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Cross theorem

by Marek Jarnicki (Kraków) and Peter Pflug (Oldenburg)

Abstract. Let $D, G \subset \mathbb{C}$ be domains, let $A \subset D$, $B \subset G$ be locally regular sets, and let $X := (D\times B)\cup(A\times G)$. Assume that A is a Borel set. Let M be a proper analytic subset of an open neighborhood of X. Then there exists a pure 1-dimensional analytic subset $\widehat{M}$ of the envelope of holomorphy $\widehat{X}$ of X such that any function separately holomorphic on $X \setminus M$ extends to a holomorphic function on $\widehat{X} \setminus \widehat{M}$. The result generalizes special cases which were studied in [Ökt 1998], [Ökt 1999], and [Sic 2000].

1. Introduction. Main result. For domains $D \subset \mathbb{C}^n$, $G \subset \mathbb{C}^m$ and non-pluripolar subsets $A \subset D$, $B \subset G$, define the *cross*

$$(*)\qquad X = \mathbb{X}(D, A; G, B) := (D \times B) \cup (A \times G)$$

(notice that X is connected). Let $U \subset D \times G$ be an open connected neighborhood of X and let M be an analytic subset of U, $M \neq U$. Put

$$M_z := \{w \in G : (z, w) \in M\}, \qquad z \in D,$$
$$M^w := \{z \in D : (z, w) \in M\}, \qquad w \in G.$$

We say that a function $f : X \setminus M \to \mathbb{C}$ is *separately holomorphic* on $X \setminus M$ ($f \in \mathcal{O}_{\mathrm{s}}(X \setminus M)$) if

$$\forall_{z\in A,\, M_z\neq G} : f(z, \cdot) \in \mathcal{O}(G \setminus M_z), \qquad \forall_{w\in B,\, M^w\neq D} : f(\cdot, w) \in \mathcal{O}(D \setminus M^w).$$

For an open set $\Omega \subset \mathbb{C}^n$ and $A \subset \Omega$ put

$$h_{A,\Omega} := \sup\{u : u \in \mathcal{PSH}(\Omega),\ u \le 1 \text{ on } \Omega,\ u \le 0 \text{ on } A\},$$

where $\mathcal{PSH}(\Omega)$ denotes the set of all functions plurisubharmonic on Ω.

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Define

$$\omega_{A,\Omega} := \lim_{k\to\infty} h^*_{A\cap\Omega_k,\Omega_k},$$

where $(\Omega_k)_{k=1}^\infty$ is a sequence of relatively compact open sets $\Omega_k \subset \Omega_{k+1} \subset\subset \Omega$ with $\bigcup_{k=1}^\infty \Omega_k = \Omega$ (h^* denotes the upper semicontinuous regularization of h). Note that the definition is independent of the approximation sequence $(\Omega_k)_{k=1}^\infty$.

For a cross (∗) put

$$(**) \qquad \widehat{X} := \{(z,w) \in D \times G : \omega_{A,D}(z) + \omega_{B,G}(w) < 1\}.$$

We say that a subset $A \subset \mathbb{C}^n$ is *locally pluriregular* if $h^*_{A\cap\Omega,\Omega}(a) = 0$ for any $a \in A$ and for any open neighborhood Ω of a (in particular, $A \cap \Omega$ is non-pluripolar). As always, if $n = 1$, then we say that A is locally "regular" instead of "pluriregular".

The main result of the paper is the following

THEOREM 1. *Let $D, G \subset \mathbb{C}$ be domains, let $A \subset D$, $B \subset G$ be locally regular sets, and let $X := (D \times B) \cup (A \times G)$. Assume that A is a Borel set. Let M be a proper analytic subset of an open connected neighborhood U of X. Then there exists a pure 1-dimensional analytic subset $\widehat{M}$ of $\widehat{X}$ ($\widehat{X}$ is given by (∗∗)) such that for any $f \in \mathcal{O}_s(X \setminus M)$ there exists exactly one $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ with $\widehat{f} = f$ on $X \setminus (M \cup \widehat{M})$.*

Moreover, if $U = \widehat{X}$ and M is pure 1-dimensional, then the above condition is satisfied by $\widehat{M} := M$.

REMARK. Consider the following general problem. Let $D_j \subset \mathbb{C}^{n_j}$ be a domain of holomorphy and let $A_j \subset D_j$ be a locally pluriregular Borel set, $j = 1, \ldots, N$. Define the *generalized cross*

$$X := (A_1 \times \ldots \times A_{N-1} \times D_N) \cup \ldots \cup (D_1 \times A_2 \times \ldots \times A_N) \subset \mathbb{C}^{n_1} \times \ldots \times \mathbb{C}^{n_N}.$$

Let $U \subset D_1 \times \ldots \times D_N$ be a connected neighborhood of X and let $M \subset U$ be a proper analytic set. A function $f : X \setminus M \to \mathbb{C}$ is said to be *separately holomorphic* ($f \in \mathcal{O}_s(X \setminus M)$) if for any $(a_1, \ldots, a_N) \in A_1 \times \ldots \times A_N$ and $k \in \{1, \ldots, N\}$ the function $f(a_1, \ldots, a_{k-1}, \cdot, a_{k+1}, \ldots, a_N)$ is holomorphic in the domain $\{z_k \in D_k : (a_1, \ldots, a_{k-1}, z_k, a_{k+1}, \ldots, a_N) \notin M\}$. Define

$$\widehat{X} := \{(z_1, \ldots, z_N) \in D_1 \times \ldots \times D_N : \omega_{A_1,D_1}(z_1) + \ldots + \omega_{A_N,D_N}(z_N) < 1\}.$$

CONJECTURE ([1]). There exists a pure 1-codimensional analytic subset $\widehat{M} \subset \widehat{X}$ such that for any $f \in \mathcal{O}_s(X \setminus M)$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ with $\widehat{f} = f$ on $X \setminus (M \cup \widehat{M})$. Moreover, $\widehat{M} = M$ if $U = \widehat{X}$ and M is pure 1-codimensional. Compare also [Ökt 1999] (for $N = 2$ and $U = \widehat{X}$).

([1]) Added in proof: Cf. M. Jarnicki, P. Pflug, *An extension theorem for separately holomorphic functions with singularities*, IMUJ Preprint 2001/27 (2001).

Theorem 1 solves the case $N = 2$, $n_1 = n_2 = 1$.

J. Siciak [Sic 2000] solved the following case: $n_1 = \ldots = n_N = 1$, $D_1 = \ldots = D_N = \mathbb{C}$, $M = P^{-1}(0)$, where P is a non-zero polynomial of N complex variables; the special subcase $N = 2$, $P(z,w) := z - w$ had been previously studied in [Ökt 1998].

The case studied in [Sic 2000] is the only known case with $n_1 + \ldots + n_N > 2$. In the general case, the answer is not known even if $U = \widehat{X}$ and M is pure 1-codimensional.

2. Auxiliary results. The following lemma gathers a few standard results which will be used in what follows.

Lemma 2 (cf. [Kli 1991], [Jar-Pfl 2000], §3.5). (a) *Let $\Omega \subset \mathbb{C}^n$ be a bounded open set and let $A \subset \Omega$. Then*:

- *If $P \subset \mathbb{C}^n$ is pluripolar, then $h^*_{A\setminus P,\Omega} = h^*_{A,\Omega}$.*
- *$h^*_{A_k\cap\Omega_k,\Omega_k} \searrow h^*_{A,\Omega}$ (pointwise on Ω) for any sequence of open sets $\Omega_k \nearrow \Omega$ and any sequence $A_k \nearrow A$.*
- *$\omega_{A,\Omega} = h^*_{A,\Omega}$.*
- *The following two conditions are equivalent*:

  (i) *for any connected component S of Ω the set $A\cap S$ is non-pluripolar*;
  (ii) *$h^*_{A,\Omega}(z) < 1$ for any $z \in \Omega$.*

- *If A is non-pluripolar, $0 < \alpha < 1$, and $\Omega_\alpha := \{z \in \Omega : h^*_{A,\Omega}(z) < \alpha\}$, then for any connected component S of Ω_α the set $A \cap S$ is non-pluripolar (in particular, $A \cap S \neq \emptyset$).*

(b) *Let $\Omega \subset \mathbb{C}^n$ be an open set and let $A \subset \Omega$. Then*:

- *$\omega_{A,\Omega} \in \mathcal{PSH}(\Omega)$.*
- *If A is locally pluriregular, then $\omega_{A,\Omega}(a) = 0$ for any $a \in A$.*
- *If $P \subset \mathbb{C}^n$ is pluripolar, then $\omega_{A\setminus P,\Omega} = \omega_{A,\Omega}$.*
- *If A is locally pluriregular and $P \subset \mathbb{C}^n$ is pluripolar, then $A \setminus P$ is locally pluriregular.*

(c) *Let $X = \mathbb{X}(D,A;G,B)$ be a cross as in $(*)$. Then*:

- *If A and B are locally pluriregular, then $X \subset \widehat{X}$.*
- *If D and G are domains of holomorphy, then $\widehat{X}$ is a region of holomorphy.*

Lemma 3. *Let $X = \mathbb{X}(D,A;G,B)$ be a cross as in $(*)$. If A and B are locally pluriregular, then $\widehat{X}$ is a domain.*

Proof. It suffices to show that for any approximation sequences $D_k \nearrow D$, $G_k \nearrow G$ of relatively compact subdomains with $A \cap D_k \neq \emptyset$, $B \cap G_k \neq \emptyset$,

$k \in \mathbb{N}$, the sets

$$\widehat{X}_k := \{(z,w) \in D_k \times G_k : h^*_{A\cap D_k, D_k}(z) + h^*_{B\cap G_k, G_k}(w) < 1\}, \quad k \in \mathbb{N},$$

are connected. Thus, we may assume that D and G are bounded. Since the cross X is connected and contained in $\widehat{X}$, we only need to prove that for any $(z_0, w_0) \in \widehat{X}$, each connected component of the fiber

$$\widehat{X}^{w_0} := \{z \in D : (z, w_0) \in \widehat{X}\} = \{z \in D : h^*_{A,D}(z) < 1 - h^*_{B,G}(w_0)\}$$

intersects A. If $h^*_{B,G}(w_0) = 0$, then $\widehat{X}^{w_0} = D$. If $h^*_{B,G}(w_0) > 0$, then we apply Lemma 2(a). ■

Theorem 4 (Classical cross theorem, cf. [Ngu-Zer 1991]). *Let* $X = \mathbb{X}(D,A;G,B)$ *be as in* $(*)$. *Assume that*:

- D, G *are domains of holomorphy*,
- A, B *are locally pluriregular*,
- A *is a Borel set.*

Then for any $f \in \mathcal{O}_s(X)$ *there exists exactly one* $\widehat{f} \in \mathcal{O}(\widehat{X})$ *with* $\widehat{f} = f$ *on* X.

Theorem 5 (Dloussky–Grauert–Remmert theorem, cf. [Jar-Pfl 2000], §3.4). *Let* $\Omega \subset \mathbb{C}^n$ *be a domain and let* M *be an analytic subset of* Ω. *Let* $\widehat{\Omega}$ *be the envelope of holomorphy of* Ω (*univalent or not*). *Then there exists a pure* 1-*codimensional analytic subset* $\widehat{M} \subset \widehat{\Omega}$ *uch that for any* $g \in \mathcal{O}(\Omega \setminus M)$ *there exists* $\widehat{g} \in \mathcal{O}(\widehat{\Omega} \setminus \widehat{M})$ *with* $\widehat{g} = g$ *on* $\Omega \setminus (M \cup \widehat{M})$.

If, *moreover*, $M = \Omega \cap \widetilde{M}$, *where* $\widetilde{M}$ *is a pure* 1-*codimensional analytic subset of* $\widehat{\Omega}$, *then the above condition is satisfied by* $\widehat{M} := \widetilde{M}$.

Lemma 6. *Let* $D, G \subset \mathbb{C}$ *be domains*, *let* $A \subset D$, $B \subset G$ *be locally regular sets*, *and let* $X := \mathbb{X}(D,A;G,B)$. *Let* M *be a proper analytic subset of an open connected neighborhood* U *of* X. *Assume that* $A' \subset A$, $B' \subset B$ *are such that*:

- $A \setminus A'$ *and* $B \setminus B'$ *are polar* (*in particular*, A', B' *are also locally regular*),
- $M_z \neq G$ *for any* $z \in A'$ *and* $M^w \neq D$ *for any* $w \in B'$.

  (a) *If* $f \in \mathcal{O}_s(X \setminus M)$ *and* $f = 0$ *on* $(A' \times B') \setminus M$, *then* $f = 0$ *on* $X \setminus M$.

  (b) *If* $g \in \mathcal{O}(U \setminus M)$ *and* $g = 0$ *on* $(A' \times B') \setminus M$, *then* $g = 0$ *on* $U \setminus M$.

Proof. (a) Take a point $(a_0, b_0) \in X \setminus M$. We may assume that $a_0 \in A$. Since $A \setminus A'$ is polar, there exists a sequence $(a_k)_{k=1}^{\infty} \subset A'$ such that $a_k \to a_0$. The set $Q := \bigcup_{k=0}^{\infty} M_{a_k}$ is at most countable. Consequently, the set $B'' := B' \setminus Q$ is non-polar. We have $f(a_k, w) = 0$ for all $w \in B''$, $k = 1, 2, \ldots$ Hence $f(a_0, w) = 0$ for any $w \in B''$. Finally, $f(a_0, w) = 0$ on $G \setminus M_{a_0} \ni b_0$.

(b) Take an $a_0 \in A'$. Since $M_{a_0} \neq G$, there exists a $b_0 \in B' \setminus M_{a_0}$. Let $P = \Delta_{a_0}(r) \times \Delta_{b_0}(r) \subset U \setminus M$ ($\Delta_{z_0}(r)$ denotes the disc with center z_0 and radius r). Then $g(\cdot, w) = 0$ on $A' \cap \Delta_{a_0}(r)$ for any $w \in B' \cap \Delta_{b_0}(r)$. The set $A' \cap \Delta_{a_0}(r)$ is non-polar. Hence $g(\cdot, w) = 0$ on $\Delta_{a_0}(r)$ for any $w \in B' \cap \Delta_{b_0}(r)$. By the same argument for the second variable we get $g = 0$ on P and, consequently, on U. ∎

3. Proof of the main theorem

STEP 1. Fix sequences $D_k \nearrow D$, $G_k \nearrow G$ of relatively compact subdomains with $D_k \subset\subset D_{k+1}$, $A \cap D_k \neq \emptyset$, $G_k \subset\subset G_{k+1}$, $B \cap G_k \neq \emptyset$, $k \in \mathbb{N}$.

For any $a \in A$ such that $M_a \neq G$ we perform the following construction:

Fix a $k \in \mathbb{N}$, $k \geq 2$. Let $M_a \cap G_k = \{b_1, \ldots, b_N\}$. Fix domains $G' = G'_{a,k}$, $G'' = G''_{a,k}$ such that $G_{k-1} \subset\subset G'' \subset\subset G' \subset\subset G_k$ and $b_1, \ldots, b_N \in G''$. Take positive numbers δ, ε, $\eta > \varepsilon$ such that

- $\Delta_a(\delta) \subset\subset D$,
- $\Delta_{b_j}(\eta) \subset\subset G''$, $j = 1, \ldots, N$,
- $\overline{\Delta}_{b_i}(\eta) \cap \overline{\Delta}_{b_j}(\eta) = \emptyset$, $i, j = 1, \ldots, N$, $i \neq j$,
- $M \cap (\Delta_a(\delta) \times G') \subset \bigcup_{j=1}^N \Delta_a(\delta) \times \Delta_{b_j}(\varepsilon)$,
- $B \cap V'' \neq \emptyset$, where $V'' := G'' \setminus \bigcup_{j=1}^N \overline{\Delta}_{b_j}(\eta)$.

Define $V' := G' \setminus \bigcup_{j=1}^N \overline{\Delta}_{b_j}(\varepsilon)$. Note that $V'' \subset\subset V'$. Consider the cross

$$Y = Y_{a,k} := \mathbb{X}(\Delta_a(\delta), A \cap \Delta_a(\delta); V', B \cap V').$$

Fix an $f \in \mathcal{O}_s(X \setminus M)$. Then $f \in \mathcal{O}_s(Y)$. By Theorem 4, the function f extends holomorphically to $\widehat{Y} \supset \{a\} \times V'$. Consequently, there exists $0 < \widehat{\delta} < \delta$ such that f is holomorphic in $\Delta_a(\widehat{\delta}) \times V''$.

STEP 2. Suppose that for some $j \in \{1, \ldots, N\}$ we have

$$M \cap (\Delta_a(\delta) \times \Delta_{b_j}(\varepsilon)) \subset \{(z, \varphi_j(z)) : z \in \Delta_a(\delta)\},$$

where $\varphi_j : \Delta_a(\delta) \to \Delta_{b_j}(\varepsilon)$ is holomorphic.

We will prove that for sufficiently small $\delta' > 0$ the function f extends holomorphically to $(\Delta_a(\delta') \times \Delta_{b_j}(\eta)) \setminus \{(z, \varphi_j(z)) : z \in \Delta_a(\delta')\}$.

Indeed, by Step 1, there exists $\eta' > \eta$ such that the function f extends holomorphically to $\Delta_a(\widehat{\delta}) \times (\Delta_{b_j}(\eta') \setminus \overline{\Delta}_{b_j}(\eta))$. Using the biholomorphism

$$\Delta_a(\delta) \times \mathbb{C} \ni (z, w) \mapsto (z, w - \varphi_j(z)) \in \Delta_a(\delta) \times \mathbb{C},$$

we reduce the problem to the case where $\varphi_j \equiv 0$. Thus we have the following problem:

Let $\Delta(r) := \Delta_0(r)$. Given a function f holomorphic on $\Delta_a(\varrho) \times P$, where $P := \Delta(R) \setminus \overline{\Delta}(r)$, such that $f(z, \cdot) \in \mathcal{O}(\Delta(R) \setminus \{0\})$ for any $z \in A \cap \Delta_a(\varrho)$, we prove that f extends holomorphically to $\Delta_a(\varrho) \times (\Delta(R) \setminus \{0\})$.

Indeed, consider the cross

$$Y := \mathbb{X}(\varDelta_a(\varrho), A \cap \varDelta_a(\varrho); \varDelta(R) \setminus \{0\}, P).$$

By Theorem 4, the function f extends to $\widehat{Y}$. It remains to observe that $\widehat{Y} = \varDelta_a(\varrho) \times (\varDelta(R) \setminus \{0\})$ (because $h^*_{P,\varDelta(R)\setminus\{0\}} \equiv 0$).

In particular, if

$$M \cap (\varDelta_a(\delta) \times \varDelta_{b_j}(\varepsilon)) = \{(z, \varphi_j(z)) : z \in \varDelta_a(\delta)\},$$

where $\varphi_j : \varDelta_a(\delta) \to \varDelta_{b_j}(\varepsilon)$ is holomorphic, for all $j = 1, \ldots, N$, then there exists $\delta' > 0$ such that f extends holomorphically to $(\varDelta_a(\delta') \times G'') \setminus M \supset (\varDelta_a(\delta') \times G_{k-1}) \setminus M$.

STEP 3. Suppose that for some $j \in \{1, \ldots, N\}$ we have

$$M \cap (\varDelta_a(\delta) \times \varDelta_{b_j}(\varepsilon)) = \{(a, b_j)\}.$$

By Step 2 (with $\varphi_j \equiv b_j$) the function f extends holomorphically to $\varDelta_a(\delta') \times (\varDelta_{b_j}(\eta) \setminus \{b_j\})$ for some small $\delta' > 0$. On the other hand, we know that f is separately holomorphic on $Z := (\varDelta_a(\delta) \setminus \{a\}) \times \varDelta_{b_j}(\varepsilon)$. Consequently, f is holomorphic on Z. Hence f is holomorphic on $(\varDelta_a(\delta') \times \varDelta_{b_j}(\varepsilon)) \setminus \{(a, b_j)\}$. Thus (a, b_j) is a removable singularity of f.

In virtue of the above remark, we may assume that M is pure 1-dimensional.

STEP 4. Let A' denote the set of all $a \in A$ such that for each $k \geq 2$ either there exists $\delta > 0$ such that $M \cap (\varDelta_a(\delta) \times G_k) = \emptyset$ or the construction from Step 1 may be performed in such a way that for each $j \in \{1, \ldots, N\}$,

$$M \cap (\varDelta_a(\delta) \times \varDelta_{b_j}(\varepsilon)) = \{(z, \varphi_j(z)) : z \in \varDelta_a(\delta)\},$$

where $\varphi_j : \varDelta_a(\delta) \to \varDelta_{b_j}(\varepsilon)$ is holomorphic (cf. Step 2). Then $A \setminus A'$ is at most countable. Indeed, write

$$M = \bigcup_{j=1}^{\infty} \{(z, w) \in P_j : g_j(z, w) = 0\},$$

where $P_j \subset\subset U$ is a polydisc and g_j is a defining function for $M \cap P_j$ (cf. [Chi 1989], § 2.9). Put $S_j := \{(z, w) \in P_j : g_j(z, w) = \frac{\partial g_j}{\partial w}(z, w) = 0\}$. Observe that if $(z_0, w_0) \in (M \cap P_j) \setminus S_j$, then there exists a small polydisc $Q = Q' \times Q'' \subset\subset P_j$ with center at (z_0, w_0) such that $M \cap Q$ is the graph of a holomorphic function $\varphi : Q' \to Q''$.

The projection $\mathrm{pr}_z(S_j)$ is at most countable. Indeed, we only need to prove that $\mathrm{pr}_z(S'_j)$ is at most countable, where S'_j is the union of all 1-dimensional irreducible components of S_j. Let S be such an irreducible component. We will show that S projects onto one point. Take $(z_1, w_1), (z_2, w_2) \in S$. We want to show that $z_1 = z_2$. It suffices to consider the case where $(z_1, w_1), (z_2, w_2)$ are regular points of S. Let $\psi = (\psi_1, \psi_2) : [0, 1] \to \mathrm{Reg}(S)$

be a $\mathcal{C}^1$-curve with $\psi(0) = (z_1, w_1)$, $\psi(1) = (z_2, w_2)$. Note that $\frac{\partial g_j}{\partial z}(z, w) \neq 0$ for $(z, w) \in \operatorname{Reg}(S)$ (because g_j is a defining function). We have

$$0 = \frac{\partial(g_j \circ \psi)}{\partial t}(t) = \frac{\partial g_j}{\partial z}(\psi(t))\psi_1'(t), \qquad t \in [0, 1].$$

Thus $\psi_1' \equiv 0$. In particular, $z_1 = z_2$.

Consequently, $A \setminus A' \subset \bigcup_{j=1}^{\infty} \operatorname{pr}_z(S_j)$ is at most countable.

STEP 5. Let B' be constructed analogously to A' with respect to the second variable. Put $X' := \mathbb{X}(D, A'; G, B')$.

By Step 2 (and Lemma 6), for any $k \in \mathbb{N}$ and any $\xi = (a, b) \in (A' \cap D_k) \times (B' \cap G_k)$ there exists $\varrho = \varrho_{\xi,k} > 0$ such that for each $f \in \mathcal{O}_{\mathrm{s}}(X \setminus M)$ there exists $\widetilde{f} = \widetilde{f}_{\xi,k} \in \mathcal{O}(\Omega_{\xi,k} \setminus M)$ with $\widetilde{f} = f$ on $X \cap \Omega_{\xi,k} \setminus M$, where

$$\begin{aligned} \Omega_{\xi,k} &:= \mathbb{X}(D_k, \Delta_a(\varrho); G_k, \Delta_b(\varrho)) \\ &= (\Delta_a(\varrho) \times G_k) \cup (D_k \times \Delta_b(\varrho)) \subset U \cap (D_k \times G_k). \end{aligned}$$

We may always assume that $\varrho_{\xi,k+1} \le \varrho_{\xi,k}$. By Lemma 6, $\widetilde{f}_{\xi,k+1} = \widetilde{f}_{\xi,k}$ on $\Omega_{\xi,k+1} \cap \Omega_{\xi,k} \setminus M$. Define

$$\Omega := \bigcup_{k=1}^{\infty} \bigcup_{\xi \in (A' \cap D_k) \times (B' \cap G_k)} \Omega_{\xi,k}.$$

It is clear that Ω is a connected neighborhood of X'. We will show that the functions $\widetilde{f}_{\xi,k}$, $\xi \in (A' \cap D_k) \times (B' \cap G_k)$, $k \in \mathbb{N}$, can be glued together. We only need to check that $\widetilde{f}_{\xi,k} = \widetilde{f}_{\eta,k}$ on $\Omega_{\xi,k} \cap \Omega_{\eta,k} \setminus M$, $\xi = (a, b)$, $\eta = (c, d)$. Let $\varrho' := \varrho_{\xi,k}$, $\varrho'' := \varrho_{\eta,k}$, $f' := \widetilde{f}_{\xi,k}$, $f'' := \widetilde{f}_{\eta,k}$. Observe that

$$\begin{aligned} \Omega_{\xi,k} \cap \Omega_{\eta,k} &= (\Delta_a(\varrho') \times \Delta_d(\varrho'')) \cup (\Delta_c(\varrho'') \times \Delta_b(\varrho')) \\ &\quad \cup ((\Delta_a(\varrho') \cap \Delta_c(\varrho'')) \times G_k) \cup (D_k \times (\Delta_b(\varrho') \cap \Delta_d(\varrho''))) \\ &=: W_1 \cup W_2 \cup W_3 \cup W_4. \end{aligned}$$

To prove that $f' = f''$ on $W_1 \setminus M$ it suffices to observe that $f' = f''$ on $(A' \cap \Delta_a(\varrho')) \times (B' \cap \Delta_d(\varrho'')) \setminus M$ (and use Lemma 6). The same argument solves the problem on $W_2 \setminus M$.

If $W_3 \neq \emptyset$, then the equality holds on a non-empty set $W_3 \cap W_1 \setminus M$ and we only need to use the identity principle. The same argument works on $W_4 \setminus M$.

STEP 6. Recall that the sets A', B' are locally regular and A' is a Borel set. Moreover, $h^*_{A',D} = h^*_{A,D}$ and $h^*_{B',G} = h^*_{B,G}$. Hence $\widehat{X}' = \widehat{X}$.

First we prove that $\widehat{X}$ is the envelope of holomorphy of Ω. We only need to show that any function $g \in \mathcal{O}(\Omega)$ extends holomorphically to $\widehat{X}$. Fix a $g \in \mathcal{O}(\Omega)$. By Theorem 4 (applied to the cross X'), there exists a $\widehat{g} \in \mathcal{O}(\widehat{X})$ (recall that $\widehat{X} = \widehat{X}'$) such that $\widehat{g} = g$ on X'. By Lemma 6, $\widehat{g} = g$ on Ω.

By Theorem 5 there exists a pure 1-dimensional analytic subset $\widehat{M}$ of $\widehat{X}$ such that for any $g \in \mathcal{O}(\Omega \setminus M)$ there exists a $\widehat{g} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ with $\widehat{g} = g$ on $\Omega \setminus (M \cup \widehat{M})$. We also know that if $U = \widehat{X}$ and M is pure 1-dimensional, then we can take $\widehat{M} = M$.

Now take an $f \in \mathcal{O}_{\mathrm{s}}(X \setminus M)$ and let $\widetilde{f} \in \mathcal{O}(\Omega \setminus M)$ be such that $\widetilde{f} = f$ on $X' \setminus M$ (Step 5). Let $\widehat{f} \in \mathcal{O}(\widehat{X} \setminus \widehat{M})$ be such that $\widehat{f} = \widetilde{f}$ in $\Omega \setminus (M \cup \widehat{M})$. In particular, $\widehat{f} = f$ on $X' \setminus (M \cup \widehat{M})$. By Lemma 6, $\widehat{f} = f$ on $X \setminus (M \cup \widehat{M})$.

Using Lemma 6 once again, we conclude that the function $\widehat{f}$ is uniquely determined.

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Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: jarnicki@im.uj.edu.pl

Fachbereich Mathematik
Carl von Ossietzky Universität Oldenburg
Postfach 2503
D-26111 Oldenburg, Germany
E-mail: pflug@mathematik.uni-oldenburg.de

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