s)}{2^*(a,s)-2}}.$$

Proof. We consider the functions

$$g(t) = I(tv_\varepsilon) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(a,s)}}{2^*(a,s)} - \int\limits_\Omega \frac{F(x,tv_\varepsilon)}{|x|^\sigma}\,dx,$$
$$\widetilde{g}(t) = \frac{t^2}{2}\|v_\varepsilon\|^2 - \frac{t^{2^*(a,s)}}{2^*(a,s)}.$$

Note that $g(t) \to -\infty$ as $t \to \infty$, $g(0) = 0$, $g(t) > 0$ as $t \to 0^+$, so $\sup_{t\ge 0} g(t)$ is attained for some $t_0 > 0$. Since

$$0 = g'(t_0) = t_0\bigg(\|v_\varepsilon\|^2 - t_0^{2^*(a,s)-2} - \frac{1}{t_0}\int\limits_\Omega \frac{f(x,t_0v_\varepsilon)v_\varepsilon}{|x|^\sigma}\,dx\bigg),$$

we have

$$\|v_\varepsilon\|^2 = t_0^{2^*(a,s)-2} + \frac{1}{t_0}\int\limits_\Omega \frac{f(x,t_0v_\varepsilon)v_\varepsilon}{|x|^\sigma}\,dx \ge t_0^{2^*(a,s)-2},$$

which, together with (2.11), shows that

$$t_0 \le \|v_\varepsilon\|^{\frac{2}{2^*(a,s)-2}} \triangleq t^0 \le (2A)^{\frac{1}{2^*(a,s)-2}}.$$

By (f$_1$), for any $\widetilde{\varepsilon} > 0$, there exists $a(\widetilde{\varepsilon}) > 0$ such that

$$|f(x,t)| \le \widetilde{\varepsilon}|t|^{r-1} + a(\widetilde{\varepsilon})|t| \quad \text{for all } x \in \overline{\Omega} \text{ and } t > 0.$$

Hence, we obtain

$$\|v_\varepsilon\|^2 \le t_0^{2^*(a,s)-2} + \widetilde{\varepsilon}\int_\Omega \frac{|t_0|^{r-2}|v_\varepsilon|^r}{|x|^\sigma}\,dx + a(\widetilde{\varepsilon})\int_\Omega \frac{|v_\varepsilon|^2}{|x|^\sigma}\,dx.$$

Set $\widetilde{\varepsilon} = (4C_2(2A)^{\frac{r-2}{2^*(a,s)-2}}(2A)^{r/2})^{-1}A$. By (2.11)–(2.13), for ε small,

$$A \le \|v_\varepsilon\|^2 \le t_0^{2^*(a,s)-2} + \widetilde{\varepsilon}C_2(2A)^{\frac{r-2}{2^*(a,s)-2}}(2A)^{r/2} + \tfrac{1}{4}A = t_0^{2^*(a,s)-2} + \tfrac{1}{2}A,$$

that is,

$$t_0^{2^*(a,s)-2} \ge A/2. \tag{2.14}$$

On the one hand, from (2.11) we will deduce that

$$\|v_\varepsilon\|^{\frac{2\cdot 2^*(a,s)}{2^*(a,s)-2}} \le A^{\frac{2^*(a,s)}{2^*(a,s)-2}} + C_7\varepsilon^{\frac{2}{2^*(a,s)-2}}. \tag{2.15}$$

In order to prove this, we first prove the following inequality:

$$(a+b)^\lambda \le a^\lambda + \lambda(a+1)^{\lambda-1}b, \qquad a>0,\ 0\le b\le 1,\ \lambda\ge 1. \tag{2.16}$$

In fact, set

$$h(x) = (a+x)^\lambda - a^\lambda - \lambda(a+1)^{\lambda-1}x, \qquad a>0,\ 0\le x\le 1,\ \lambda\ge 1.$$

Clearly, $h'(x) < 0$, $x\in(0,1)$, so $h(b)\le h(0)=0$; then (2.16) holds. Let $a=A$, $b = C_3\varepsilon^{\frac{2}{2^*(a,s)-2}}$, $\lambda = \frac{2^*(a,s)}{2^*(a,s)-2}$; then (2.15) holds.

On the other hand, the function $\widetilde{g}(t)$ attains its maximum at t^0, and is increasing in the interval $[0,t^0]$; together with (2.11), (2.14), (2.15) and $F(x,t)\ge C_8|t|^\rho$ which is directly derived from (f_2), we deduce that

$$\begin{aligned}
g(t_0) &\le \widetilde{g}(t^0) - \int_\Omega \frac{F(x,t_0v_\varepsilon)}{|x|^\sigma}\,dx\\
&= \frac{2^*(a,s)-2}{2\cdot 2^*(a,s)}\|v_\varepsilon\|^{\frac{2\cdot 2^*(a,s)}{2^*(a,s)-2}} - \int_\Omega \frac{F(x,t_0v_\varepsilon)}{|x|^\sigma}\,dx\\
&\le \frac{2^*(a,s)-2}{2\cdot 2^*(a,s)}A^{\frac{2^*(a,s)}{2^*(a,s)-2}} + C_9\varepsilon^{\frac{2}{2^*(a,s)-2}} - \int_\Omega \frac{F(x,t_0v_\varepsilon)}{|x|^\sigma}\,dx\\
&\le \frac{2^*(a,s)-2}{2\cdot 2^*(a,s)}A^{\frac{2^*(a,s)}{2^*(a,s)-2}} + C_9\varepsilon^{\frac{2}{2^*(a,s)-2}} - C_8\int_\Omega \frac{t_0^\rho|v_\varepsilon|^\rho}{|x|^\sigma}\,dx\\
&\le \frac{2^*(a,s)-2}{2\cdot 2^*(a,s)}A^{\frac{2^*(a,s)}{2^*(a,s)-2}} + C_9\varepsilon^{\frac{2}{2^*(a,s)-2}} - C_8\left(\frac{A}{2}\right)^{\frac{\rho}{2^*(a,s)-2}}\int_\Omega \frac{|v_\varepsilon|^\rho}{|x|^\sigma}\,dx,
\end{aligned}$$

where $C_9 = C_7\frac{2^*(a,s)-2}{2\cdot 2^*(a,s)}$. Furthermore, from (2.12), we get

$$\int_\Omega \frac{|v_\varepsilon|^\rho}{|x|^\sigma}\,dx \ge C_5\varepsilon^{\frac{N-\sigma-\rho(\sqrt{\mu}-a)}{(2^*(a,s)-2)\beta}}.$$

By (1.5), we obtain

$$\frac{2}{2^*(a,s)-2} > \frac{N-\sigma-\rho(\sqrt{\mu}-a)}{(2^*(a,s)-2)\beta}.$$

Choosing ε small enough, we have

$$\sup_{t\ge 0} I(tv_\varepsilon) = g(t_0) < \frac{2^*(a,s)-2}{2\cdot 2^*(a,s)} A^{\frac{2^*(a,s)}{2^*(a,s)-2}}. \ \blacksquare$$

Proof of Theorem 1.1. By (f_1), for any $\varepsilon > 0$, there exists $b(\varepsilon) > 0$ such that

$$|f(x,t)| \le \varepsilon|t| + b(\varepsilon)|t|^{r-1} \qquad \text{for } (x,t) \in \overline{\Omega}\times(0,\infty),$$
$$|F(x,t)| \le \frac{1}{2}\varepsilon|t|^2 + \frac{b(\varepsilon)}{r}|t|^r \qquad \text{for } (x,t) \in \overline{\Omega}\times(0,\infty).$$

Combining this with the Hardy–Sobolev inequality, (1.2) and (2.2), we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2^*(a,s)}\int_\Omega \frac{(u^+)^{2^*(a,s)}}{|x|^s}\,dx - \int_\Omega \frac{F(x,u^+)}{|x|^\sigma}\,dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{(C_{a,s})^{2^*(a,s)/2}}{2^*(a,s)}\|u^+\|^{2^*(a,s)} - \frac{\varepsilon}{2}\int_\Omega \frac{|u|^2}{|x|^\sigma}\,dx - \frac{b(\varepsilon)}{r}\int_\Omega \frac{|u|^r}{|x|^\sigma}\,dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{(C_{a,s})^{2^*(a,s)/2}}{2^*(a,s)}\|u^+\|^{2^*(a,s)} - \frac{C\varepsilon}{2}\|u\|^2 - \frac{C_2 b(\varepsilon)}{r}\|u\|^r \end{aligned}$$

for ε small enough. So there exists $\alpha > 0$ such that $I(u) \ge \alpha$ for all $u \in \partial B_R = \{u \in H_a : \|u\| = R\}$, where $R > 0$ is small enough. By Lemma 2.2, there exists $u_0 \in H_a$ such that $u_0 \not\equiv 0$ and

$$\sup_{t\ge 0} I(tu_0) < \frac{2^*(a,s)-2}{2\cdot 2^*(a,s)} A^{\frac{2^*(a,s)}{2^*(a,s)-2}}.$$

From the nonnegativity of $F(x,t)$, we obtain

$$\begin{aligned} I(tu_0) &= \frac{1}{2}t^2\|u_0\|^2 - \frac{t^{2^*(a,s)}}{2^*(a,s)}\int_\Omega \frac{(u_0^+)^{2^*(a,s)}}{|x|^s}\,dx - \int_\Omega \frac{F(x,tu_0^+)}{|x|^\sigma}\,dx \\ &\le \frac{1}{2}t^2\|u_0\|^2 - \frac{t^{2^*(a,s)}}{2^*(a,s)}\int_\Omega \frac{(u_0^+)^{2^*(a,s)}}{|x|^s}\,dx, \end{aligned}$$

which implies that $I(tu_0) \to -\infty$ as $t \to \infty$. Hence we can choose $t_1 > 0$ such that $\|t_1u_0\| > R$ and $I(t_1u_0) \le 0$. Applying the Mountain Pass Lemma of [Ra], there is a sequence $\{u_n\} \subset H_a$ satisfying

$$I(u_n) \to c \ge \alpha, \qquad I'(u_n) \to 0,$$

where

$$c = \inf_{h\in\tau}\max_{t\in[0,1]} I(h(t)) \quad \text{and} \quad \tau = \{h \in C([0,1],H_a) : h(0) = 0,\ h(1) = t_1u_0\}.$$

Note that

$$\begin{aligned} 0 < \alpha \le c = \inf_{h\in\tau}\max_{t\in[0,1]} I(h(t)) &\le \max_{t\in[0,1]} I(tt_1u_0) \le \sup_{t\ge 0} I(tu_0) \\ &< \frac{2^*(a,s)-2}{2\cdot 2^*(a,s)} A^{\frac{2^*(a,s)}{2^*(a,s)-2}}. \end{aligned}$$

Now Lemma 2.1 suggests $\{u_n\} \subset H_a$ has a convergent subsequence, still denoted by $\{u_n\}$. Assume that $\{u_n\}$ converges to some $u \in H_a$. From the continuity of I', we know that u is a weak solution of problem (2.1). Hence $u \geq 0$ from $\langle I'(u), u^- \rangle = 0$, where $u^- = \min\{u, 0\}$.

Moreover, suppose $u \equiv 0$. By (2.9) and (2.10), if $k = 0$, we get $c = I(0) = 0$, which contradicts $c > 0$; if $k > 0$, we obtain

$$c = \left(\frac{1}{2} - \frac{1}{2^*(a,s)}\right)k \geq \frac{2^*(a,s) - 2}{2 \cdot 2^*(a,s)} A^{\frac{2^*(a,s)}{2^*(a,s)-2}},$$

which contradicts $c < \frac{2^*(a,s)-2}{2 \cdot 2^*(a,s)} A^{\frac{2^*(a,s)}{2^*(a,s)-2}}$.

Therefore, $u \not\equiv 0$ and u is a nontrivial solution of problem (1.1). By the Strong Maximum Principle, u is a positive solution of problem (1.1), so Theorem 1.1 holds. ■

Proof of Theorem 1.3. By Theorem 1.1 problem (1.1) has a positive solution u_1. Set $g(x,t) = -f(x,-t)$ for $t \in \mathbb{R}$. It follows from Theorem 1.1 that the equation

$$-\operatorname{div}(|x|^{-2a}\nabla u) - \mu \frac{u}{|x|^{2(1+a)}} = \frac{|u|^{2^*(a,s)-2}}{|x|^s} u + \frac{g(x,u)}{|x|^\sigma}$$

has a positive solution v. Let $u_2 = -v$. Then u_2 is a solution of the equation

$$-\operatorname{div}(|x|^{-2a}\nabla u) - \mu \frac{u}{|x|^{2(1+a)}} = \frac{|u|^{2^*(a,s)-2}}{|x|^s} u + \frac{f(x,u)}{|x|^\sigma}.$$

It is obvious that $u_1 \neq 0$, $u_2 \neq 0$ and $u_1 \neq u_2$. So problem (1.1) has at least two nontrivial solutions. Therefore, Theorem 1.3 holds. ■

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