ANNALES
POLONICI MATHEMATICI
LXXVII.3 (2001)

The kernel theorem for Laplace ultradistributions

by Sławomir Michalik (Warszawa)

Abstract. A kernel theorem for spaces of Laplace ultradistributions supported by an n-dimensional cone of product type is stated and proved.

Introduction. Laurent Schwartz showed in [5] that for every continuous linear map $A : D(\Omega) \to D'(\Omega)$ there exists a unique distribution $K \in D'(\Omega \times \Omega)$, called the *distributional kernel* of the operator A, such that

$$A[\varphi][\psi] = K[\varphi \otimes \psi] \quad \text{for } \varphi, \psi \in D(\Omega). \tag{1}$$

In this paper we give the kernel theorem for the space $L'^{(M_p)}_{(\omega)}(\Gamma)$ of Laplace ultradistributions supported by an n-dimensional cone Γ of product type (i.e. $\Gamma = v + (\overline{\mathbb{R}}_+)^n$). Namely for any continuous linear map $A : L^{(M_p)}_{(\omega_1)}(\Gamma_1) \to L'^{(M_p)}_{(\omega_2)}(\Gamma_2)$ there exists $K \in L'^{(M_p)}_{(\omega_1,\omega_2)}(\Gamma_1 \times \Gamma_2)$ such that (1) holds for all $\varphi \in L^{(M_p)}_{(\omega_1)}(\Gamma_1)$, $\psi \in L^{(M_p)}_{(\omega_2)}(\Gamma_2)$. The proof of this theorem is based on the proof of the S'-version of the kernel theorem given in [7].

Notation. We use the vector notation. In particular, if $a, b, v \in \mathbb{R}^n$ then $a < b$ means $a_i < b_i$ for $i = 1, \ldots, n$, $[v, \infty)$ means $[v_1, \infty) \times \ldots \times [v_n, \infty)$ and x^z means $x_1^{z_1} \ldots x_n^{z_n}$ for $x \in \mathbb{R}^n_+$, $z \in \mathbb{C}^n$.

Let $\Gamma \subseteq U \subseteq \mathbb{R}^n$ be such that U is open in $\mathbb{R}^n$, Γ is relatively closed in U and $\Gamma \subseteq \overline{\mathrm{int}\Gamma}$ (i.e. Γ is a *fat set*). Then for $k \in \mathbb{N}_0 \cup \{\infty\}$,

$$C^k(\Gamma) := \{f : \Gamma \to \mathbb{C} : \text{there exists } g \in C^k(U) \text{ such that } g|_\Gamma = f\}.$$

We write D for the differential operator d/dx.

Let $\{P_\tau\}_{\tau \in T}$ be a family of multinormed vector spaces. Then $\varinjlim_{\tau \in T} P_\tau$ (resp. $\varprojlim_{\tau \in T} P_\tau$) denotes the inductive limit (resp. projective limit) of P_τ, $\tau \in T$.

2000 *Mathematics Subject Classification*: Primary 46F05.
Key words and phrases: Laplace ultradistributions, kernel theorem.

Laplace ultradistributions. Let $(M_p)_{p\in\mathbb{N}_0}$ be a sequence of positive numbers satisfying the conditions (see [2]):

(M.0) $M_0 = M_1 = 1$.
(M.1) $M_p^2 \le M_{p-1}M_{p+1}$ for $p \in \mathbb{N}$.
(M.2) There are constants A, H such that

$$M_p \le AH^p \min_{0\le q\le p} M_q M_{p-q} \quad \text{for } p \in \mathbb{N}_0.$$

(M.3) There is a constant A such that

$$\sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \le Ap\frac{M_p}{M_{p+1}} \quad \text{for } p \in \mathbb{N}_0.$$

The *associated function* M of the sequence (M_p) is defined by

$$M(\varrho) := \sup_{p\in\mathbb{N}_0} \log \frac{\varrho^p}{M_p} \quad \text{for } \varrho > 0.$$

An *ultradifferential operator* $P(D)$ of class (M_p) is defined by

$$P(D) := \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha D^\alpha,$$

where the $a_\alpha \in \mathbb{C}$ satisfy the condition: there are constants $K, C < \infty$ such that

$$|a_\alpha| \le C\frac{K^{|\alpha|}}{M_{|\alpha|}} \quad \text{for } \alpha \in \mathbb{N}_0^n.$$

The entire function $\mathbb{C}^n \ni z \mapsto P(z)$ is called a *symbol of class* (M_p).

Definition 1 (see [3]). Let $v \in \mathbb{R}^n$, $\Gamma := v + (\overline{\mathbb{R}}_+)^n = [v,\infty)$, $\omega \in (\mathbb{R}\cup\{\infty\})^n$. The space $L'^{(M_p)}_{(\omega)}(\Gamma)$ of *Laplace ultradistributions* is defined as the dual space of

$$L^{(M_p)}_{(\omega)}(\Gamma) := \varinjlim_{a<\omega} L^{(M_p)}_a(\Gamma),$$

where for any $a \in \mathbb{R}^n$,

$$L^{(M_p)}_a(\Gamma) := \varprojlim_{h>0} L^{(M_p)}_{a,h}(\Gamma),$$

and for any $h > 0$,

$$L^{(M_p)}_{a,h}(\Gamma) := \left\{\varphi \in C^\infty(\Gamma) : q^{(M_p)}_{a,h,\Gamma}(\varphi) := \sup_{y\in\Gamma}\sup_{\alpha\in\mathbb{N}_0^n} \frac{|e^{-ay}D^\alpha\varphi(y)|}{h^{|\alpha|}M_{|\alpha|}} < \infty\right\}.$$

Fix $\varepsilon > 0$. We will construct a linear continuous extension mapping

$$E_\varepsilon : L^{(M_p)}_a(\Gamma) \to L^{(M_p)}_a(-\varepsilon + \Gamma)$$

such that $\operatorname{supp}(E_\varepsilon\varphi) \subset -\varepsilon/2 + \Gamma$ for every $\varphi \in L^{(M_p)}_a(\Gamma)$.

Without loss of generality we can assume that $\varepsilon < 1$. For $k \in \mathbb{N}_0^n$ let

$$U_k := \{x \in \mathbb{R}^n : -\varepsilon < x_i - v_i - k_i < 1 + \varepsilon \text{ for } i = 1, \ldots, n\}$$

be a covering of $\Gamma = v + (\overline{\mathbb{R}}_+)^n$. Let $\{\psi_k\}_{k\in\mathbb{N}_0^n}$ be a locally finite partition of unity (see Proposition 5.2 in [2]) subordinate to $\{U_k\}_{k\in\mathbb{N}_0^n}$ such that:

1) $\psi_k \in L_0^{(M_p)}(-\varepsilon + \Gamma)$;
2) the family $\{\psi_k\}_{k\in\mathbb{N}_0^n}$ is equibounded in $L_0^{(M_p)}(-\varepsilon + \Gamma)$;
3) $\operatorname{supp} \psi_k \subset U_k$;
4) $\sum \psi_k(x) = 1$ on Γ.

Furthermore, let $\widetilde{E}_{\varepsilon,k}$ be a linear continuous extension operator for ultradifferentiable functions on the compact set $\overline{U}_k \cap \Gamma$ (see Theorem 3.1 in [4]):

$$\widetilde{E}_{\varepsilon,k} : \mathcal{E}^{(M_p)}(\overline{U}_k \cap \Gamma) \to \mathcal{E}^{(M_p)}(\mathbb{R}^n),$$

such that:

1) $\operatorname{supp}(\widetilde{E}_{\varepsilon,k}\psi) \subset (-\varepsilon/2, \varepsilon/2]^n + U_k \cap \Gamma$ for every $\psi \in \mathcal{E}^{(M_p)}(\overline{U}_k \cap \Gamma)$;

2) if $\psi \in \mathcal{E}^{(M_p)}(\overline{U}_k \cap \Gamma)$ and $\operatorname{supp} \psi \subset U_k \cap \Gamma$ then $\operatorname{supp}(\widetilde{E}_{\varepsilon,k}\psi) \cap \Gamma = \operatorname{supp} \psi$.

Observe that for every $k \in \mathbb{N}_0^n$ there exists $j \in \{0, \ldots, n\}$ such that $\overline{U}_k \cap \Gamma$ is isometric to $[-\varepsilon, 1+\varepsilon]^j \times [0, 1+\varepsilon]^{n-j}$. Hence we may assume that:

3) the family $\{\widetilde{E}_{\varepsilon,k}\}_{k\in\mathbb{N}_0^n}$ of operators is equicontinuous.

Now we define E_ε by

$$E_\varepsilon(\varphi) := \sum_{k\in\mathbb{N}_0^n} \widetilde{E}_{\varepsilon,k}(\psi_k\varphi) \quad \text{for } \varphi \in L_a^{(M_p)}(\Gamma).$$

By the properties of the functions $\{\psi_k\}_{k\in\mathbb{N}_0^n}$ and the mappings $\{\widetilde{E}_{\varepsilon,k}\}_{k\in\mathbb{N}_0^n}$, E_ε is an extension operator and we may estimate pseudonorms of $E_\varepsilon(\varphi)$ by appropriate pseudonorms of φ. Therefore E_ε is a continuous linear extension mapping.

Following the proof of Proposition 5.1 in [7] and using the mapping E_ε we conclude that the space $L_a^{(M_p)}(\Gamma)$ is complete.

Let $v_1 \in \mathbb{R}^{n_1}$, $v_2 \in \mathbb{R}^{n_2}$, $\Gamma_1 := [v_1, \infty)$, $\Gamma_2 := [v_2, \infty)$, $\omega_1 \in (\mathbb{R} \cup \{\infty\})^{n_1}$, $\omega_2 \in (\mathbb{R} \cup \{\infty\})^{n_2}$. We denote by $L'^{(M_p)}_{(\omega_1)}(\Gamma_1, L'^{(M_p)}_{(\omega_2)}(\Gamma_2))$ the space of *Laplace ultradistributions* on Γ_1 with values in $L'^{(M_p)}_{(\omega_2)}(\Gamma_2)$, i.e.

$$A \in L'^{(M_p)}_{(\omega_1)}(\Gamma_1, L'^{(M_p)}_{(\omega_2)}(\Gamma_2))$$

if for any $\varphi \in L^{(M_p)}_{(\omega_1)}(\Gamma_1)$ we have $A[\varphi] \in L'^{(M_p)}_{(\omega_2)}(\Gamma_2)$ and the mapping

$$L^{(M_p)}_{(\omega_1)}(\Gamma_1) \ni \varphi \mapsto A[\varphi] \in L'^{(M_p)}_{(\omega_2)}(\Gamma_2)$$

is linear and continuous.

We say that a sequence $(A_\nu)_{\nu\in\mathbb{N}}$, where $A_\nu \in L'^{(M_p)}_{(\omega_1)}(\Gamma_1, L'^{(M_p)}_{(\omega_2)}(\Gamma_2))$, converges to zero in $L'^{(M_p)}_{(\omega_1)}(\Gamma_1, L'^{(M_p)}_{(\omega_2)}(\Gamma_2))$ if

$$\lim_{\nu\to\infty} A_\nu[\varphi][\psi] = 0 \quad \text{for every } \varphi \in L^{(M_p)}_{(\omega_1)}(\Gamma_1),\ \psi \in L^{(M_p)}_{(\omega_2)}(\Gamma_2).$$

Analogously, we say that a sequence $(\widetilde{A}_\nu)_{\nu\in\mathbb{N}}$, where $\widetilde{A}_\nu \in L'^{(M_p)}_{(\omega_1,\omega_2)}(\Gamma_1\times\Gamma_2)$, converges to zero in $L'^{(M_p)}_{(\omega_1,\omega_2)}(\Gamma_1\times\Gamma_2)$ if

$$\lim_{\nu\to\infty} \widetilde{A}_\nu[\Phi] = 0 \quad \text{for every } \Phi \in L^{(M_p)}_{(\omega_1,\omega_2)}(\Gamma_1\times\Gamma_2).$$

The kernel theorem

THEOREM 1 (The kernel theorem). *The mapping*

$$\mathcal{I}_{M_p} : L'^{(M_p)}_{(\omega_1,\omega_2)}(\Gamma_1\times\Gamma_2) \to L'^{(M_p)}_{(\omega_1)}(\Gamma_1, L'^{(M_p)}_{(\omega_2)}(\Gamma_2))$$

such that for any $\widetilde{A} \in L'^{(M_p)}_{(\omega_1,\omega_2)}(\Gamma_1\times\Gamma_2)$,

$$\mathcal{I}_{M_p}(\widetilde{A})[\varphi][\psi] := \widetilde{A}[\varphi\otimes\psi] \quad \textit{for } \varphi \in L^{(M_p)}_{(\omega_1)}(\Gamma_1),\ \psi \in L^{(M_p)}_{(\omega_2)}(\Gamma_2), \tag{2}$$

is a linear topological isomorphism of the space $L'^{(M_p)}_{(\omega_1,\omega_2)}(\Gamma_1\times\Gamma_2)$ *onto* $L'^{(M_p)}_{(\omega_1)}(\Gamma_1, L'^{(M_p)}_{(\omega_2)}(\Gamma_2))$.

The proof is based on the Mazur–Orlicz theorem on the separate continuity of 2-linear functionals.

THEOREM 2 (Mazur–Orlicz; Theorem 4.7.1 of [1]). *Let* E^1, E^2 *be multinormed complete vector spaces with the topologies given by non-decreasing sequences of pseudonorms* q^j_k $(j = 1,2;\ k = 0,1,\ldots)$. *Then each separately continuous bilinear form* $\Phi : E^1\times E^2 \to \mathbb{C}$ *is continuous*, *i.e. there exist constants* $C < \infty$ *and* $k\in\mathbb{N}_0$ *such that*

$$|\Phi(\zeta_1,\zeta_2)| \le C q^1_k(\zeta_1) q^2_k(\zeta_2) \quad \textit{for } \zeta_1\in E^1,\ \zeta_2\in E^2. \tag{3}$$

Furthermore, we have

THEOREM 3 (see Theorem 1.3 in [8]). *Let* E^j_k $(j = 1,2;\ k = 0,1,\ldots)$ *be a Banach space with norm* q^j_k *such that* $E^j_{k+1} \subseteq E^j_k$ *and* $q^j_k(\zeta_j) \le q^j_{k+1}(\zeta_j)$ *for* $\zeta_j \in E^j_{k+1}$. *Let* $E^j := \varprojlim_{k\in\mathbb{N}_0} E^j_k$. *Assume that* $K^j_{k+1} := \{\zeta_j \in E^j_{k+1} : q^j_{k+1}(\zeta_j) \le 1\}$ *is precompact in* E^j_k. *Let* $\Phi_\nu : E^1\times E^2 \to \mathbb{C}$ $(\nu = 1,2,\ldots)$ *be separately continuous bilinear forms converging to zero*, *i.e.*

$$\lim_{\nu\to\infty} \Phi_\nu(\zeta_1,\zeta_2) = 0 \quad \textit{for every } \zeta_1\in E^1,\ \zeta_2\in E^2.$$

Then there exist $k\in\mathbb{N}_0$ *and a sequence* $\varepsilon_\nu \to 0_+$ *such that*

$$|\Phi_\nu(\zeta_1,\zeta_2)| \le \varepsilon_\nu q^1_{k+1}(\zeta_1) q^2_{k+1}(\zeta_2) \quad \textit{for } \zeta_1\in E^1_{k+1},\ \zeta_2\in E^2_{k+1},\ \nu = 1,2,\ldots$$

It is easily seen that the spaces $E^1 := L_{a_1}^{(M_p)}(\mathit{\Gamma}_1)$ and $E^2 := L_{a_2}^{(M_p)}(\mathit{\Gamma}_2)$ satisfy the assumptions of Theorems 2 and 3.

In the proof of the kernel theorem we shall use a lemma which generalizes a theorem on the change of order of integration. The lemma is analogous to Theorem 18.11 of [6], so we omit its proof.

LEMMA 1. *Let* $g : \mathbb{R}^n \times \mathit{\Gamma} \to \mathbb{R}$, *where* $\mathit{\Gamma} := [v, \infty)$, $v \in \mathbb{R}^n$, *and let* $a \in \mathbb{R}^n$, $h > 0$. *Put* $g_s(x) := g(s, x)$, *where* $s \in \mathbb{R}^n$, $x \in \mathit{\Gamma}$. *Assume that* g *satisfies*:

1. *For any* $\alpha \in \mathbb{N}_0^n$, $D_x^\alpha g(s, x)$ *is continuous on* $\mathbb{R}^n \times \mathit{\Gamma}$.
2. *For any* $s \in \mathbb{R}^n$, $g_s \in L_{a,h}^{(M_p)}(\mathit{\Gamma})$.
3. *For any* $s_0 \in \mathbb{R}^n$, $\lim_{s \to s_0} g_s = g_{s_0}$ *in* $L_{a,h}^{(M_p)}(\mathit{\Gamma})$.

Let $\gamma \in C_0^0(\mathbb{R}^n)$ *and* $u \in L_{a,h}'^{(M_p)}(\mathit{\Gamma})$. *Then*

$$\int_{\mathbb{R}^n} \gamma(s) u[g_s]\, ds = u\Big[\int_{\mathbb{R}^n} \gamma(s) g_s\, ds\Big]. \tag{4}$$

Let $\gamma \in C^0(\mathbb{R}^n)$ be such that $|\gamma(s)| q_{a,h,\mathit{\Gamma}}^{(M_p)}(g_s)(1+|s_1|)^2 \ldots (1+|s_n|)^2 < C$. Choose a sequence of functions $\gamma_\nu \in C_0^0(\mathbb{R}^n)$ such that $\gamma_\nu \to \gamma$ in $C^0(\mathbb{R}^n)$, $|\gamma_\nu| \le |\gamma|$ and pass to the limit in the already proved formula for $\gamma_\nu \in C_0^0(\mathbb{R}^n)$. Then we have

LEMMA 2. *Under the conditions of Lemma* 1, (4) *holds for functions* $\gamma \in C^0(\mathbb{R}^n)$ *such that* $|\gamma(s)| q_{a,h,\mathit{\Gamma}}^{(M_p)}(g_s)(1+|s_1|)^2 \ldots (1+|s_n|)^2 < C$ *for* $s \in \mathbb{R}^n$ *with some* $C < \infty$.

Proof of Theorem 1. We first observe that the transformation $\mathcal{I}_{M_p}$ is well defined. Indeed, let $\widetilde{A} \in L_{(\omega_1,\omega_2)}'^{(M_p)}(\mathit{\Gamma}_1 \times \mathit{\Gamma}_2)$. Then for any $a_j < \omega_j$ $(j = 1, 2)$ there exist $h > 0$ and $c < \infty$ such that

$$\begin{aligned} |\mathcal{I}_{M_p}(\widetilde{A})[\varphi][\psi]| &= |\widetilde{A}[\varphi \otimes \psi]| \\ &\le c \sup_{x_1 \in \mathit{\Gamma}_1} \sup_{\alpha_1 \in \mathbb{N}_0^{n_1}} \frac{|e^{-a_1 x_1} D_{x_1}^{\alpha_1} \varphi(x_1)|}{h^{|\alpha_1|} M_{|\alpha_1|}} \sup_{x_2 \in \mathit{\Gamma}_2} \sup_{\alpha_2 \in \mathbb{N}_0^{n_2}} \frac{|e^{-a_2 x_2} D_{x_2}^{\alpha_2} \psi(x_2)|}{h^{|\alpha_2|} M_{|\alpha_2|}} \end{aligned} \tag{5}$$

for $\varphi \in L_{a_1}^{(M_p)}(\mathit{\Gamma}_1)$ and $\psi \in L_{a_2}^{(M_p)}(\mathit{\Gamma}_2)$. Thus $\mathcal{I}_{M_p}(\widetilde{A}) \in L_{(\omega_1)}'^{(M_p)}(\mathit{\Gamma}_1, L_{(\omega_2)}'^{(M_p)}(\mathit{\Gamma}_2))$. If we have a sequence $(\widetilde{A}_\nu)_{\nu \in \mathbb{N}}$ convergent to zero in $L_{(\omega_1,\omega_2)}'^{(M_p)}(\mathit{\Gamma}_1 \times \mathit{\Gamma}_2)$ then the sequence of the corresponding numbers c_ν in (5) is also convergent to zero and consequently the sequence $(\mathcal{I}_{M_p}(\widetilde{A}_\nu))_{\nu \in \mathbb{N}}$ is convergent to zero in $L_{(\omega_1)}'^{(M_p)}(\mathit{\Gamma}_1, L_{(\omega_2)}'^{(M_p)}(\mathit{\Gamma}_2))$. Thus the operator $\mathcal{I}_{M_p}$ is continuous.

Now we construct a continuous inverse transformation

$$\mathcal{I}_{M_p}^{-1} : L'^{(M_p)}_{(\omega_1)}(\Gamma_1, L'^{(M_p)}_{(\omega_2)}(\Gamma_2)) \to L'^{(M_p)}_{(\omega_1,\omega_2)}(\Gamma_1 \times \Gamma_2)$$

such that $\mathcal{I}_{M_p}\mathcal{I}_{M_p}^{-1} = \mathrm{Id}$ and $\mathcal{I}_{M_p}^{-1}\mathcal{I}_{M_p} = \mathrm{Id}$.

Fix $A \in L'^{(M_p)}_{(\omega_1)}(\Gamma_1, L'^{(M_p)}_{(\omega_2)}(\Gamma_2))$ and take any a_1, a_2, d_1, d_2 such that $a_j < d_j < \omega_j$ $(j = 1, 2)$. By Theorem 2 there exist $c_A < \infty$, $h > 0$ such that

$$|A[\varphi][\psi]| \le c_A q^{(M_p)}_{d_1,h,\Gamma_1}(\varphi) q^{(M_p)}_{d_2,h,\Gamma_2}(\psi) \qquad \text{for } \varphi \in L^{(M_p)}_{d_1}(\Gamma_1), \psi \in L^{(M_p)}_{d_2}(\Gamma_2). \tag{6}$$

By the Hahn–Banach theorem, (6) holds for $\varphi \in L^{(M_p)}_{d_1,h}(\Gamma_1)$, $\psi \in L^{(M_p)}_{d_2,h}(\Gamma_2)$. Put $\zeta_j := b_j + i\eta_j$ where $b_j \in \mathbb{R}^{n_j}$ with $a_j < b_j < d_j$ and $\eta_j \in \mathbb{R}^{n_j}$ $(j = 1, 2)$. Since there exist c_j, k_j $(k_j := (1 + |b_j|)/h)$ such that

$$q^{(M_p)}_{d_j,h,\Gamma_j}(e^{x_j\zeta_j}) \le c_j \exp M(k_j(1 + |\eta_j|)),$$

where $\exp M(k_j(1 + |\eta_j|)) := \prod_{i=1}^{n_j} \exp M(k_j(1 + |\eta_j^i|))$, the function $\Gamma_j \ni x_j \mapsto e^{x_j\zeta_j}$ belongs to $L^{(M_p)}_{d_j,h}(\Gamma_j)$ $(j = 1, 2)$. So we conclude from (6) that

$$|A[e^{x_1\zeta_1}][e^{x_2\zeta_2}]| \le c_A c_1 c_2 \exp M(k_1(1 + |\eta_1|)) \exp M(k_2(1 + |\eta_2|)). \tag{7}$$

Let $\Phi \in L^{(M_p)}_{a_1,a_2}(\Gamma_1 \times \Gamma_2)$. Then the Laplace transform $\mathcal{L}\Phi$ given by

$$\mathcal{L}\Phi(\zeta) := \int_{\Gamma} \Phi(x) e^{-\zeta x}\, dx \qquad \text{for } \operatorname{Re}\zeta > \mathrm{a}$$

satisfies

$$|\mathcal{L}\Phi(\zeta_1, \zeta_2)| \le c q^{(M_p)}_{(a_1,a_2),1,\Gamma_1\times\Gamma_2}(\Phi) =: c_\Phi < \infty. \tag{8}$$

Put $Q(\zeta_1, \zeta_2) := Q_1(\zeta_1)Q_2(\zeta_2)$ with

$$\begin{aligned} Q_j(\zeta_j) &:= (\zeta_j - d_j - 1)^{p_0+1} \prod_{p=p_0}^{\infty} \left(1 - \frac{k_j\zeta_j}{m_p}\right) \\ &:= \prod_{i=1}^{n_j} (\zeta_j^i - d_j^i - 1)^{p_0+1} \prod_{p=p_0}^{\infty} \left(1 - \frac{k_j\zeta_j^i}{m_p}\right), \end{aligned}$$

where $m_p := M_p/M_{p-1}$, p_0 is such that $m_p > 2k_j|b_j| + k_j$ and $|m_p - k_j\zeta_j| \ge k_j|\zeta_j|$ for $p \ge p_0$, $j = 1, 2$. By the Hadamard factorization theorem (Propositions 4.5 and 4.6 in [2]), Q is a symbol of class (M_p) and it satisfies the inequality (see [3], Lemma 3)

$$\frac{\exp M(k_1|\zeta_1|) \exp M(k_2|\zeta_2|)}{|Q(\zeta_1, \zeta_2)|} \le \frac{K'}{(1 + |\eta_1|)^2(1 + |\eta_2|)^2} \tag{9}$$

with some $K' < \infty$, where $(1 + |\eta_j|)^2 := \prod_{i=1}^{n_j} (1 + |\eta_j^i|)^2$ $(j = 1, 2)$.

Now we can write the mapping $\mathcal{I}_{M_p}^{-1}$:

$$(10)\qquad \mathcal{I}_{M_p}^{-1}(A)[\Phi] := \left(\frac{1}{2\pi i}\right)^{n_1+n_2} Q(D_{x_1}, D_{x_2}) \int\limits_{b_1+i\mathbb{R}^{n_1}} \int\limits_{b_2+i\mathbb{R}^{n_2}} A[e^{x_1\zeta_1}][e^{x_2\zeta_2}] \times \frac{\mathcal{L}\Phi(\zeta_1,\zeta_2)}{Q(\zeta_1,\zeta_2)}\, d\zeta_1\, d\zeta_2.$$

From (7)–(9) we obtain

$$\begin{aligned} &\left| A[e^{x_1\zeta_1}][e^{x_2\zeta_2}] \frac{\mathcal{L}\Phi(\zeta_1,\zeta_2)}{Q(\zeta_1,\zeta_2)} \right| \\ &\qquad \le c_A c_1 c_2 c_\Phi \frac{\exp M(k_1(1+|\eta_1|)) \exp M(k_2(1+|\eta_2|))}{|Q(\zeta_1,\zeta_2)|} \\ &\qquad \le \frac{K}{(1+|\eta_1|)^2(1+|\eta_2|)^2} \end{aligned}$$

with some $K < \infty$. Therefore the integral in (10) is convergent (vector notation!).

Since the ultradifferential operator

$$Q(D_{x_1}, D_{x_2}) : L_a^{(M_p)}(\Gamma) \to L_a^{(M_p)}(\Gamma)$$

is continuous (cf. Th. 2.12 in [2]), for $h > 0$ sufficiently small we have

$$|\mathcal{I}_{M_p}^{-1}(A)[\Phi]| \le C c_A q^{(M_p)}_{(a_1,a_2),h,\Gamma_1\times\Gamma_2}(\Phi)$$

with some $C < \infty$. Thus $\mathcal{I}_{M_p}^{-1}(A) \in L'^{(M_p)}_{(\omega_1,\omega_2)}(\Gamma_1 \times \Gamma_2)$.

If a sequence $(A_\nu)_{\nu\in\mathbb{N}}$ is convergent to zero in $L'^{(M_p)}_{(\omega_1)}(\Gamma_1, L'^{(M_p)}_{(\omega_2)}(\Gamma_2))$ then by Theorem 3 the sequence of the corresponding numbers c_{A_ν} in (6) converges to zero. Thus the sequence $(\mathcal{I}_{M_p}^{-1}(A_\nu))_{\nu\in\mathbb{N}}$ is convergent to zero in $L'^{(M_p)}_{(\omega_1,\omega_2)}(\Gamma_1 \times \Gamma_2)$ and we conclude that the operator $\mathcal{I}_{M_p}^{-1}$ is continuous.

Next we show that $\mathcal{I}_{M_p}^{-1}$ is the inverse mapping to $\mathcal{I}_{M_p}$. To this end we apply the operator $Q(D_x)$ to the inversion formula for the Laplace transformation (see [9]). For $\varphi \in L_a^{(M_p)}(\Gamma)$ we have

$$\varphi(x) = Q(D_x)\left(\frac{1}{2\pi i}\right)^{n} \int\limits_{b+i\mathbb{R}^n} e^{x\zeta} \frac{\mathcal{L}\varphi(\zeta)}{Q(\zeta)}\, d\zeta, \qquad \text{where } x \in \Gamma.$$

From the above equality and Lemma 2 we derive that

$$\mathcal{I}_{M_p}(\mathcal{I}_{M_p}^{-1}(A))[\varphi][\psi] = \mathcal{I}_{M_p}^{-1}(A)[\varphi\otimes\psi]$$

is equal to

$$
\begin{aligned}
\left(\frac{1}{2\pi i}\right)^{n_1+n_2} & Q_1(D_{x_1})Q_2(D_{x_2}) \int\limits_{b_1+i\mathbb{R}^{n_1}} \int\limits_{b_2+i\mathbb{R}^{n_2}} A[e^{x_1\zeta_1}][e^{x_2\zeta_2}] \\
& \times \frac{\mathcal{L}\varphi(\zeta_1)}{Q_1(\zeta_1)} \frac{\mathcal{L}\psi(\zeta_2)}{Q_2(\zeta_2)}\, d\zeta_1\, d\zeta_2 \\
& = \left(\frac{1}{2\pi i}\right)^{n_1} Q_1(D_{x_1}) \int\limits_{b_1+\mathbb{R}^{n_1}} \frac{\mathcal{L}\varphi(\zeta_1)}{Q_1(\zeta_1)} \\
& \quad \times A[e^{x_1\zeta_1}]\left[\left(\frac{1}{2\pi i}\right)^{n_2} Q_2(D_{x_2}) \int\limits_{b_2+i\mathbb{R}^{n_2}} e^{x_2\zeta_2} \frac{\mathcal{L}\psi(\zeta_2)}{Q_2(\zeta_2)}\, d\zeta_2\right] d\zeta_1 \\
& = \left(\frac{1}{2\pi i}\right)^{n_1} Q_1(D_{x_1}) \int\limits_{b_1+i\mathbb{R}^{n_1}} \frac{\mathcal{L}\varphi(\zeta_1)}{Q_1(\zeta_1)} A[e^{x_1\zeta_1}][\psi]\, d\zeta_1 \\
& = A\left[\left(\frac{1}{2\pi i}\right)^{n_1} Q_1(D_{x_1}) \int\limits_{b_1+i\mathbb{R}^{n_1}} e^{x_1\zeta_1} \frac{\mathcal{L}\varphi(\zeta_1)}{Q_1(\zeta_1)}\, d\zeta_1\right][\psi] \\
& = A[\varphi][\psi].
\end{aligned}
$$

Similarly we obtain

$$
\begin{aligned}
& \mathcal{I}_{M_p}^{-1}(\mathcal{I}_{M_p}(\widetilde{A}))[\Phi] \\
& = \left(\frac{1}{2\pi i}\right)^{n_1+n_2} Q(D_{x_1}, D_{x_2}) \\
& \quad \times \int\limits_{b_1+i\mathbb{R}^{n_1}} \int\limits_{b_2+i\mathbb{R}^{n_2}} \frac{\mathcal{L}\Phi(\zeta_1,\zeta_2)}{Q(\zeta_1,\zeta_2)} \mathcal{I}_{M_p}(\widetilde{A})[e^{x_1\zeta_1}][e^{x_2\zeta_2}]\, d\zeta_1 d\zeta_2 \\
& = \widetilde{A}\left[\left(\frac{1}{2\pi i}\right)^{n_1+n_2} Q(D_{x_1}, D_{x_2}) \int\limits_{b_1+i\mathbb{R}^{n_1}} \int\limits_{b_2+i\mathbb{R}^{n_2}} e^{x_1\zeta_1+x_2\zeta_2} \frac{\mathcal{L}\Phi(\zeta_1,\zeta_2)}{Q(\zeta_1,\zeta_2)}\, d\zeta_1 d\zeta_2\right] \\
& = \widetilde{A}[\Phi].
\end{aligned}
$$

This completes the proof.

Acknowledgments. The author gratefully acknowledges many helpful suggestions of Grzegorz Łysik during the preparation of the paper.

References

[1] J. Horváth, *Topological Vector Spaces and Distributions I*, Addison-Wesley, Reading, MA, 1966.

[2] H. Komatsu, *Ultradistributions*, *I. Structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo 20 (1973), 25–105.

[3] G. Łysik, *Laplace ultradistributions on a half line and a strong quasi-analyticity principle*, Ann. Polon. Math. 63 (1996), 13–33.
[4] R. Meise and A. Taylor, *Linear extension operators for ultradifferentiable functions of Beurling type on compact sets*, Amer. J. Math. 111 (1989), 309–337.
[5] L. Schwartz, *Théorie des noyaux*, in: Proc. Internat. Congress Math. Cambridge 1950, I, 220–230.
[6] Z. Szmydt, *Fourier Transformation and Linear Differential Equations*, PWN–Polish Sci. Publ., Warszawa, and Reidel, Dordrecht, 1977.
[7] Z. Szmydt and B. Ziemian, *The Mellin Transformation and Fuchsian Type Partial Differential Equations*, Kluwer, Dordrecht, 1992.
[8] —, —, *Topological imbedding of Laplace distributions in Laplace hyperfunctions*, Dissertationes Math. 376 (1998).
[9] D. V. Widder, *The Laplace Transform*, Princeton Univ. Press, Princeton NJ, 1946.

Institute of Mathematics
Polish Academy of Sciences
P.O. Box 137
Śniadeckich 8
00-950 Warszawa, Poland
E-mail: slawek@impan.gov.pl

Reçu par la Rédaction le 19.11.1998
Révisé le 26.2.2001 (1021)