ANNALES
POLONICI MATHEMATICI
110.2 (2014)

Existence of solutions for impulsive fractional partial neutral integro-differential inclusions with state-dependent delay in Banach spaces

by ZUOMAO YAN and HONGWU ZHANG (Zhangye)

Abstract. We study the existence of mild solutions for a class of impulsive fractional partial neutral integro-differential inclusions with state-dependent delay. We assume that the undelayed part generates an α-resolvent operator and transform it into an integral equation. Sufficient conditions for the existence of solutions are derived by means of the fixed point theorem for discontinuous multi-valued operators due to Dhage and properties of the α-resolvent operator. An example is given to illustrate the theory.

1. Introduction. The theory of impulsive differential or integro-differential systems has become an active area of investigation due to their applications in fields such as mechanics, electrical engineering, medicine, biology, ecology and so on. One can refer to [BH], [HC], [LB] and the references therein. Several authors have established results on the existence of mild solutions for these equations (see [AA], [HG], [HL], [Y1] and references therein). Nonlinear fractional differential or integro-differential equations has recently been an object of increasing interest because of their wide applicability in nonlinear oscillations of earthquakes and other physical phenomena; see the monographs of Kilbas et al. [KS], Miller and Ross [MR], Podlubny [PO] and the papers [BO], [GN], [MS]. The existence of solutions for fractional semilinear differential or integro-differential equations has been extensively studied by many authors (see [E1], [E2], [Y2], [ZJ] and the references therein). On the other hand, functional differential equations with state-dependent delay can be met in various applications. Some recent applications can be found in [AA], [BE], [CN], [HG], [RB], [S]. The problem of the existence of solutions for fractional functional differential equations with

2010 *Mathematics Subject Classification*: Primary 34A37, 34A60; Secondary 35R12, 26A33.

Key words and phrases: impulsive integro-differential inclusions, fractional partial neutral integro-differential inclusions, multi-valued map, state-dependent delay, α-resolvent operator.

DOI: 10.4064/ap110-2-4

state-dependent delay in Banach spaces has attracted considerable interest among researchers [AS], [DN], [SA].

The existence, uniqueness and other quantitative and qualitative properties of solutions to various impulsive semilinear fractional differential and integrodifferential equations have been extensively studied in Banach spaces. For example, Mophou [M] obtained the existence and uniqueness of mild solutions for semilinear impulsive fractional differential equations. Shu et al. [SL] investigated the existence and uniqueness of mild solutions for a class of impulsive fractional partial semilinear differential equations and corrected some errors in [M]. Chauhan et al. [CD] extended the results of [SL] to impulsive fractional order semilinear evolution equations with nonlocal conditions. Balachandran et al. [BK1], [BK2] discussed some fractional-order impulsive integrodifferential equations. The existence of solutions of fractional differential equation of Sobolev type with impulse effect in Banach spaces was also considered in [BK2]. Debbouche and Baleanu [DB] proved the controllability of a class of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems. Further, Dabas et al. [DC] dealt with the existence and uniqueness of mild solution for semilinear fractional-order functional evolution differential equations with infinite delay.

However, many systems arising from realistic models can be described as partial fractional differential or integro-differential inclusions (see [AM], [Y3] and references therein), so it is natural to extend the concept of mild solution for impulsive fractional evolution equations to impulsive systems represented by fractional differential or integro-differential inclusions.

In this paper, we consider a class of impulsive fractional partial neutral integro-differential inclusions with state-dependent delay in Banach spaces of the form

$$(1.1)\qquad {}^cD^\alpha N(x_t) \in AN(x_t) + \int_0^t Q(t-s)N(x_s)\,ds + F(t, x_{\rho(t,x_t)}),$$
$$t\in J=[0,b], \qquad t\neq t_k,\ k=1,\dots,m,$$

$$(1.2)\qquad x_0=\varphi\in\mathcal{B}, \qquad x'(0)=0,$$

$$(1.3)\qquad \Delta x(t_k)=I_k(x_{t_k}), \qquad k=1,\dots,m,$$

where the unknown $x(\cdot)$ takes values in the Banach space X with norm $\|\cdot\|$, ${}^cD^\alpha$ is the Caputo fractional derivative of order $\alpha\in(1,2)$, A and $(Q(t))_{t\geq 0}$ are closed linear operators defined on a common domain which is dense in $(X,\|\cdot\|)$, and $D_t^\alpha\xi(t)$ represents the Caputo derivative of order $\alpha>0$ defined by

$$D_t^\alpha\xi(t)=\int_0^t g_{n-\alpha}(t-s)\frac{d^n}{ds^n}\xi(s)\,ds,$$

where n is the smallest integer greater than or equal to α and $g_\beta(t) := t^{\beta-1}/\Gamma(\beta), t > 0, \beta \geq 0$. The time history $x_t : (-\infty, 0] \to X$ given by $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; and $F : J \times \mathcal{B} \to \mathcal{P}(X)$ is a bounded closed convex-valued multi-valued map, $\mathcal{P}(X)$ is the family of all nonempty subsets of X, $G : J \times \mathcal{B} \to X$, $N(\psi) = \psi(0) + G(t, \psi)$ for $\psi \in \mathcal{B}$, and $I_k : \mathcal{B} \to X$ $(k = 1, \ldots, m)$, $\rho : J \times \mathcal{B} \to (-\infty, b]$, are functions subject to some additional conditions. Moreover, let $0 < t_1 < \cdots < t_m < b$ be given points, and $\Delta x(t_k) := x(t_k^+) - x(t_k^-)$, where $x(t_k^-)$ and $x(t_k^+)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively.

To the best of our knowledge, there is no work reported on the existence of mild solutions for impulsive fractional partial neutral integro-differential inclusions with state-dependent delay of the form (1.1)–(1.3), and the aim of this paper is to close this gap. Motivated by the previously mentioned papers, we will study this interesting problem. Sufficient conditions for the existence are given by means of a fixed point theorem for multi-valued mapping due to Dhage [D] with the α-resolvent operator combined with approximation techniques. In particular, the results of [M], [SL], [CD], [BK1], [BK2], [DB], [DC] are generalized to the fractional multi-valued setting and to the case of infinite delay.

2. Preliminaries. Let $(X, \|\cdot\|)$ be a Banach space. $C(J, X)$ is the Banach space of all continuous functions from J into X with the norm $\|x\|_\infty = \sup\{\|x(t)\| : t \in J\}$ and $L(X)$ denotes the Banach space of bounded linear operators from X to X. A measurable function $x : J \to X$ is *Bochner integrable* if and only if $\|x\|$ is Lebesgue integrable. For properties of the Bochner integral see Yosida [YO]. $L^1(J, X)$ denotes the Banach space of measurable functions $x : J \to X$ which are Bochner integrable, normed by $\|x\|_{L^1} = \int_0^b \|x(t)\|\, dt$ for all $x \in L^1(J, X)$. Furthermore, for appropriate functions $\mathcal{K} : [0, \infty) \to X$ the notation $\widehat{\mathcal{K}}$ denotes the Laplace transform of $\mathcal{K}$. The notation $B_r(x, X)$ stands for the closed ball with center at x and radius $r > 0$ in X.

$\mathcal{P}(X)$ denotes the family of nonempty subsets of X. Let us introduce the following notations:

$$\mathcal{P}_{\mathrm{cl}}(X) = \{x \in \mathcal{P}(X) : x \text{ is closed}\}, \quad \mathcal{P}_{\mathrm{bd}}(X) = \{x \in \mathcal{P}(X) : x \text{ is bounded}\},$$
$$\mathcal{P}_{\mathrm{cv}}(X) = \{x \in \mathcal{P}(X) : x \text{ is convex}\}, \quad \mathcal{P}_{\mathrm{cp}}(X) = \{x \in \mathcal{P}(X) : x \text{ is compact}\}.$$

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}^+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\Big\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\Big\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{\mathrm{bd,cl}}(X), H_d)$ is a metric space and $(\mathcal{P}_{\mathrm{cl}}(X), H_d)$ is a generalized metric space.

A multi-valued map $\Phi : X \to \mathcal{P}(X)$ is *convex* (resp. *closed*) *valued* if $G(X)$ is convex (resp. closed) for all $x \in X$; and Φ is *bounded on bounded sets* if $\Phi(B) = \bigcup_{x\in B} \Phi(x)$ is bounded in X for any bounded set B of X, that is, $\sup_{x\in B} \sup\{\|y\| : y \in \Phi(x)\} < \infty$.

Φ is called *upper semicontinuous* (u.s.c., for short) on X if for any $x \in X$, the set $\Phi(x)$ is a nonempty closed subset of X, and if for each open set B of X containing $\Phi(x)$, there exists an open neighborhood N of x such that $\Phi(N) \subseteq B$.

Φ is said to be *completely continuous* if $\Phi(D)$ is relatively compact for every bounded subset D of X. If the multi-valued map Φ is completely continuous with nonempty compact values, then Φ is u.s.c. if and only if Φ has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in \Phi(x_n)$ imply $y_* \in \Phi(x_*)$.

A multi-valued map $\Phi : J \to \mathcal{P}_{\mathrm{bd,cl,cv}}(X)$ is said to be *measurable* if for each $x \in X$, the function $Y : J \to \mathbb{R}^+$ defined by $Y(t) = d(x, \Phi(t)) = \inf\{d(x,z) : z \in \Phi(t)\}$ is measurable.

Φ has a fixed point if there is $x \in X$ such that $x \in \Phi(x)$.

For more details on multi-valued maps we refer the reader to the books of Deimling [DE], and Hu and Papageorgiou [HP].

In this paper, we assume that the phase space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into X, and satisfying the following fundamental axioms due to Hale and Kato (see, e.g., [HK]):

(A) If $x : (-\infty, \sigma+b] \to X$, $b > 0$, is such that $x|_{[\sigma,\sigma+b]} \in C([\sigma,\sigma+b], X)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma+b]$:

  (i) x_t is in $\mathcal{B}$;
  (ii) $\|x(t)\| \le \tilde{H}\|x_t\|_{\mathcal{B}}$;
  (iii) $\|x_t\|_{\mathcal{B}} \le K(t-\sigma)\sup\{\|x(s)\| : \sigma \le s \le t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$,

  where $\tilde{H} \ge 0$ is a constant; $K, M : [0,\infty) \to [1,\infty)$, K is continuous and M locally bounded; $\tilde{H}, K, M$ are independent of $x(\cdot)$.

(B) For $x(\cdot)$ in (A), $t \mapsto x_t$ is continuous from $[\sigma, \sigma+b]$ into $\mathcal{B}$.

(C) The space $\mathcal{B}$ is complete.

To describe appropriately our problems we say that a function $x : [\mu,\tau] \to X$ is a *normalized piecewise continuous function* on $[\mu,\tau]$ if x is piecewise continuous and left continuous on $(\mu,\tau]$. We denote by $\mathcal{PC}([\mu,\tau], X)$ the space of normalized piecewise continuous functions from $[\mu,\tau]$ into X. In particular, we introduce the space $\mathcal{PC}$ of all functions $x : [0,b] \to X$ such that x is continuous at $t \ne t_k$, $x(t_k) = x(t_k^-)$ and $x(t_k^+)$ exists for $k = 1,\dots,m$. In this paper, we always assume that $\mathcal{PC}$ is endowed with the norm $\|x\|_{\mathcal{PC}} = \sup_{t\in[0,b]} \|x(t)\|$. Then $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space.

To simplify the notations, we put $t_0 = 0$, $t_{m+1} = b$ and for $x \in \mathcal{PC}$, we denote by $\hat{x}_k \in C([t_k, t_{k+1}], X)$, $k = 0, 1, \ldots, m$, the function given by

$$\hat{x}_k(t) := \begin{cases} x(t) & \text{for } t \in (t_k, t_{k+1}], \\ x(t_k^+) & \text{for } t = t_k. \end{cases}$$

Moreover, for $B \subseteq \mathcal{PC}$ we set $\hat{B}_k = \{\hat{x}_k : x \in B\}$, $k = 0, 1, \ldots, m$.

Let us recall the following definitions and facts.

DEFINITION 2.1 ([SA]). A one-parameter family of bounded linear operators $(\mathcal{R}_\alpha(t))_{t\geq 0}$ on X is called an *α-resolvent operator* for

$$ {}^cD^\alpha x(t) = Ax(t) + \int_0^t Q(t-s)x(s)\,ds, \tag{2.1}$$

$$ x_0 = \varphi \in X, \qquad x'(0) = 0, \tag{2.2}$$

if the following conditions hold:

(a) $\mathcal{R}_\alpha(\cdot) : [0, \infty) \to L(X)$ is strongly continuous and $\mathcal{R}_\alpha(0)x = x$ for all $x \in X$ and $\alpha \in (1, 2)$.
(b) For $x \in D(A)$, $\mathcal{R}_\alpha(\cdot)x \in C([0, \infty), [D(A)]) \cap C^1((0, \infty), X)$, we have

$$D_t^\alpha \mathcal{R}_\alpha(t)x = A\mathcal{R}_\alpha(t)x + \int_0^t Q(t-s)\mathcal{R}_\alpha(s)x\,ds,$$

$$D_t^\alpha \mathcal{R}_\alpha(t)x = \mathcal{R}_\alpha(t)Ax + \int_0^t \mathcal{R}_\alpha(t-s)Q(s)x\,ds,$$

for every $t \geq 0$.

In this work we will consider the following conditions:

(P1) $A : D(A) \subseteq X \to X$ is a closed linear operator with $[D(A)]$ dense in X. Let $\alpha \in (1, 2)$. For some $\phi_0 \in (0, \pi/2]$, for each $\phi < \phi_0$ there is a positive constant $C_0 = C_0(\phi)$ such that

$$\Sigma_{0,\alpha\vartheta} := \{\lambda \in \mathbb{C} : \lambda \neq 0,\ |\arg(\lambda)| < \alpha\vartheta\} \subset p(A),$$

where $\vartheta = \phi + \pi/2$ and $\|R(\lambda, A)\| \leq C_0/|\lambda|$ for all $\lambda \in \Sigma_{0,\alpha\vartheta}$.

(P2) For all $t \geq 0$, $Q(t) : D(Q(t)) \subseteq X \to X$ is a closed linear operator, $D(A) \subseteq D(Q(t))$, and $Q(\cdot)x$ is strongly measurable on $(0, \infty)$ for each $x \in D(A)$. There exists $b(\cdot) \in L^1_{\text{loc}}(\mathbb{R}^+)$ such that $\widehat{b}(\lambda)$ exists for $\operatorname{Re}(\lambda) > 0$ and $\|Q(t)x\| \leq b(t)\|x\|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\widehat{Q} : \Sigma_{0,\pi/2} \to L([D(A)], X)$ has an analytical extension (still denoted by $\widehat{Q}$) to $\Sigma_{0,\vartheta}$ such that $\|\widehat{Q}(\lambda)x\| \leq \|\widehat{Q}(\lambda)\|\,\|x\|_1$ for all $x \in D(A)$, and $\|\widehat{Q}(\lambda)\| = O(1/|\lambda|)$ as $|\lambda| \to \infty$.

(P3) There exists a subspace $D \subseteq D(A)$ dense in $[D(A)]$ and a positive constant $\widetilde{C}$ such that $A(D) \subseteq D(A)$, $\widehat{Q}(\lambda)(D) \subseteq D(A)$, and $\|A\widehat{Q}(\lambda)x\| \leq \widetilde{C}\|x\|$, for every $x \in D$ and all $\lambda \in \Sigma_{0,\vartheta}$.

For $r > 0$ and $\theta \in (\pi/2, \vartheta)$, we set

$$\Sigma_{r,\theta} = \{\lambda \in \mathbb{C} : |\lambda| > r,\ |\arg(\lambda)| < \theta\},$$

and we consider the paths

$$\Gamma^1_{r,\theta} = \{te^{i\theta} : t \geq r\}, \quad \Gamma^2_{r,\theta} = \{te^{i\xi} : |\xi| \leq \theta\}, \quad \Gamma^3_{r,\theta} = \{te^{-i\theta} : t \geq r\},$$

with $\Gamma_{r,\theta} = \bigcup_{i=1}^3 \Gamma^i_{r,\theta}$ oriented counterclockwise. In addition,

$$\rho_\alpha(G_\alpha) = \{\lambda \in \mathbb{C} : G_\alpha(\lambda) := \lambda^{\alpha-1}(\lambda^\alpha I - A - \widehat{Q}(\lambda))^{-1} \in L(X)\}.$$

We now define an operator family $(\mathcal{R}_\alpha(t))_{t\geq 0}$ by

$$\mathcal{R}_\alpha(t) := \begin{cases} (2\pi i)^{-1} \int_{\Gamma_{r,\theta}} e^{\lambda t} G_\alpha(\lambda)\, d\lambda, & t > 0, \\ I, & t = 0. \end{cases}$$

Lemma 2.2 ([SA]). *Assume that conditions* (P1)–(P3) *hold. Then there exists a unique α-resolvent operator for problem* (2.1)–(2.2).

Lemma 2.3 ([SA]). *The function $\mathcal{R}_\alpha : [0,\infty) \to L(X)$ is strongly continuous and $\mathcal{R}_\alpha : (0,\infty) \to L(X)$ is uniformly continuous.*

Definition 2.4 ([SA]). For $\alpha \in (1,2)$, we define a family $(\mathcal{S}_\alpha(t))_{t\geq 0}$ by

$$\mathcal{S}_\alpha(t)x := \int_0^t g_{\alpha-1}(t-s)\mathcal{R}_\alpha(s)\, ds \quad \text{for each } t \geq 0.$$

Lemma 2.5 ([SA]). *If $\mathcal{R}_\alpha(\cdot)$ is exponentially bounded in $L(X)$, then so is $\mathcal{S}_\alpha(\cdot)$.*

Lemma 2.6 ([SA]). *If $\mathcal{R}_\alpha(\cdot)$ is exponentially bounded in $L([D(A)])$, then so is $\mathcal{S}_\alpha(\cdot)$.*

Lemma 2.7 ([SA]). *If $R(\lambda_0^\alpha, A)$ is compact for some $\lambda_0^\alpha \in \rho(A)$, then so are $\mathcal{R}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$, for all $t > 0$.*

Definition 2.8. A function $x : (-\infty, b] \to X$ is called a *mild solution* of the problem (1.1)–(1.3) if $x_0 = \varphi$, $x_{\rho(s,x_s)} \in \mathcal{B}$ for every $s \in J$, $\Delta x(t_k) = I_k(x_{t_k})$, $k = 1,\ldots,m$, the restriction of x to $(t_k, t_{k+1}]$ $(k = 0,1,\ldots,m)$ is continuous, and

$$x(t) \in \begin{cases} \mathcal{R}_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t,x_t) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s)F(s,x_{\rho(s,x_s)})\,ds, & t \in [0,t_1], \\ \mathcal{R}_\alpha(t-t_1)[x(t_1^-) + I_1(x_{t_1}) - G(t_1, x_{t_1^+})] \\ \quad + G(t,x_t) + \int_{t_1}^t \mathcal{S}_\alpha(t-s)F(s,x_{\rho(s,x_s)})\,ds, & t \in (t_1,t_2], \\ \vdots \\ \mathcal{R}_\alpha(t-t_m)[x(t_m^-) + I_m(x_{t_m}) - G(t_m, x_{t_m^+})] \\ \quad + G(t,x_t) + \int_{t_m}^t \mathcal{S}_\alpha(t-s)F(s,x_{\rho(s,x_s)})\,ds, & t \in (t_m,b]. \end{cases}$$

LEMMA 2.9. *A set $B \subseteq \mathcal{PC}$ is relatively compact in $\mathcal{PC}$ if, and only if, the set $\hat{B}_k$ is relatively compact in $C([t_k,t_{k+1}],X)$ for every $k = 0,1,\ldots,m$.*

LEMMA 2.10 (Dhage's fixed point theorem [D]). *Let X be a Banach space, and $\Phi_1 : X \to \mathcal{P}_{\mathrm{cl,cv,bd}}(X)$ and $\Phi_2 : X \to \mathcal{P}_{\mathrm{cp,cv}}(X)$ be two multi-valued operators such that*

(a) *Φ_1 is a contraction, and*
(b) *Φ_2 is completely continuous.*

Then either

(i) *the operator inclusion $x \in \Phi_1 x + \Phi_1 x$ has a solution, or*
(ii) *the set $G = \{x \in X : x \in \lambda\Phi_1 x + \lambda\Phi_2 x\}$ is unbounded for $\in (0,1)$.*

3. Main results. In this section we shall present and prove our main results. Assume that $\rho : J \times \mathcal{B} \to (-\infty, b]$ is continuous. In addition, we make the following hypotheses:

(H1) The operator families $\mathcal{R}_\alpha(t)$ and $\mathcal{S}_\alpha(t)$ are compact for all $t > 0$, and there exist constants M and δ such that $\|\mathcal{R}_\alpha(t)\|_{L(X)} \le Me^{\delta t}$ and $\|\mathcal{S}_\alpha(t)\|_{L(X)} \le Me^{\delta t}$ for every $t \in J$.
(H2) The function $t \mapsto \varphi_t$ is continuous from $\mathcal{R}(\rho^-) = \{(s,\psi) \in J \times \mathcal{B} : \rho(s,\psi) \le 0\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $J^\varphi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \le J^\varphi(t)\|\varphi\|_{\mathcal{B}}$ for each $t \in \mathcal{R}(\rho^-)$.
(H3) The multi-valued map $F : J \times \mathcal{B} \to \mathcal{P}_{\mathrm{bd,cl,cv}}(X)$ is such that for each $t \in J$, the function $F(t,\cdot) : \mathcal{B} \to \mathcal{P}_{\mathrm{bd,cl,cv}}(X)$ is u.s.c. and for each $\psi \in \mathcal{B}$, the function $F(\cdot,\psi)$ is measurable; for each fixed $\psi \in \mathcal{B}$, the set

$$S_{F,\psi} = \{f \in L^1(J,X) : f(t) \in F(t,\psi) \text{ for a.e. } t \in J\}$$

is nonempty.

(H4) There exist a continuous function $m : J \to [0, \infty)$ and a continuous nondecreasing function $\Theta : [0, \infty) \to (0, \infty)$ such that

$$\begin{aligned}\|F(t,\psi)\| &= \sup\{\|f\| : f \in F(t,\psi)\}\\ &\le m(t)\Theta(\|\psi\|_{\mathcal{B}}), \quad t \in J,\ \psi \in \mathcal{B},\end{aligned}$$

with

$$\int_1^\infty \frac{1}{s+\Theta(s)}\,ds = \infty.$$

(H5) The function $G : J \times \mathcal{B} \to X$ is continuous and there exists a $L > 0$ such that

$$\begin{aligned}&\|G(t,\psi_1) - G(t,\psi_2)\|\\ &\qquad \le L[|t_1 - t_2| + \|\psi_1 - \psi_2\|_{\mathcal{B}}], \quad t_1, t_2 \in J,\ \psi_1, \psi_2 \in \mathcal{B},\\ &\qquad \|G(t,\psi)\| \le L(\|\psi\|_{\mathcal{B}} + 1), \quad t \in J,\ \psi \in \mathcal{B}.\end{aligned}$$

(H6) The functions $I_k : \mathcal{B} \to X$ are continuous and there exist constants c_k such that

$$0 \le \limsup_{\|\psi\|_{\mathcal{B}} \to \infty} \frac{\|I_k(\psi)\|}{\|\psi\|_{\mathcal{B}}} \le c_k, \quad \psi \in \mathcal{B},\ k = 1, \ldots, m.$$

LEMMA 3.1 ([HG]). *Let $x : (-\infty, b] \to X$ be such that $x_0 = \varphi$ and $x|_{[0,b]} \in \mathcal{PC}(J, X)$. Then*

$$\begin{aligned}\|x_s\|_{\mathcal{B}} &\le (M_b + J_0^\varphi)\|\varphi\|_{\mathcal{B}}\\ &\quad + K_b \sup\{\|x(\theta)\| : \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup J,\end{aligned}$$

where $J_0^\varphi = \sup_{t \in \mathcal{R}(\rho^-)} J^\varphi(t)$, $M_b = \sup_{t \in J} M(t)$, and $K_b = \sup_{t \in J} K(t)$.

LEMMA 3.2 ([LO]). *Let J be a compact interval and X be a Banach space. Let F be a multi-valued map satisfying* (H3) *and let P be a continuous linear operator from $L^1(J, X)$ to $C(J, X)$. Then the operator*

$$P \circ S_F : C(J, X) \to \mathcal{P}_{\mathrm{cp,cv}}(X), \quad x \mapsto (P \circ S_F)(x) := P(S_{F,x}),$$

has a closed graph in $C(J, X) \times C(J, X)$.

THEOREM 3.3. *If the assumptions* (H1)–(H6) *are satisfied with $\rho(t, \psi) \le t$ for every $(t, \psi) \in J \times \mathcal{B}$, then the problem* (1.1)–(1.3) *has a mild solution on J, provided that*

$$\max_{1 \le k \le m} \{M_* N_* [1 + K_b M (c_k + L)] + K_b L M\} < 1, \tag{3.1}$$

where $M = M \max\{1, e^{\delta b}\}$, $N_* = \max\{1, e^{-\delta b}\}$.

Proof. Consider the space $\mathcal{BPC} = \{x : (-\infty, b] \to X : x_0 = 0,\ x|_J \in \mathcal{PC}(J, X)\}$ endowed with the uniform convergence topology and define a multi-valued map $\Phi : \mathcal{BPC} \to \mathcal{P}(\mathcal{BPC})$ by letting Φx be the set of $h \in \mathcal{BPC}$ such that

$$h(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \mathcal{R}_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t,\bar{x}_t) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s) f(s)\, ds, & t \in [0, t_1], \\ \mathcal{R}_\alpha(t-t_1)[\bar{x}(t_1^-) + I_1(\bar{x}_{t_1}) - G(t_1, \bar{x}_{t_1^+})] \\ \quad + G(t,\bar{x}_t) + \int_{t_1}^t \mathcal{S}_\alpha(t-s) f(s)\, ds, & t \in (t_1, t_2], \\ \vdots \\ \mathcal{R}_\alpha(t-t_m)[\bar{x}(t_m^-) + I_m(\bar{x}_{t_m}) - G(t_m, \bar{x}_{t_m^+})] \\ \quad + G(t,\bar{x}_t) + \int_{t_m}^t \mathcal{S}_\alpha(t-s) f(s)\, ds, & t \in (t_m, b], \end{cases}$$

where $f \in S_{F,\bar{x}_\rho} = \{f \in L^1(J,X) : f(t) \in F(t, \bar{x}_{\rho(s,\bar{x}_t)}) \text{ a.e. } t \in J\}$, and $\bar{x} : (-\infty, 0] \to X$ is such that $\bar{x}_0 = \varphi$ and $\bar{x} = x$ on J. We aim to show that Φ has a fixed point, which is a solution of (1.1)–(1.3).

Let $\{\sigma_n : n \in \mathbb{N}\}$ be a decreasing sequence in $(0, t_1) \subset (0, b)$ such that $\lim_{n\to\infty} \sigma_n = 0$. We consider the following problem:

$$(3.2)\qquad {}^cD^\alpha \widetilde{N}(x_t) \in A\widetilde{N}(x_t) + \int_0^t Q(t-s)\widetilde{N}(x_s)\, ds + F(t, x_{\rho(t,x_t)}),$$
$$t \in J = [0,b],\ t \neq t_k,\ k = 1, \ldots, m,$$

$$(3.3)\qquad x_0 = \varphi \in \mathcal{B}, \qquad x'(0) = 0,$$

$$(3.4)\qquad \Delta x(t_k) = \mathcal{R}_\alpha(\sigma_n) I_k(x_{t_k}), \qquad k = 1, \ldots, m,$$

where $\widetilde{N}(x_t) = \varphi(0) + \mathcal{R}_\alpha(\sigma_n) G(t, x_t)$. We shall show that this problem has a mild solution $x_n \in \mathcal{BPC}$.

For fixed $n \in \mathbb{N}$, define a multi-valued map $\Phi_n : \mathcal{BPC} \to \mathcal{P}(\mathcal{BPC})$ by letting $\Phi_n x$ be the set of $h_n \in \mathcal{BPC}$ such that

$$h_n(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \mathcal{R}_\alpha(t)[\varphi(0) - \mathcal{R}_\alpha(\sigma_n) G(0,\varphi)] + \mathcal{R}_\alpha(\sigma_n) G(t,\bar{x}_t) \\ \quad + \int_0^t \mathcal{S}_\alpha(t-s) f(s)\, ds, & t \in [0, t_1], \\ \mathcal{R}_\alpha(t-t_1)\big[\bar{x}(t_1^-) + \mathcal{R}_\alpha(\sigma_n) I_1(\bar{x}_{t_1}) - \mathcal{R}_\alpha(\sigma_n) G(t_1, \bar{x}_{t_1^+})\big] \\ \quad + \mathcal{R}_\alpha(\sigma_n) G(t,\bar{x}_t) + \int_{t_1}^t \mathcal{S}_\alpha(t-s) f(s)\, ds, & t \in (t_1, t_2], \\ \vdots \\ \mathcal{R}_\alpha(t-t_m)\big[\bar{x}(t_m^-) + \mathcal{R}_\alpha(\sigma_n) I_m(\bar{x}_{t_m}) - \mathcal{R}_\alpha(\sigma_n) G(t_m, \bar{x}_{t_m^+})\big] \\ \quad + \mathcal{R}_\alpha(\sigma_n) G(t,\bar{x}_t) + \int_{t_m}^t \mathcal{S}_\alpha(t-s) f(s)\, ds, & t \in (t_m, b], \end{cases}$$

where $f \in S_{F,\bar{x}_\rho}$. It is easy to see that each fixed point of Φ_n is a mild solution of the Cauchy problem (3.2)–(3.4).

Let $\bar{\varphi} : (-\infty, 0) \to X$ be the extension of $(-\infty, 0]$ such that $\bar{\varphi}(\theta) = \varphi(0) = 0$ on J and $J_0^{\varphi} = \sup\{J^{\varphi}(s) : s \in \mathcal{R}(\rho^-)\}$. Now, we consider the multi-valued operators Φ_n^1 and Φ_n^2 defined by

$$(\Phi_n^1 x)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ -\mathcal{R}_\alpha(t)\mathcal{R}_\alpha(\sigma_n)G(0,\varphi) + \mathcal{R}_\alpha(\sigma_n)G(t,\bar{x}_t), & t \in [0, t_1], \\ -\mathcal{R}_\alpha(t-t_1)\mathcal{R}_\alpha(\sigma_n)G(t_1,\bar{x}_{t_1^+}) + \mathcal{R}_\alpha(\sigma_n)G(t,\bar{x}_t), & t \in (t_1, t_2], \\ \vdots \\ -\mathcal{R}_\alpha(t-t_m)\mathcal{R}_\alpha(\sigma_n)G(t_m,\bar{x}_{t_m^+}) + \mathcal{R}_\alpha(\sigma_n)G(t,\bar{x}_t), & t \in (t_m, b], \end{cases}$$

and

$$(\Phi_n^2 x)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \mathcal{R}_\alpha(t)\varphi(0) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)\,ds, & t \in [0, t_1], \\ \mathcal{R}_\alpha(t-t_1)[\bar{x}(t_1^-) + \mathcal{R}_\alpha(\sigma_n)I_1(\bar{x}_{t_1})] \\ \quad + \int_{t_1}^t \mathcal{S}_\alpha(t-s)f(s)\,ds, & t \in (t_1, t_2], \\ \vdots \\ \mathcal{R}_\alpha(t-t_m)[\bar{x}(t_m^-) + \mathcal{R}_\alpha(\sigma_n)I_m(\bar{x}_{t_m})] \\ \quad + \int_{t_m}^t \mathcal{S}_\alpha(t-s)f(s)\,ds, & t \in (t_m, b]. \end{cases}$$

It is clear that $\Phi_n = \Phi_n^1 + \Phi_n^2$. The problem of finding mild solutions of (1.1)–(1.3) is reduced to finding solutions of the operator inclusion $x \in \Phi_n^1(x) + \Phi_n^2(x)$. We shall show that Φ_n^1 and Φ_n^2 satisfy the conditions of Lemma 2.10.

STEP 1. *Φ_n^1 is a contraction on $\mathcal{BPC}$.*

Let $t \in [0, t_1]$ and $v^*, v^{**} \in \mathcal{BPC}$. From (H5) and Lemma 3.1, we have

$$\begin{aligned} &\|(\Phi_n^1 v^*)(t) - (\Phi_n^2 v^{**})(t)\| \\ &\qquad \le \|\mathcal{R}_\alpha(\sigma_n)[G(t,\overline{v^*}_t) - G(t,\overline{v^{**}}_t)]\| \le LMe^{\delta\sigma_n}\|\overline{v^*}_t - \overline{v^{**}}_t\|_{\mathcal{B}} \\ &\qquad \le LMe^{\delta\sigma_n}K_b \sup\{\|\overline{v^*}(\tau) - \overline{v^{**}}(\tau)\| : 0 \le \tau \le t\} \\ &\qquad \le LMe^{\delta\sigma_n}K_b \sup_{s\in[0,b]} \|\overline{v^*}(s) - \overline{v^{**}}(s)\| \\ &\qquad = LMe^{\delta\sigma_n}K_b \sup_{s\in[0,b]} \|v^*(s) - v^{**}(s)\| \quad (\text{since } \bar{v} = v \text{ on } J) \\ &\qquad = LMe^{\delta\sigma_n}K_b\|v^* - v^{**}\|_{\mathcal{PC}}. \end{aligned}$$

Similarly, for any $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$, we have

$$\begin{aligned} &\|(\Phi_n^1 v^*)(t) - (\Phi_n^1 v^{**})(t)\| \\ &\qquad = \|\mathcal{R}_\alpha(t-t_k)[\mathcal{R}_\alpha(\sigma_n)(-G(t_k,\overline{v^*}_{t_k^+}) + G(t_k,\overline{v^{**}}_{t_k^+}))]\| \\ &\qquad\quad + \|\mathcal{R}_\alpha(\sigma_n)[G(t,\overline{v^*}_t) - G(t,\overline{v^{**}}_t)]\| \end{aligned}$$

$$
\begin{aligned}
&\le Me^{\delta(t-t_k)}LMe^{\delta\sigma_n}\|\overline{v^*}_{t_k^+}-\overline{v^{**}}_{t_k^+}\|_{\mathcal{B}}+LMe^{\delta\sigma_n}\|\overline{v^*}_t-\overline{v^{**}}_t\|_{\mathcal{B}}\\
&\le M_*N_*LMe^{\delta\sigma_n}K_b\sup\{\|\overline{v^*}(\tau)-\overline{v^{**}}(\tau)\|:0\le\tau\le b\}\\
&\quad+LMe^{\delta\sigma_n}K_b\sup\{\|\overline{v^*}(\tau)-\overline{v^{**}}(\tau)\|:0\le\tau\le t\}\\
&\le (M_*N_*+1)LMe^{\delta\sigma_n}K_b\sup_{s\in[0,b]}\|\overline{v^*}(s)-\overline{v^{**}}(s)\|\\
&=(M_*N_*+1)LMe^{\delta\sigma_n}K_b\sup_{s\in[0,b]}\|v^*(s)-v^{**}(s)\|\quad(\text{since }\bar v=v\text{ on }J)\\
&=(M_*N_*+1)LMe^{\delta\sigma_n}K_b\|v^*-v^{**}\|_{\mathcal{PC}},
\end{aligned}
$$

where $M_*=M\max\{1,e^{\delta b}\}$ and $N_*=\max\{1,e^{-\delta b}\}$. Thus, for all $t\in[0,b]$,

$$\|(\Phi_n^1v^*)(t)-(\Phi_n^1v^{**})(t)\|\le(M_*N_*+1)LMe^{\delta\sigma_n}K_b\|v^*-v^{**}\|_{\mathcal{PC}}.$$

Since $\lim_{n\to\infty}\sigma_n=0$, it follows that

$$\|(\Phi_n^1v^*)(t)-(\Phi_n^1v^{**})(t)\|\le L_0\|v^*-v^{**}\|_{\mathcal{PC}}.$$

Taking supremum over t gives

$$\|\Phi_n^1v^*-\Phi_n^1v^{**}\|_{\mathcal{PC}}\le L_0\|v^*-v^{**}\|_{\mathcal{PC}},$$

where $L_0=(M_*N_*+1)LMK_b$. By (3.1), we see that $L_0<1$. Hence, Φ_n^1 is a contraction on $\mathcal{BPC}$.

STEP 2. *Φ_n^2 has compact, convex values and it is completely continuous.*

(1) *Φ_n^2x is convex for each $x\in\mathcal{BPC}$.*

In fact, if $\tilde h_n^1,\tilde h_n^2$ belong to Φ_n^2x, then there exist $f_1,f_2\in S_{F,\bar x_\rho}$ such that

$$\tilde h_n^i(t)=\mathcal{R}_\alpha(t)\varphi(0)+\int_0^t\mathcal{S}_\alpha(t-s)f_i(s)\,ds,\qquad t\in[0,t_1],\ i=1,2.$$

Let $0\le\lambda\le1$. For each $t\in[0,t_1]$ we have

$$(\lambda\tilde h_n^1+(1-\lambda)\tilde h_n^2)(t)=\mathcal{R}_\alpha(t)\varphi(0)+\int_0^t\mathcal{S}_\alpha(t-s)[\lambda f_1(s)+(1-\lambda)f_2(s)]\,ds.$$

Similarly, for any $t\in(t_k,t_{k+1}]$, $k=1,\dots,m$, we have

$$\tilde h_n^i(t)=\mathcal{R}_\alpha(t-t_k)[\bar x(t_k^-)+\mathcal{R}_\alpha(\sigma_n)I_k(\bar x_{t_k})]+\int_{t_k}^t\mathcal{S}_\alpha(t-s)f_i(s)\,ds,\qquad i=1,2.$$

Let $0\le\lambda\le1$. For each $t\in(t_k,t_{k+1}]$, $k=1,\dots,m$, we have

$$
\begin{aligned}
(\lambda\tilde h_n^1+(1-\lambda)\tilde h_n^2)(t)&=\mathcal{R}_\alpha(t-t_k)[\bar x(t_k^-)+\mathcal{R}_\alpha(\sigma_n)I_k(\bar x_{t_k})]\\
&\quad+\int_{t_k}^t\mathcal{S}_\alpha(t-s)[\lambda f_1(s)+(1-\lambda)f_2(s)]\,ds.
\end{aligned}
$$

Since $S_{F,\bar{x}_\rho}$ is convex (as F has convex values) we have $(\lambda\tilde{h}_n^1 + (1-\lambda)\tilde{h}_n^2) \in \Phi_n^2 x$.

(2) *Φ_n^2 maps bounded sets into bounded sets in $\mathcal{BPC}$.*

Indeed, it is enough to show that there exists a positive constant $\mathcal{L}$ such that for each $\tilde{h}_n \in \Phi_n^2 x$ with $x \in B_r(0, \mathcal{BPC}) = \{x \in \mathcal{BPC} : \|x\|_{\mathcal{PC}} \le r\}$, one has $\|\tilde{h}_n\|_{\mathcal{PC}} \le \mathcal{L}$. If $\tilde{h}_n \in \Phi_n^2 x$, then there exists $f \in S_{F,\bar{x}_\rho}$ such that, for each $t \in [0, t_1]$,

$$\tilde{h}_n(t) = \mathcal{R}_\alpha(t)\varphi(0) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)\,ds. \tag{3.5}$$

If $x \in B_r(0, \mathcal{BPC})$, from Lemma 3.1 it follows that

$$\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathcal{B}} \le (M_b + J_0^\varphi)\|\varphi\|_{\mathcal{B}} + K_b r := r^*.$$

By (H1), (H4), from (3.5) we have, for $t \in [0, t_1]$,

$$\begin{aligned}
\|\tilde{h}_n(t)\| &\le \|\mathcal{R}_\alpha(t)\varphi(0)\| + \left\| \int_0^t \mathcal{S}_\alpha(t-s)f(s)\,ds \right\| \\
&\le Me^{\delta t}\tilde{H}\|\varphi\|_{\mathcal{B}} + Me^{\delta t}\int_0^t e^{-\delta s}m(s)\Theta(\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathcal{B}})\,ds \\
&\le M_*\tilde{H}\|\varphi\|_{\mathcal{B}} + M_*\Theta(r^*)\int_0^{t_1} e^{-\delta s}m(s)\,ds =: \mathcal{L}_0.
\end{aligned}$$

Similarly, for any $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$, we have

$$\tilde{h}_n(t) = \mathcal{R}_\alpha(t-t_k)[\bar{x}(t_k^-) + \mathcal{R}_\alpha(\sigma_n)I_k(\bar{x}_{t_k})] + \int_0^t \mathcal{S}_\alpha(t-s)f(s)\,ds. \tag{3.6}$$

However, on the other hand, from the condition (H6), we conclude that there exist positive constants ϵ_k $(k = 1, \ldots, m), \gamma_1$ such that, for all $\|\psi\|_{\mathcal{B}} > \gamma_1$,

$$\|I_k(\psi)\| \le (c_k + \epsilon_k)\|\psi\|_{\mathcal{B}},$$

$$\max_{1\le k\le m}\{M_*N_*[1 + K_bM(c_k + \epsilon_k + L)] + K_bLM\} < 1. \tag{3.7}$$

Let

$$\begin{aligned}
&F_1 = \{\psi : \|\psi\|_{\mathcal{B}} \le \gamma_1\}, \qquad F_2 = \{\psi : \|\psi\|_{\mathcal{B}} > \gamma_1\}, \\
&C_1 = \max\{\|I_k(\psi)\| : x \in F_1\}.
\end{aligned}$$

Then

$$\|I_k(\psi)\| \le C_1 + (c_k + \epsilon_k)\|\psi\|_{\mathcal{B}}. \tag{3.8}$$

By (H1), (H4), (3.8), from (3.6) we have for $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$,

$$\begin{aligned}
\|\tilde{h}_n(t)\| &\le \|\mathcal{R}_\alpha(t-t_k)[\bar{x}(t_k^-) + \mathcal{R}_\alpha(\sigma_n)I_k(\bar{x}_{t_k})]\| + \Big\| \int_{t_k}^{t} \mathcal{S}_\alpha(t-s)f(s)\,ds \Big\| \\
&\le Me^{\delta(t-t_k)}\{\|\bar{x}(t_k^-)\| + Me^{\delta\sigma_n}[C_1 + (c_k+\epsilon_k)\|\bar{x}_{t_k}\|_{\mathcal{B}}]\} \\
&\quad + Me^{\delta t}\int_{t_k}^{t} e^{-\delta s} m(s)\Theta(\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathcal{B}})\,ds \\
&\le M_*N_*\{r + Me^{\delta\sigma_n}[C_1 + (c_k+\epsilon_k)r^*]\} \\
&\quad + M_*\Theta(r^*)\int_{t_k}^{t_{k+1}} e^{-\delta s}m(s)\,ds := \mathcal{L}_k.
\end{aligned}$$

Take $\mathcal{L} = \max_{0\le k\le m}\{\mathcal{L}_k\}$. Then for each $\tilde{h}_n \in \Phi_n^2 x$, we have $\|\tilde{h}_n\|_{\mathcal{PC}} \le \mathcal{L}$.

(3) *Φ_n^2 maps bounded sets into equicontinuous sets of $\mathcal{BPC}$.*

Let $0 < \tau_1 < \tau_2 \le t_1$. For each $x \in B_r(0, \mathcal{BPC})$ and $\tilde{h}_n \in \Phi_n^2 x$, there exists $f \in S_{F,\bar{x}_\rho}$ such that

$$\tilde{h}_n(t) = \mathcal{R}_\alpha(t)\varphi(0) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)\,ds. \tag{3.9}$$

Then

$$\begin{aligned}
&\|\tilde{h}_n(\tau_2) - \tilde{h}_n(\tau_1)\| \\
&\quad\le \|[\mathcal{R}_\alpha(\tau_2) - \mathcal{R}_\alpha(\tau_1)]\varphi(0)\| + \Big\| \int_0^{\tau_1-\varepsilon} [\mathcal{S}_\alpha(\tau_2-s) - \mathcal{S}_\alpha(\tau_1-s)]f(s)\,ds \Big\| \\
&\qquad + \Big\| \int_{\tau_1-\varepsilon}^{\tau_1} [\mathcal{S}_\alpha(\tau_2-s) - \mathcal{S}_\alpha(\tau_1-s)]f(s)\,ds \Big\| + \Big\| \int_{\tau_1}^{\tau_2} \mathcal{S}_\alpha(\tau_2-s)f(s)\,ds \Big\| \\
&\quad\le \|[\mathcal{R}_\alpha(\tau_2) - \mathcal{R}_\alpha(\tau_1)]\varphi(0)\| \\
&\qquad + \Theta(r^*)\int_0^{\tau_1-\varepsilon} \|\mathcal{S}_\alpha(\tau_2-s) - \mathcal{S}_\alpha(\tau_1-s)\| m(s)\,ds \\
&\qquad + 2M_*\Theta(r^*)\int_{\tau_1-\varepsilon}^{\tau_1} e^{-\delta s}m(s)\,ds + Me^{\delta\tau_2}\Theta(r^*)\int_{\tau_1}^{\tau_2} e^{-\delta s}m(s)\,ds.
\end{aligned}$$

Similarly, for any $\tau_1, \tau_2 \in (t_k, t_{k+1}]$, $\tau_1 < \tau_2$, $k = 1, \ldots, m$, we have

$$\tilde{h}_n(t) = \mathcal{R}_\alpha(t-t_k)[\bar{x}(t_k^-) + \mathcal{R}_\alpha(\sigma_n)I_k(\bar{x}_{t_k})] + \int_{t_k}^{t} \mathcal{S}_\alpha(t-s)f(s)\,ds. \tag{3.10}$$

Then

$$
\begin{aligned}
&\|\tilde{h}_n(\tau_2) - \tilde{h}_n(\tau_1)\| \\
&\qquad \le \|[\mathcal{R}_\alpha(\tau_2) - \mathcal{R}_\alpha(\tau_1)][\bar{x}(t_k^-) + \mathcal{R}_\alpha(\sigma_n) I_k(\bar{x}_{t_k})]\| \\
&\qquad\quad + \Big\| \int_{t_k}^{\tau_1-\varepsilon} [\mathcal{S}_\alpha(\tau_2 - s) - \mathcal{S}_\alpha(\tau_1 - s)] f(s)\, ds \Big\| \\
&\qquad\quad + \Big\| \int_{\tau_1-\varepsilon}^{\tau_1} [\mathcal{S}_\alpha(\tau_2 - s) - \mathcal{S}_\alpha(\tau_1 - s)] f(s)\, ds \Big\| + \Big\| \int_{\tau_1}^{\tau_2} \mathcal{S}_\alpha(\tau_2 - s) f(s)\, ds \Big\| \\
&\qquad \le \|[\mathcal{R}_\alpha(\tau_2) - \mathcal{R}_\alpha(\tau_1)][\bar{x}(t_k^-) + \mathcal{R}_\alpha(\sigma_n) I_k(\bar{x}_{t_k})]\| \\
&\qquad\quad + \Theta(r^*) \int_{t_k}^{\tau_1-\varepsilon} \|\mathcal{S}_\alpha(\tau_2 - s) - \mathcal{S}_\alpha(\tau_1 - s)\| m(s)\, ds \\
&\qquad\quad + 2M_* \Theta(r^*) \int_{\tau_1-\varepsilon}^{\tau_1} e^{-\delta s} m(s)\, ds + M e^{\delta \tau_2} \Theta(r^*) \int_{\tau_1}^{\tau_2} e^{-\delta s} m(s)\, ds.
\end{aligned}
$$

The right-hand side of the above inequality is independent of $x \in B_r(0, \mathcal{BPC})$ and tends to zero as $\tau_2 \to \tau_1$, with ε sufficiently small, since the compactness of $\mathcal{R}_\alpha(t)$, $\mathcal{S}_\alpha(t)$ for $t > 0$ implies the continuity in the uniform operator topology and the set $\{\mathcal{R}_\alpha(\sigma_n) I_k(\bar{x}_{t_k}) : x \in B_r(0, \mathcal{BPC}),\ k = 1, \ldots, m\}$ is relatively compact in X.

It remains to prove that the functions $\Phi_n^2 x$, $x \in B_r(0, \mathcal{BPC})$, are equicontinuous at $t = 0$. Indeed, this is true since $\mathcal{R}_\alpha(\sigma_n)$ is a compact operator. Thus, the set $\{\Phi_n^2 x : x \in B_r(0, \mathcal{BPC})\}$ is equicontinuous.

(4) *Φ_n^2 is a compact multi-valued map.*

From the above claims, we see that the family $\Phi_n^2 B_r(0, \mathcal{BPC})$ is uniformly bounded and equicontinuous. Therefore, it suffices to show by the Arzelà–Ascoli theorem that Φ_n^2 maps $B_r(0, \mathcal{BPC})$ into a relatively compact set in X.

To this end, we decompose Φ_n^2 as $\Phi_n^2 = \Gamma_n^1 + \Gamma_n^2$, where the map $\Gamma_n^1 : B_r(0, \mathcal{BPC}) \to \mathcal{P}(\mathcal{BPC})$ is defined by letting $\Gamma_n^1 x$ be the set of $\tilde{\gamma}_n^1 \in \mathcal{BPC}$ such that

$$
\tilde{\gamma}_n^1(t) = \begin{cases}
\int_0^t S_\alpha(t-s) f(s)\, ds, & t \in [0, t_1], \\
\int_{t_1}^t S_\alpha(t-s) f(s)\, ds, & t \in (t_1, t_2], \\
\vdots & \\
\int_{t_m}^t S_\alpha(t-s) f(s)\, ds, & t \in (t_m, b],
\end{cases}
$$

and the map $\Gamma_n^2 : B_r(0, \mathcal{BPC}) \to \mathcal{P}(\mathcal{BPC})$ is defined by letting $\Gamma_n^2 x$ be the set of $\tilde{\gamma}_n^2 \in \mathcal{BPC}$ such that

$$\tilde{\gamma}_n^2(t)=\begin{cases}\mathcal{R}_\alpha(t)\varphi(0), & t\in[0,t_1],\\ \mathcal{R}_\alpha(t-t_1)[\bar{x}(t_1^-)+\mathcal{R}_\alpha(\sigma_n)I_1(\bar{x}_{t_1})], & t\in(t_1,t_2],\\ \vdots & \\ \mathcal{R}_\alpha(t-t_m)[\bar{x}(t_m^-)+\mathcal{R}_\alpha(\sigma_n)I_m(\bar{x}_{t_m})], & t\in(t_m,b].\end{cases}$$

We now prove that $\Gamma_1(B_r(0,\mathcal{BPC}))(t)=\{\tilde{\gamma}_n^1(t):\tilde{\gamma}_n^1(t)\in\Gamma_n^1(B_r(0,\mathcal{BPC}))\}$ is relatively compact for every $t\in[0,b]$. Let $0<t\le s\le t_1$ be fixed, and let ε be a real number satisfying $0<\varepsilon<t$. For $x\in B_r(0,\mathcal{BPC})$, we define

$$\tilde{\gamma}_n^{1,\varepsilon}(t)=\int_0^{t-\varepsilon}\mathcal{S}_\alpha(t-s)f(s)\,ds,$$

where $f\in S_{F,\bar{x}_\rho}$. Using the compactness of $\mathcal{S}_\alpha(t)$ for $t>0$, we deduce that the set $U_\varepsilon(t)=\{\tilde{\gamma}_n^{1,\varepsilon}(t):x\in B_r(0,\mathcal{BPC})\}$ is relatively compact in X for every ε, $0<\varepsilon<t$. Moreover, for every $x\in B_r(0,\mathcal{BPC})$ we have

$$\begin{aligned}\|\tilde{\gamma}_n^1(t)-\tilde{\gamma}_n^{1,\varepsilon}(t)\|&\le\left\|\int_{t-\varepsilon}^{t}\mathcal{S}_\alpha(t-s)f(s)\,ds\right\|\\&\le M_*\Theta(r^*)\int_{t-\varepsilon}^{t}e^{-\delta s}m(s)\,ds.\end{aligned}$$

Similarly, for any $t\in(t_k,t_{k+1}]$ with $k=1,\dots,m$, let $t_k<t\le s\le t_{k+1}$ be fixed, and let ε be a real number satisfying $0<\varepsilon<t$. For $x\in B_r(0,\mathcal{BPC})$, we define

$$\tilde{\gamma}_n^{1,\varepsilon}(t)=\int_{t_k}^{t-\varepsilon}\mathcal{S}_\alpha(t-s)f(s)\,ds,$$

where $f\in S_{F,\bar{x}_\rho}$. Using the compactness of $\mathcal{S}_\alpha(t)$ for $t>0$, we deduce that the set $U_\varepsilon(t)=\{\tilde{\gamma}_n^{1,\varepsilon}(t):x\in B_r(0,\mathcal{BPC})\}$ is relatively compact in X for all $0<\varepsilon<t$. Moreover, for every $x\in B_r(0,\mathcal{BPC})$ we have

$$\begin{aligned}\|\tilde{\gamma}_n^1(t)-\tilde{\gamma}_n^{1,\varepsilon}(t)\|&\le\left\|\int_{t-\varepsilon}^{t}\mathcal{S}_\alpha(t-s)f(s)\,ds\right\|\\&\le M_*\Theta(r^*)\int_{t-\varepsilon}^{t}e^{-\delta s}m(s)\,ds.\end{aligned}$$

The right-hand side tends to zero as $\varepsilon\to0$. Since there are relatively compact sets arbitrarily close to the set $U(t)=\{\tilde{\gamma}_n^1(t):x\in B_r(0,\mathcal{BPC})\}$, the Arzelà–Ascoli theorem shows that Γ_n^2 is a compact multi-valued map.

Next, we show that

$$\Gamma_n^2(B_r(0,\mathcal{BPC}))(t)=\{\tilde{\gamma}_n^2(t):\tilde{\gamma}_n^2(t)\in\Gamma_2(B_r(0,\mathcal{BPC}))\}$$

is relatively compact for every $t \in [0,b]$. For all $t \in [0,t_1]$, since $\tilde{\gamma}_n^2(t) = \mathcal{R}_\alpha(t)\varphi(0)$, by (H1), it follows that $\{\tilde{\gamma}_n^2(t) : t \in [0,t_1],\, x \in B_r(0,\mathcal{BPC})\}$ is a compact subset of X. On the other hand, for $t \in (t_k,t_{k+1}]$, $k = 1,\dots,m$, and $x \in B_r(0,\mathcal{BPC})$, there exists $r' > 0$ such that

$$[\widehat{\tilde{\gamma}_n^2}]_k(t) \in \begin{cases} \mathcal{R}_\alpha(t-t_k)[\bar{x}(t_k^-) + \mathcal{R}_\alpha(\sigma_n)I_k(\bar{x}_{t_k})], & t\in(t_k,t_{k+1}),\ x\in B_{r'}(0,\mathcal{BPC}),\\ \mathcal{R}_\alpha(t_{k+1}-t_k)[\bar{x}(t_k^-) + \mathcal{R}_\alpha(\sigma_n)I_k(\bar{x}_{t_k})], & t = t_{k+1},\ x\in B_{r'}(0,\mathcal{BPC}),\\ \bar{x}(t_k^-) + \mathcal{R}_\alpha(\sigma_n)I_k(\bar{x}_{t_k}), & t = t_k, x\in B_{r'}(0,\mathcal{BPC}), \end{cases}$$

where $B_{r'}(0,\mathcal{BPC})$ is a closed ball of radius r'. From (3.8), $I_k(\bar{x}_{t_k})$ is bounded in X. By the compactness of $(\mathcal{R}_\alpha(t))_{t>0}$, the sets $\{\mathcal{R}_\alpha(\sigma_n)I_k(\bar{x}_{t_k}) : x \in B_{r'}(0,\mathcal{BPC}),\, k = 1,\dots,m\}$ are relatively compact in X. Also, it follows that $[\widehat{\tilde{\gamma}_n^2}]_k(t)$ is relatively compact in X, for all $t \in [t_k,t_{k+1}]$, $k = 1,\dots,m$. By Lemma 2.9, we infer that $\Gamma_n^2(B_r(0,\mathcal{BPC}))$ is relatively compact. Moreover, using the continuity of the operator $\mathcal{R}_\alpha(t)$, for all $t \in [0,b]$, we conclude that the operator Γ_n^2 is also a compact multi-valued map.

(5) *Φ_n^2 has a closed graph.*

Let $x^{(j)} \to x^*$, $\tilde{h}_n^{(j)} \in \Phi_n^2 x^{(j)}$, $x^{(j)} \in B_r(0,\mathcal{BPC})$ and $\tilde{h}_n^{(j)} \to \tilde{h}_n^*$. From Axiom (A), it is easy to see that $(\overline{x^{(j)}})_s \to \overline{x^*}_s$ uniformly for $s \in (-\infty,b]$ as $j \to \infty$. We prove that $\tilde{h}_n^* \in \Phi_n^2\overline{x^*}$. Now $\tilde{h}_n^{(j)} \in \Phi_n^2\overline{x^{(j)}}$ means that there exists $f^{(j)} \in S_{F,\overline{x^{(j)}}_\rho}$ such that, for each $t \in [0,t_1]$,

$$\tilde{h}_n^{(j)}(t) = \mathcal{R}_\alpha(t)\varphi(0) + \int_0^t \mathcal{S}_\alpha(t-s)f^{(j)}(s)\,ds, \qquad t \in [0,t_1].$$

We must prove that there exists $f^* \in S_{F,\overline{x^*}_\rho}$ such that, for each $t \in J$,

$$\tilde{h}_n^*(t) = \mathcal{R}_\alpha(t)\varphi(0) + \int_0^t \mathcal{S}_\alpha(t-s)f^*(s)\,ds, \qquad t \in [0,t_1].$$

Now, for every $t \in [0,t_1]$, we have

$$\big\|\big(\tilde{h}_n^{(j)}(t) - \mathcal{R}_\alpha(t)\varphi(0)\big) - \big(\tilde{h}_n^*(t) - \mathcal{R}_\alpha(t)\varphi(0)\big)\big\|_{\mathcal{PC}} \to 0 \qquad \text{as } j \to \infty.$$

Consider the continuous linear operator $\Psi : L^1([0,t_1],X) \to C([0,t_1],X)$,

$$\Psi(f)(t) = \int_0^t \mathcal{S}_\alpha(t-s)f(s)\,ds.$$

From Lemma 3.2, it follows that $\Psi \circ S_F$ is a closed graph operator. Also, from the definition of Ψ, for every $t \in [0,t_1]$,

$$\tilde{h}_n^{(j)}(t) - \mathcal{R}_\alpha(t)\varphi(0) \in \Psi(S_{F,\overline{x^{(j)}}_\rho}).$$

Since $\overline{x^{(j)}} \to \overline{x^*}$, for some $f^* \in S_{F,\overline{x^*}_\rho}$ and all $t \in [0,t_1]$ we have

$$\tilde{h}_n^*(t) - \mathcal{R}_\alpha(t)\varphi(0) = \int_0^t \mathcal{S}_\alpha(t-s)f^*(s)\,ds.$$

Similarly, for any $t \in (t_k,t_{k+1}]$, $k = 1,\ldots,m$,

$$\begin{aligned}\tilde{h}_n^{(j)}(t) = \mathcal{R}_\alpha(t-t_k)&[\overline{x^{(j)}}(t_k^-) + \mathcal{R}_\alpha(\sigma_n)I_k(\overline{x^{(j)}}_{t_k})]\\ &+ \int_{t_k}^t \mathcal{S}_\alpha(t-s)f^{(j)}(s)\,ds, \qquad t \in (t_k,t_{k+1}].\end{aligned}$$

We must prove that there exists $f^* \in S_{F,\overline{x^*}_\rho}$ such that, for each $t \in (t_k,t_{k+1}]$,

$$\begin{aligned}\tilde{h}_n^*(t) = \mathcal{R}_\alpha(t-t_k)&[\overline{x^*}(t_k^-) + \mathcal{R}_\alpha(\sigma_n)I_k(\overline{x^*}_{t_k})]\\ &+ \int_{t_k}^t \mathcal{S}_\alpha(t-s)f^*(s)\,ds, \qquad t \in (t_k,t_{k+1}].\end{aligned}$$

Now, for every $t \in (t_k,t_{k+1}]$, $k = 1,\ldots,m$, we have

$$\begin{aligned}\big\|\big(\tilde{h}_n^{(j)}(t) - \mathcal{R}_\alpha(t-t_k)[\overline{x^{(j)}}(t_k^-) + \mathcal{R}_\alpha(\sigma_n)I_k(\overline{x^{(j)}}_{t_k})]\big)&\\ - \big(\tilde{h}^*(t) - \mathcal{R}_\alpha(t-t_k)[\overline{x^*}(t_k^-) + I_k(\overline{x^*}_{t_k})]\big)\big\|_{\mathcal{PC}} &\to 0\end{aligned}$$

as $j \to \infty$. Consider the continuous linear operator

$$\Psi : L^1((t_k,t_{k+1}],X) \to C((t_k,t_{k+1}],X), \qquad k = 1,\ldots,m,$$

$$\Psi(f)(t) = \int_{t_k}^t \mathcal{S}_\alpha(t-s)f(s)\,ds.$$

By Lemma 3.2, $\Psi \circ S_F$ is a closed graph operator. Also, from the definition of Ψ, for every $t \in (t_k,t_{k+1}]$, $k = 1,\ldots,m$,

$$\tilde{h}_n^{(j)}(t) - \mathcal{R}_\alpha(t-t_k)[\overline{x^{(j)}}(t_k^-) + \mathcal{R}_\alpha(\sigma_n)I_k(\overline{x^{(j)}}_{t_k})] \in \Psi(S_{F,\overline{x^{(j)}}_\rho}).$$

Since $\overline{x^{(j)}} \to \overline{x^*}$, for some $f^* \in S_{F,\overline{x^*}_\rho}$ and all $t \in (t_k,t_{k+1}]$ we have

$$\tilde{h}_n^*(t) - \mathcal{R}_\alpha(t-t_k)[\overline{x^*}(t_k^-) + \mathcal{R}_\alpha(\sigma_n)I_k(\overline{x^*}_{t_k})] = \int_{t_k}^t \mathcal{S}_\alpha(t-s)f^*(s)\,ds.$$

Therefore, Φ_n^2 is a completely continuous multi-valued map, u.s.c. with convex closed, compact values.

Step 3. *The set*

$$G = \{x \in \mathcal{BPC} : x \in \lambda\Phi_n^1 x + \lambda\Phi_n^2 x \text{ for some } \lambda \in (0,1)\}$$

is bounded on J.

Let $x \in \mathcal{BPC}$. Then there exists an $f \in S_{F,\bar{x}_\rho}$ such that

$$x(t) = \begin{cases} \lambda \mathcal{R}_\alpha(t)[\varphi(0) - \mathcal{R}_\alpha(\sigma_n)G(0,\varphi)] + \lambda \mathcal{R}_\alpha(\sigma_n)G(t,\bar{x}_t) \\ \quad + \lambda \int_0^t \mathcal{S}_\alpha(t-s)f(s)\,ds, & t \in [0,t_1], \\ \lambda \mathcal{R}_\alpha(t-t_1)[\bar{x}(t_1^-) + \mathcal{R}_\alpha(\sigma_n)I_1(\bar{x}_{t_1}) - \mathcal{R}_\alpha(\sigma_n)G(t_1,\bar{x}_{t_1^+})] \\ \quad + \lambda \mathcal{R}_\alpha(\sigma_n)G(t,\bar{x}_t) + \lambda \int_{t_1}^t \mathcal{S}_\alpha(t-s)f(s)\,ds, & t \in (t_1,t_2], \\ \vdots \\ \lambda \mathcal{R}_\alpha(t-t_m)[\bar{x}(t_m^-) + \mathcal{R}_\alpha(\sigma_n)I_m(\bar{x}_{t_m}) - \mathcal{R}_\alpha(\sigma_n)G(t_m,\bar{x}_{t_m^+})] \\ \quad + \lambda \mathcal{R}_\alpha(\sigma_n)G(t,\bar{x}_t) + \lambda \int_{t_m}^t \mathcal{S}_\alpha(t-s)f(s)\,ds, & t \in (t_m,b], \end{cases}$$

for some $\lambda \in (0,1)$. Then, by (H1), (H4) and (H5), from the above equation, for $t \in [0,t_1]$ we have

$$\begin{aligned} \|x(t)\| \le{}& Me^{\delta t}[\tilde{H}\|\varphi\|_{\mathcal{B}} + LMe^{\delta\sigma_n}(\|\varphi\|_{\mathcal{B}}+1)] + LMe^{\delta\sigma_n}(\|\bar{x}_t\|_{\mathcal{B}}+1) \\ &+ Me^{\delta t}\int_0^t e^{-\delta s}m(s)\Theta(\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathcal{B}})\,ds. \end{aligned}$$

Similarly, for any $t \in (t_k,t_{k+1}]$, $k = 1,\dots,m$,

$$\begin{aligned} \|x(t)\| \le{}& Me^{\delta(t-t_k)}\big\{\|\bar{x}(t_k^-)\| + Me^{\delta\sigma_n}[C_1 + (c_k+\epsilon_k)\|\bar{x}_{t_k}\|_{\mathcal{B}}] \\ &\qquad + LMe^{\delta\sigma_n}(\|\bar{x}_{t_k^+}\|_{\mathcal{B}}+1)\big\} \\ &+ LMe^{\delta\sigma_n}(\|\bar{x}_t\|_{\mathcal{B}}+1) + Me^{\delta t}\int_{t_k}^t e^{-\delta s}m(s)\Theta(\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathcal{B}})\,ds. \end{aligned}$$

Then, for all $t \in [0,b]$,

$$\begin{aligned} \|x(t)\| \le{}& \widetilde{M}e^{\delta t} + Me^{\delta t}N_*\big[\|\bar{x}(t_k^-)\| + Me^{\delta\sigma_n}(c_k+\epsilon_k)\|\bar{x}_{t_k}\|_{\mathcal{B}} \\ &\qquad + LMe^{\delta\sigma_n}\|\bar{x}_{t_k^+}\|_{\mathcal{B}}\big] \\ &+ LMe^{\delta\sigma_n}\|\bar{x}_t\|_{\mathcal{B}} + Me^{\delta t}\int_0^t e^{-\delta s}m(s)\Theta(\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathcal{B}})\,ds, \end{aligned}$$

where

$$\begin{aligned} \widetilde{M} = \max\big\{&M[\tilde{H}\|\varphi\|_{\mathcal{B}} + LMe^{\delta\sigma_n}(\|\varphi\|_{\mathcal{B}}+1)] + LMe^{\delta\sigma_n}, \\ &MN_*[Me^{\delta\sigma_n}C_1 + LMe^{\delta\sigma_n}] + LMe^{\delta\sigma_n}\big\}. \end{aligned}$$

Since $\lim_{n\to\infty}\sigma_n = 0$, it follows that

$$\begin{aligned} \|x(t)\| \le{}& \widetilde{M}e^{\delta t} + Me^{\delta t}N_*[\|\bar{x}(t_k^-)\| + M(c_k+\epsilon_k)\|\bar{x}_{t_k}\|_{\mathcal{B}} + LM\|\bar{x}_{t_k^+}\|_{\mathcal{B}}] \\ &+ LM\|\bar{x}_t\|_{\mathcal{B}} + Me^{\delta t}\int_0^t e^{-\delta s}m(s)\Theta(\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathcal{B}})\,ds. \end{aligned}$$

By Lemma 3.1, $\rho(t, \bar{x}_t) \leq t$, $t \in [0, b]$, and

$$\|\bar{x}_{\rho(s,\bar{x}_s)}\|_{\mathcal{B}} \leq (M_b + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b\|x\|_t,$$

where $\|x\|_t = \sup_{0\leq s\leq t} \|x(s)\|$. If $v(t) = (M_b + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b\|x\|_t$, then

$$\begin{aligned} v(t) \leq (M_b + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} &+ K_b\widetilde{M}e^{\delta t} + Me^{\delta t}N_* v(t) \\ &+ K_b M e^{\delta t} N_*[M(c_k + \epsilon_k)v(t) + LMv(t)] \\ &+ K_b LMv(t) + K_b M e^{\delta t} \int_0^t e^{-\delta s} m(s)\Theta(v(s))\, ds. \end{aligned}$$

Since $\widetilde{L} = \max_{1\leq k\leq m}\{M_* N_*[1 + K_b M(c_k + \epsilon_k + L)] + K_b LM\} < 1$, we obtain

$$e^{-\delta t}v(t) \leq \frac{1}{1-\widetilde{L}}\Big[N_*(M_b + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b\widetilde{M} + K_b M \int_0^t e^{-\delta s} m(s)\Theta(v(s))\, ds\Big].$$

Denoting by $w(t)$ the right-hand side of the above inequality, we have

$$v(t) \leq e^{\delta t}w(t) \quad \text{ for all } t \in J,$$

and

$$\begin{aligned} w(0) &= \frac{1}{1-\widetilde{L}}[N_*(M_b + J_0^{\varphi})\|\varphi\|_{\mathcal{B}} + K_b\widetilde{M}], \\ w'(t) &= \frac{1}{1-\widetilde{L}} K_b M e^{-\delta t} m(t)\Theta(v(t)) \\ &\leq \frac{1}{1-\widetilde{L}} K_b M e^{-\delta t} m(t)\Theta(e^{\delta t}w(t)), \quad t \in J. \end{aligned}$$

Then for each $t \in J$ we have

$$\begin{aligned} (e^{\delta t}w(t))' &= \delta e^{\delta t}w(t) + w'(t)e^{\delta t} \\ &\leq \delta e^{\delta t}w(t) + \frac{1}{1-\widetilde{L}} K_b M m(t)\Theta(e^{\delta t}w(t)) \\ &\leq \max\Big\{\delta, \frac{1}{1-\widetilde{L}} K_b M m(t)\Big\}[e^{\delta t}w(t) + \Theta(e^{\delta t}w(t))], \quad t \in J. \end{aligned}$$

This implies that

$$\int_{w(0)}^{e^{\delta t}w(t)} \frac{du}{u + \Theta(u)} \leq \int_0^b \max\Big\{\delta, \frac{1}{1-\widetilde{L}} K_b M m(s)\Big\}\, ds < \infty.$$

This inequality shows that there is a constant $\widetilde{K}$ such that $e^{\delta t}w(t) \leq \widetilde{K}$, $t \in J$, and hence $\|x\|_{\mathcal{PC}} \leq \frac{1}{K_b}v(t) \leq \frac{1}{K_b}e^{\delta t}w(t) \leq \frac{1}{K_b}\widetilde{K}$, where $\widetilde{K}$ depends only on M, δ, b and on the functions $m(\cdot)$ and $\Theta(\cdot)$. This indicates that G

is bounded on J. Consequently, by Lemma 2.10, $\Phi_n^1 + \Phi_n^2$ has a fixed point $x \in \mathcal{BPC}$, which is a mild solution of the problem (3.2)–(3.4). Then

(3.11)
$$x_n(t) = \begin{cases} \mathcal{R}_\alpha(t)[\varphi(0) - \mathcal{R}_\alpha(\sigma_n)G(0,\varphi)] \\ \quad + \mathcal{R}_\alpha(\sigma_n)G(t,\bar{x}_{n,t}) + \int_0^t \mathcal{S}_\alpha(t-s)f_n(s)\,ds, & t \in [0,t_1], \\ \mathcal{R}_\alpha(t-t_1)[\bar{x}_n(t_1^-) + \mathcal{R}_\alpha(\sigma_n)I_1(\bar{x}_{n,t_1}) - \mathcal{R}_\alpha(\sigma_n)G(t_1,\bar{x}_{n,t_1^+})] \\ \quad + \mathcal{R}_\alpha(\sigma_n)G(t,\bar{x}_{n,t}) + \int_{t_1}^t \mathcal{S}_\alpha(t-s)f_n(s)\,ds, & t \in (t_1,t_2], \\ \vdots \\ \mathcal{R}_\alpha(t-t_m)[\bar{x}_n(t_m^-) + \mathcal{R}_\alpha(\sigma_n)I_m(\bar{x}_{n,t_m}) - \mathcal{R}_\alpha(\sigma_n)G(t_m,\bar{x}_{n,t_m^+})] \\ \quad + \mathcal{R}_\alpha(\sigma_n)G(t,\bar{x}_{n,t}) + \int_{t_m}^t \mathcal{S}_\alpha(t-s)f_n(s)\,ds, & t \in (t_m,b], \end{cases}$$

for $t \in [0,b]$, and some $f_n \in S_{F,\bar{x}_{n,\rho}}$.

Next we will show that the set $\{x_n : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{BPC}$. We consider the decomposition $x_n = x_n^1 + x_n^2$ where

$$x_n^1(t) = \begin{cases} -\mathcal{R}_\alpha(t)\mathcal{R}_\alpha(\sigma_n)G(0,\varphi) \\ \quad + \mathcal{R}_\alpha(\sigma_n)G(t,\bar{x}_{n,t}) + \int_0^t \mathcal{S}_\alpha(t-s)f_n(s)\,ds, & t \in [0,t_1], \\ -\mathcal{R}_\alpha(t-t_1)\mathcal{R}_\alpha(\sigma_n)G(t_1,\bar{x}_{n,t_1^+}) \\ \quad + \mathcal{R}_\alpha(\sigma_n)G(t,\bar{x}_{n,t}) + \int_{t_1}^t \mathcal{S}_\alpha(t-s)f_n(s)\,ds, & t \in (t_1,t_2], \\ \vdots \\ -\mathcal{R}_\alpha(t-t_m)\mathcal{R}_\alpha(\sigma_n)G(t_m,\bar{x}_{n,t_m^+}) \\ \quad + \mathcal{R}_\alpha(\sigma_n)G(t,\bar{x}_{n,t}) + \int_{t_m}^t \mathcal{S}_\alpha(t-s)f_n(s)\,ds, & t \in (t_m,b], \end{cases}$$

for some $f_n \in S_{F,\bar{x}_{n,\rho}}$, and

$$x_n^2(t) = \begin{cases} \mathcal{R}_\alpha(t)\varphi(0), & t \in [0,t_1], \\ \mathcal{R}_\alpha(t-t_1)[\bar{x}_n(t_1^-) + \mathcal{R}_\alpha(\sigma_n)I_1(\bar{x}_{n,t_1})], & t \in (t_1,t_2], \\ \vdots \\ \mathcal{R}_\alpha(t-t_m)[\bar{x}_n(t_m^-) + \mathcal{R}_\alpha(\sigma_n)I_m(\bar{x}_{n,t_m})], & t \in (t_m,b]. \end{cases}$$

STEP 4. $\{x_n^1(t) : n \in \mathbb{N}\}$ *is relatively compact in* $\mathcal{BPC}$.

(1) $\{x_n^1 : n \in \mathbb{N}\}$ *is equicontinuous on* J.

For $\varepsilon > 0$ and $x_n \in B_r(0,\mathcal{BPC})$, there exists a constant $\eta > 0$ such that for all $t \in (0,t_1]$ and $\xi \in (0,\eta)$ with $t + \xi \le t_1$, we have

$$\begin{aligned}
&\|x_n^1(t+\xi) - x_n^1(t)\| \\
&\qquad \le \|[\mathcal{R}_\alpha(t+\xi) - \mathcal{R}_\alpha(t)]\mathcal{R}_\alpha(\sigma_n)G(0,\varphi)\| \\
&\qquad\quad + \|\mathcal{R}_\alpha(\sigma_n)[G(t+\xi,\bar{x}_{n,t+\xi}) - G(t,\bar{x}_{n,t})]\| \\
&\qquad\quad + \Big\| \int_t^{t+\xi} \mathcal{S}_\alpha(t+\xi-s)f_n(s)\,ds \Big\| \\
&\qquad\quad + \Big\| \int_0^t [\mathcal{S}_\alpha(t+\xi-s) - \mathcal{S}_\alpha(t-s)]f_n(s)\,ds \Big\| \\
&\qquad \le \|[\mathcal{R}_\alpha(t+\xi) - \mathcal{R}_\alpha(t)]\mathcal{R}_\alpha(\sigma_n)G(0,\varphi)\| \\
&\qquad\quad + Me^{\delta\sigma_n}L[\xi + \|\bar{x}_{n,t+\xi} - \bar{x}_{n,t}\|_{\mathcal{B}}] + M_*\Theta(r^*)\int_t^{t+\xi} e^{-\delta s}m(s)\,ds \\
&\qquad\quad + \Theta(r^*)\int_0^t \|\mathcal{S}_\alpha(t+\xi-s) - \mathcal{S}_\alpha(t-s)\|m(s)\,ds.
\end{aligned}$$

Similarly, for any $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$,

$$\begin{aligned}
&\|x_n^1(t+\xi) - x_n^1(t)\| \\
&\qquad \le \|[\mathcal{R}_\alpha(t+\xi) - \mathcal{R}_\alpha(t)]\mathcal{R}_\alpha(\delta_n)G(t_k,\bar{x}_{n,t_k^+})\| \\
&\qquad\quad + Me^{\delta\sigma_n}L[\xi + \|\bar{x}_{n,t+\xi} - \bar{x}_{n,t}\|_{\mathcal{B}}] + M_*\Theta(r^*)\int_t^{t+\xi} e^{-\sigma s}m(s)\,ds \\
&\qquad\quad + \Theta(r^*)\int_{t_k}^t \|\mathcal{S}_\alpha(t+\xi-s) - \mathcal{S}_\alpha(t-s)\|m(s)\,ds.
\end{aligned}$$

Then, for all $t \in (0,b]$, using the compact operator property, we get either

$$\|[\mathcal{R}_\alpha(t+\xi) - \mathcal{R}_\alpha(t)]\mathcal{R}_\alpha(\delta_n)G(0,\varphi)\| < \varepsilon/4, \tag{3.12}$$

or

$$\|[\mathcal{R}_\alpha(t+\xi) - \mathcal{R}_\alpha(t)]\mathcal{R}_\alpha(\delta_n)G(t_k,\bar{x}_{n,t_k^+})\| < \varepsilon/4, \tag{3.13}$$

and

$$Me^{\delta\sigma_n}L[\xi + \|\bar{x}_{n,t+\xi} - \bar{x}_{n,t}\|_{\mathcal{B}}] < \varepsilon/4, \tag{3.14}$$

$$M_*\Theta(r^*)\int_t^{t+\xi} e^{-\delta s}m(s)\,ds < \varepsilon/4, \tag{3.15}$$

$$\Theta(r^*)\int_0^t \|\mathcal{S}_\alpha(t+\xi-s) - \mathcal{S}_\alpha(t-s)\|m(s)\,ds < \varepsilon/4. \tag{3.16}$$

By (3.12)–(3.16) one has $\|x_n^1(t+\xi)-x_n^1(t)\|<\varepsilon$. Therefore, $\{x_n^1(t):n\in\mathbb{N}\}$ is equicontinuous for $t\in(0,b]$. Clearly $\{x_n^1(0):n\in\mathbb{N}\}$ is equicontinuous.

(2) $\{x_n^1(t):n\in\mathbb{N}\}$ *is relatively compact in* X.

Let $t\in(0,t_1]$, $\varepsilon>0$, $x_n\in B_r(0,\mathcal{BPC})$. There exists $\xi\in(0,t)$ such that

$$\|x_n^1(t)-x_n^\xi(t)\|\le\int_{t-\xi}^{t}\|\mathcal{S}(t-s)f_n(s)\|\,ds\le M_*\Theta(r^*)\int_{t-\xi}^{t}e^{-\delta s}m(s)\,ds<\varepsilon,$$

where

$$x_n^\xi(t)=-\mathcal{R}_\alpha(t)\mathcal{R}_\alpha(\sigma_n)G(0,\varphi)+\mathcal{R}_\alpha(\sigma_n)G(t,\bar{x}_{n,t})+\int_0^{t-\xi}\mathcal{S}_\alpha(t-s)f_n(s)\,ds$$

for some $f_n\in S_{F,\bar{x}_{n,\rho}}$.

Similarly, for any $t\in(t_k,t_{k+1}]$, $k=1,\dots,m$, $\varepsilon>0$, $x_n\in B_r(0,\mathcal{BPC})$, there exists $\xi\in(0,t)$ such that

$$\|x_n^1(t)-x_n^\xi(t)\|\le\int_{t-\xi}^{t}\|\mathcal{S}(t-s)f_n(s)\|\,ds\le M_*\Theta(r^*)\int_{t-\xi}^{t}e^{-\delta s}m(s)\,ds<\varepsilon,$$

where

$$\begin{aligned}x_n^\xi(t)=&-\mathcal{R}_\alpha(t-t_k)\mathcal{R}_\alpha(\sigma_n)G(t_k,\bar{x}_{n,t_k^+})+\mathcal{R}_\alpha(\sigma_n)G(t,\bar{x}_{n,t})\\&+\int_{t_k}^{t-\xi}\mathcal{S}_\alpha(t-s)f_n(s)\,ds\end{aligned}$$

for some $f_n\in S_{F,\bar{x}_{n,\rho}}$. From (H5), we infer that $G(t_k,\bar{x}_{n,t_k^+})$ and $G(t,\bar{x}_{n,t})$ are bounded in X. By the compactness of $\mathcal{R}_\alpha(t)$, $\mathcal{S}_\alpha(t)$ for $t>0$, we see that $\{x_n^\xi(t):n\in\mathbb{N}\}$ is relatively compact in X. Combining the above shows that $\{x_n^1(t):n\in\mathbb{N}\}$ is relatively compact in X.

Step 5. $\{x_n^2(t):n\in\mathbb{N}\}$ *is relatively compact in* $\mathcal{BPC}$.

(1) $\{x_n^2:n\in\mathbb{N}\}$ *is equicontinuous on* J.

Fix $\varepsilon>0$ and $0<t<t_1$. Since $\mathcal{R}_\alpha(\sigma_n)$ is a compact operator, the set $W_1=\{\mathcal{R}_\alpha(\sigma_n)G(0,\varphi)\}$ is relatively compact in X. From the strong continuity of $(\mathcal{R}_\alpha(t))_{t\ge0}$, we can choose $0<\eta<b-t$ such that

$$\|(\mathcal{R}_\alpha(t+\xi)-\mathcal{R}_\alpha(t))\nu\|<\varepsilon,\qquad \nu\in W_1,$$

when $|\xi|<\eta$. For each $x_n\in B_r$,

$$\|x_n^2(t+\xi)-x_n^2(t)\|\le\|[\mathcal{R}_\alpha(t+\xi)-\mathcal{R}_\alpha(t)]\mathcal{R}_\alpha(\delta_n)G(0,\varphi)\|<\varepsilon.$$

The same holds for any $t\in(t_k,t_{k+1}]$, $k=1,\dots,m$, and $\varepsilon>0$. Since $\mathcal{R}_\alpha(\sigma_n)$ is a compact operator, the set $W_2=\{\mathcal{R}_\alpha(\sigma_n)I_k(\bar{x}_{n,t_k}):x_n\in B_r(0,\mathcal{BPC})\}$ is relatively compact in X. From the strong continuity of $(\mathcal{R}_\alpha(t))_{t\geq 0}$, for $\varepsilon > 0$ we can choose $0 < \eta < b - t$ such that

$$\|(\mathcal{R}_\alpha(t+\xi) - \mathcal{R}_\alpha(t))\nu\| < \varepsilon, \qquad \nu \in W_2,$$

when $|\xi| < \eta$. For each $x_n \in B_r$, $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$, we have

$$\|x_n^2(t+\xi) - x_n^2(t)\| \leq \|[\mathcal{R}_\alpha(t+\xi-t_k) - \mathcal{R}_\alpha(t-t_k)]\mathcal{R}_\alpha(\sigma_n)I_k(\bar{x}_{n,t_k})\| < \varepsilon.$$

As $\xi \to 0$ and ε sufficiently small, the right-hand side of the above inequality tends to zero independently of x_n, so $\widehat{[x_n^2]}_k$, $k = 1, \dots, m$, are equicontinuous.

(2) $\{x_n^2(t) : n \in \mathbb{N}\}$ *is relatively compact in* X.

Let $t \in (0, t_1]$ and $x_n \in B_r(0, \mathcal{BPC})$. By (H1), $\{x_n^2(t) : t \in [0, t_1],\ x_n \in B_r(0, \mathcal{BPC})\}$ is a compact subset of X. Using similar arguments to those in Step 2, for $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$, and $x_n \in B_r(0, \mathcal{BPC})$, we find that

$$\widehat{[x_n^2]}_k(t) \in \begin{cases} \mathcal{R}_\alpha(t-t_k)[\bar{x}_n(t_k^-) + \mathcal{R}_\alpha(\sigma_n)I_k(\bar{x}_{n,t_k})], \\ \qquad\qquad\qquad\qquad t \in (t_k, t_{k+1}),\ x_n \in B_{r'}(0, \mathcal{BPC}), \\ \mathcal{R}_\alpha(t_{k+1}-t_k)[\bar{x}_n(t_k^-) + \mathcal{R}_\alpha(\sigma_n)I_k(\bar{x}_{n,t_k})], \\ \qquad\qquad\qquad\qquad t = t_{k+1},\ x_n \in B_{r'}(0, \mathcal{BPC}), \\ \bar{x}_n(t_k^-) + \mathcal{R}_\alpha(\sigma_n)I_k(\bar{x}_{n,t_k}), \qquad t = t_k,\ x_n \in B_{r'}(0, \mathcal{BPC}), \end{cases}$$

where $B_{r'}(0, \mathcal{BPC})$ is a closed ball of radius r'. One sees that $\widehat{[x_n^2]}_k(t)$, $k = 1, \dots, m$, is relatively compact for every $t \in [t_k, t_{k+1}]$, and $\{x_n^2(t) : n \in \mathbb{N}\}$ is relatively compact in X.

Thus, the set $\{x_n : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{BPC}$. We may suppose that

$$x_n \to x_* \in \mathcal{BPC} \qquad \text{as } n \to \infty.$$

Obviously, $x_* \in \mathcal{BPC}$, and taking limits in (3.11) one has

$$x_*(t) = \begin{cases} \mathcal{R}_\alpha(t)[\varphi(0) - G(0,\varphi)] \\ \quad + G(t, \bar{x}_{*,t}) + \int_0^t \mathcal{S}_\alpha(t-s)f_*(s)\,ds, & t \in [0, t_1], \\ \mathcal{R}_\alpha(t-t_1)[\bar{x}_*(t_1^-) + I_1(\bar{x}_{*,t_1}) - G(t_1, \bar{x}_{*,t_1^+})] \\ \quad + G(t, \bar{x}_{*,t}) + \int_{t_1}^t \mathcal{S}_\alpha(t-s)f_*(s)\,ds, & t \in (t_1, t_2], \\ \vdots \\ \mathcal{R}_\alpha(t-t_m)[\bar{x}_n(t_m^-) + I_m(\bar{x}_{*,t_m}) - G(t_m, \bar{x}_{*,t_m^+})] \\ \quad + G(t, \bar{x}_{*,t}) + \int_{t_m}^t \mathcal{S}_\alpha(t-s)f_*(s)\,ds, & t \in (t_m, b], \end{cases}$$

for $t \in [0, b]$, and some $f_* \in S_{F,\bar{x}_{*,\rho}}$, which implies that x_* is a mild solution of the problem (1.1)–(1.3), and the proof of Theorem 3.3 is complete. ∎

4. Application. Consider the following impulsive fractional partial neutral functional integro-differential inclusions:

$$
\begin{aligned}
(4.1)\quad & D_t^\alpha\Big[z(t,x)-\int_{-\infty}^{t} b_1(s-t)z(s,x)\,ds\Big]\\
&\in \frac{\partial^2}{\partial x^2}\Big[z(t,x)-\int_{-\infty}^{t} b_1(s-t)z(s,x)\,ds\Big]\\
&\quad+\int_0^t (t-s)^\sigma e^{-\mu(t-s)}\frac{\partial^2}{\partial x^2}\Big[z(s,x)-\int_{-\infty}^{s} b_1(\tau-s)z(\tau,x)\,d\tau\Big]\,ds\\
&\quad+\int_{-\infty}^{t} b_2\big(t,s-t,x,z(s-\rho_1(t)\rho_2(\|z(t)\|),x)\big),\\
&\qquad\qquad 0\le t\le b,\ 0\le x\le\pi,
\end{aligned}
$$

$$(4.2)\quad z(t,0)=z(t,\pi)=0,\qquad 0\le t\le b,$$

$$(4.3)\quad z_t(0,x)=0,\qquad 0\le x\le\pi,$$

$$(4.4)\quad z(\tau,x)=\varphi(\tau,x),\qquad \tau\le 0,\ 0\le x\le\pi,$$

$$(4.5)\quad \triangle z(t_k,x)=\int_{-\infty}^{t_k}\eta_k(s-t_k)z(s,x)\,ds,\qquad k=1,\dots,m,$$

where D_t^α is the Caputo fractional partial derivative of order $\alpha\in(1,2)$, and σ, μ are positive numbers. Let $X=L^2([0,\pi])$ with the norm $\|\cdot\|$, and define the operator $A:D(A)\subseteq X\to X$ by $A\omega=\omega''$ with the domain

$$D(A):=\{\omega\in X:\omega,\omega'\text{ are absolutely continuous, }\omega''\in X,\ \omega(0)=\omega(\pi)=0\}.$$

Then

$$A\omega=\sum_{n=1}^{\infty}n^2\langle\omega,\omega_n\rangle\omega_n,\qquad \omega\in D(A),$$

where $\omega_n(x)=\sqrt{2/\pi}\sin(nx)$, $n=1,2,\dots$, is the orthogonal set of eigenvectors of A. It is well known that A generates a strongly continuous semigroup $T(t)$, $t\ge 0$, which is compact, analytic and self-adjoint in X; moreover, A is sectorial and (P1) is satisfied. The operator $Q(t):D(A)\subseteq X\to X$, $t\ge 0$, is given by $Q(t)x=t^\sigma e^{-\omega t}x''$ for $x\in D(A)$. Moreover, it is easy to see that conditions (P2) and (P3) in Section 2 are satisfied with $b(t)=t^\sigma e^{-\mu t}$ and $D=C_0^\infty([0,\pi])$, where $C_0^\infty([0,\pi])$ is the space of infinitely differentiable functions that vanish at $x=0$ and $x=\pi$.

In the next applications, $\mathcal{B}$ will be the phase space $\mathcal{PC}_0\times L^2(h,X)$ (see [HG]). Additionally, we will assume that

(i) The functions $\rho_i:[0,\infty)\to[0,\infty)$, $i=1,2$, are continuous.

(ii) The function $b_1 : \mathbb{R} \to \mathbb{R}$ is continuous, and $L_G = \big(\int_{-\infty}^0 \frac{(b_1(s))^2}{h(s)}\, ds\big)^{1/2} < \infty$.

(iii) The function $b_2 : \mathbb{R}^4 \to \mathbb{R}$ is continuous and there exist continuous functions $a_1, a_2 : \mathbb{R} \to \mathbb{R}$ such that

$$|b_2(t,s,x,y)| \le a_1(t)a_2(s)|y|, \qquad (t,s,x,y) \in \mathbb{R}^4,$$

with $L_F = \big(\int_{-\infty}^0 \frac{(a_2(s))^2}{h(s)}\, ds\big)^{1/2} < \infty$.

(iv) The functions $\eta_k : \mathbb{R} \to \mathbb{R}$, $k = 1,\dots,m$, are continuous, and $L_k = \big(\int_{-\infty}^0 \frac{(\eta_k(s))^2}{h(s)}\, ds\big)^{1/2} < \infty$ for every $k = 1,\dots,m$,

Take $\varphi \in \mathcal{B}$ with $\varphi(s)(s) = \varphi(s,\tau)$. Let $N, G : \mathcal{B} \to X$, $F : [0,b] \times \mathcal{B} \to \mathcal{P}_{\mathrm{bd,cl,cv}}(X)$ and $\rho : [0,b] \times \mathcal{B} \to \mathbb{R}$ be the operators defined by

$$N(\psi)(x) = \psi(0,x) + G(\psi)(x), \qquad G(\psi)(x) = \int_{-\infty}^{0} b_1(s)\psi(s,x)\, ds,$$

$$F(t,\psi)(x) = \int_{-\infty}^{0} b_2(t,s,x,\psi(s,x))\, ds,$$

$$\rho(t,\psi) = \rho_1(t)\rho_2(\|\psi(0)\|), \qquad I_k(\psi)(x) = \int_{-\infty}^{0} \eta_k(s)\psi(s,x)\, ds.$$

Using these definitions, we can represent the system (4.1)–(4.5) in the abstract form (1.1)–(1.3). Moreover, G, I_k are bounded linear operators on $\mathcal{B}$ with $\|G\| \le L_G$, $\|I_k\| \le L_k$, $k = 1,\dots,m$. Using (iii), we see that F is continuous and $\|F(t,\psi)\| \le a(t)\|\psi\|_{\mathcal{B}}$ for all $(t,\psi) \in J \times \mathcal{B}$, where $a(t) = L_F a_1(t)$, $t \in [0,b]$. Further, under suitable conditions on the above-defined functions the assumptions of Theorem 3.3 are satisfied, and we can conclude that system (4.1)–(4.5) has at least one mild solution on $[0,b]$.

References

[AM] R. P. Agarwal, M. Belmekki and M. Benchohra, *A survey on semilinear differential equations and inclusions involving Riemann–Liouville fractional derivative*, Adv. Difference Equations 2009, art. ID 981728, 47 pp.

[AS] R. P. Agarwal, B. De Andrade and G. Siracusa, *On fractional integro-difierential equations with state-dependent delay*, Comput. Math. Appl. 62 (2011), 1143–1149.

[AA] A. Anguraj, M. M. Arjunan and E. Hernández, *Existence results for an impulsive neutral functional differential equation with state-dependent delay*, Appl. Anal. 86 (2007), 861–872.

[BK1] K. Balachandran, S. Kiruthika and J. J. Trujillo, *Existence results for fractional impulsive integrodifferential equations in Banach spaces*, Comm. Nonlinear Sci. Numer. Simul. 16 (2011), 1970–1977.

[BK2] K. Balachandran, S. Kiruthika and J. J. Trujillo, *On fractional impulsive equations of Sobolev type with nonlocal condition in Banach spaces*, Comput. Math. Appl. 62 (2011), 1157–1165.

[BE] M. Benchohra and B. Hedia, *Functional differential equations with state-dependent delay on unbounded domains in a Banach space*, Comm. Math. Anal. 12 (2012), 117–133.

[BH] M. Benchohra, J. Henderson and S. K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Hindawi Publ., New York, 2006.

[BO] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahab, *Existence results for fractional functional differential inclusions with infinite delay and application to control theory*, Fract. Calc. Appl. Anal. 11 (2008), 35–56.

[CD] A. Chauhan and J. Dabas, *Existence of mild solutions for impulsive fractional order semilinear evolution equations with nonlocal conditions*, Electron. J. Differential Equations 2011, no. 107, 10 pp.

[CN] C. Cuevas, G. M. N'Guérékata and M. Rabelo, *Mild solutions for impulsive neutral functional differential equations with state-dependent delay*, Semigroup Forum 80 (2010), 375–390.

[DC] J. Dabas, A. Chauhan and M. Kumar, *Existence of the mild solutions for impulsive fractional equations with infinite delay*, Int. J. Differential Equations 2011, art. ID 793023, 20 pp.

[DN] M. A. Darwish and S. K. Ntouyas, *Functional differential equations of fractional order with state-dependent delay*, Dynam. Systems Appl. 18 (2009), 539–550.

[DB] A. Debbouche and D. Baleanu, *Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems*, Comput. Math. Appl. 62 (2011), 1442–1450.

[DE] K. Deimling, *Multi-Valued Differential Equations*, de Gruyter, Berlin, 1992.

[D] B. C. Dhage, *Fixed-point theorems for discontinuous multi-valued operators on ordered spaces with applications*, Comput. Math. Appl. 51 (2006), 589–604.

[E1] M. M. El-Borai, *Some probability densities and fundamental solutions of fractional evolution equations*, Chaos Solitons Fractals 14 (2002), 433–440.

[E2] M. M. El-Borai, *Semigroups and some nonlinear fractional differential equations*, Appl. Math. Comput. 149 (2004), 823–831.

[GN] W. G. Glockle and T. F. Nonnemacher, *A fractional calculus approach of self-similar protein dynamics*, Biophys. J. 68 (1995), 46–53.

[HC] W. M. Haddad, V. Chellaboina and S. G. Nersesov, *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control*, Princeton Univ. Press, Princeton, NJ, 2006.

[HK] J. K. Hale and J. Kato, *Phase spaces for retarded equations with infinite delay*, Funkcial. Ekvac. 21 (1978), 11–41.

[HG] E. Hernández, M. Pierri and G. Goncalves, *Existence results for an impulsive abstract partial differential equation with state-dependent delay*, Comput. Math. Appl. 52 (2006), 411–420.

[HL] J. Hu and X. Liu, *Existence results of impulsive partial neutral integrodifferential inclusions with infinity delay*, Nonlinear Anal. 71 (2009), e1132–e1138.

[HP] S. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis*, Kluwer, Dordrecht, 1997.

[KS] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Math. Stud. 204, Elsevier, Amsterdam, 2006.

[LB] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Sci., Singapore, 1989.

[LO] A. Lasota and Z. Opial, *An application of the Kakutani–Ky Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 13 (1965), 781–786.

[MS] F. Metzler, W. Schick, H. G. Kilian and T. F. Nonnemacher, *Relaxation in filled polymers: A fractional calculus approach*, J. Chem. Phys. 103 (1995), 7180–7186.

[MR] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, Wiley, New York, 1993.

[M] G. M. Mophou, *Existence and uniqueness of mild solutions to impulsive fractional differential equations*, Nonlinear Anal. 72 (2010), 1604–1615.

[PO] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[RB] B. Radhakrishnan and K. Balachandran, *Controllability of neutral evolution integrodifferential systems with state-dependent delay*, J. Optim. Theory Appl. 153 (2012), 85–97.

[SA] J. P. C. Santos, M. M. Arjunan and C. Cuevas, *Existence results for fractional neutral integro-differential equations with state-dependent delay*, Comput. Math. Appl. 62 (2011), 1275–1283.

[SL] X.-B. Shu, Y. Lai and Y. Chen, *The existence of mild solutions for impulsive fractional partial differential equations*, Nonlinear Anal. 74 (2011), 2003–2011.

[S] E. Stumpf, *On a differential equation with state-dependent delay*, J. Dynam. Differential Equations 24 (2012), 197–248.

[Y1] Z. Yan, *Existence of solutions for nonlocal impulsive partial functional integrodifferential equations via fractional operators*, J. Comput. Appl. Math. 235 (2011), 2252–2262.

[Y2] Z. Yan, *Existence results for fractional functional integrodifferential equations with nonlocal conditions in Banach spaces*, Ann. Polon. Math. 97 (2010), 285–299.

[Y3] Z. Yan, *On a nonlocal problem for fractional integrodifferential inclusions in Banach spaces*, Ann. Polon. Math. 101 (2011), 87–103.

[YO] K. Yosida, *Functional Analysis*, 6th ed., Springer, Berlin, 1980.

[ZJ] Y. Zhou and F. Jiao, *Existence of mild solutions for fractional neutral evolution equations*, Comput. Math. Appl. 59 (2010), 1063–1077.

Zuomao Yan, Hongwu Zhang
Department of Mathematics
Hexi University
Zhangye, Gansu 734000, P.R. China
E-mail: yanzuomao@163.com

Received 10.11.2012
and in final form 7.1.2013 (2945)