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Separately superharmonic functions in product networks

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Abstract. Let $X \times Y$ be the Cartesian product of two locally finite, connected networks that need not have reversible conductance. If X, Y represent random walks, it is known that if $X \times Y$ is recurrent, then X, Y are both recurrent. This fact is proved here by non-probabilistic methods, by using the properties of separately superharmonic functions. For this class of functions on the product network $X \times Y$, the Dirichlet solution, balayage, minimum principle etc. are obtained. A unique integral representation is given for any function that belongs to a restricted subclass of positive separately superharmonic functions in $X \times Y$.

1. Introduction. Simple random walks in both $\mathbb{R}$ and $\mathbb{R}^2$ are recurrent. Considering $\mathbb{R}^2$ as the Cartesian product $\mathbb{R} \times \mathbb{R}$, there is a more general result: if X, Y are two infinite (not necessarily reversible) random walks and if their Cartesian product $X \times Y$ is recurrent, then both X, Y must be recurrent. In this note, we consider this result in a non-probabilistic cadre by using potential-theoretic methods: Let $\{X, t(x,y)\}$ be an infinite network that is connected and locally finite, where the conductance $t(x,y)$ need not be reversible, that is, there may not be any function $\varphi(x) > 0$ on X such that $\varphi(x)t(x,y) = \varphi(y)t(y,x)$ for all pairs of vertices x, y in X. Suppose Y is another such infinite network. We show that if there is no positive potential in $X \times Y$, then neither X nor Y has any positive potential.

A function $s(x,y)$ in $X \times Y$ is said to be *separately superharmonic* if $s(x,y)$ is superharmonic in each variable when the other is fixed. It is more interesting to develop the theory of separately superharmonic functions in the discrete space $X \times Y$ than to study potentials in product spaces. Thus, we take a closer look at separately superharmonic functions in $X \times Y$, proving a minimum principle for such functions and discussing the separately harmonic Dirichlet solution on subsets of the product network. We introduce the notion of doubly supermedian and doubly median functions in $X \times Y$ which

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play an important role in the integral representation of positive separately superharmonic functions in the product network.

In the context of integral representations of separately superharmonic functions in the continuous case, we have the following important developments: In 1965, Gowrisankaran [12] showed that there is a unique integral representation for any positive separately harmonic function in $\Omega\times\Omega'$ where Ω, Ω' are harmonic spaces with countable bases satisfying the Brelot axioms 1–3 [5], by using the Choquet theorem on integral representation [8]. In 1968, Cairoli [7] gave a unique integral representation for a subfamily of separately excessive functions. In 1972, Drinkwater [9] obtained an integral representation for positive separately superharmonic functions in $\Omega \times \Omega'$, where as before Ω, Ω' are Brelot harmonic spaces and in addition each has a countable base of completely determining regular domains, but without establishing whether such a representation is unique. In 1973, Gowrisankaran [13] showed that such an integral representation is unique for a subclass of positive separately superharmonic functions in $\Omega \times \Omega'$ (similar to Cairoli's subclass).

Here we introduce the family of doubly supermedian functions in a product network $X \times Y$ as a discrete analogue of the subclasses considered by Cairoli and Gowrisankaran, and prove that every positive doubly supermedian function in $X \times Y$ has a unique integral representation; details about the extremal elements and the unique representing measure are also given.

2. Preliminaries. Let X be an infinite network that is *locally finite* (that is, every vertex has only a finite number of neighbours), *connected* (any two vertices can be joined by a path), and without self-loops. If the vertices x and z are neighbours, we write $x \sim z$. If E is a subset of X, a vertex x is said to be an *interior vertex* of E if x and all its neighbours are in E. Denote the set of all the interior vertices of E by $\mathring{E}$. Write $\partial E = E \setminus \mathring{E}$ to denote the boundary of E. Let $t : X\times X \to [0,\infty)$ be a conductance function on X; it has the following properties: $t(x,z) \geq 0$ for any pair of vertices x and z; $t(x,z) > 0$ if and only if $x \sim z$; consequently, $t(x) = \sum_{x\sim z} t(x,z) > 0$ for every x in X. Importantly, we do not assume that $t(x,z)$ is reversible. (Recall that t being reversible means that there exists a function $\varphi(x) > 0$ on X such that $\varphi(x)t(x,z) = \varphi(z)t(z,x)$ for every pair of vertices x and z. When $\varphi(x) = 1$ for all x, we say that the conductance is *symmetric*.) Since the conductance may not be reversible, we do not have access to the Green formula, inner product, Dirichlet norm etc. in X.

Let $\{X, t_1\}$ and $\{Y, t_2\}$ be two such infinite networks. Define their Cartesian product network as $\{X \times Y, t\}$ such that the neighbours of (x, y) are (x_i, y) and (x, y_j) where $x \sim x_i$ in X and $y \sim y_j$ in Y; and $t\{(x, y), (x_i, y)\} = t_1(x, x_i)$ and $t\{(x, y), (x, y_j)\} = t_2(y, y_j)$. Note that if the conductance were symmetric or reversible, we could introduce the notion of Dirichlet norm and use functional analysis techniques to prove theorems in the case of product networks as well. For example, Lyons and Peres [18, Chapter 9] observe that if X, Y are infinite locally finite networks with unit conductance, then there are no non-constant Dirichlet finite harmonic functions in $X \times Y$. However in the absence of reversible conductance, we proceed as follows:

Let $u(x)$ be a real-valued function on a subset E of $\{X, t_1\}$. If $x \in \mathring{E}$, define

$$\Delta_1 u(x) = \sum_{z \sim x} t_1(x, z)[u(z) - u(x)].$$

The function u is said to be Δ_1-*superharmonic* at the vertex x if $\Delta_1 u(x) \le 0$, and Δ_1-*harmonic* at x if $\Delta_1 u(x) = 0$; u is said to be Δ_1-*superharmonic on* E if it is superharmonic at each vertex in $\mathring{E}$. The network $\{X, t_1\}$ is said to be *hyperbolic* (or *transient*) if there exists a positive Δ_1-superharmonic function on X that is not Δ_1-harmonic at least at one vertex in X; otherwise $\{X, t_1\}$ is said to be *parabolic* (or *recurrent*). In other words, $\{X, t_1\}$ is parabolic if and only if every positive Δ_1-superharmonic function on X is constant. Similar definitions are given for the network $\{X, t_2\}$.

Let $f(x, y)$ be a function defined on a subset ω in $X \times Y$. Write $\omega_y = \{x \in X : (x, y) \in \omega\}$ for a fixed $y \in Y$, if this subset of X is non-empty; similarly, $\omega^x = \{y \in Y : (x, y) \in \omega\}$ if this subset of Y is non-empty. When $f(x, y)$ is considered as a function of $x \in \omega_y$, it is denoted by $f_y(x)$. Similarly, for a fixed $x \in X$, $f^x(y) = f(x, y)$ is a function of $y \in \omega^x$. Write

$$\Delta f(x, y) = \sum_{(x,y) \sim (a,b)} t\{(x, y), (a, b)\}[f(a, b) - f(x, y)].$$

Then

$$\begin{aligned}
\Delta f(x, y) &= \sum_{(x,y) \sim (x_i, y)} t\{(x, y), (x_i, y)\}[f(x_i, y) - f(x, y)] \\
&\quad + \sum_{(x,y) \sim (x, y_j)} t\{(x, y), (x, y_j)\}[f(x, y_j) - f(x, y)] \\
&= \sum_{x \sim x_i} t_1(x, x_i)[f_y(x_i) - f_y(x)] + \sum_{y \sim y_j} t_2(y, y_j)[f^x(y_j) - f^x(y)] \\
&= \Delta_1 f_y(x) + \Delta_2 f^x(y).
\end{aligned}$$

3. Separately superharmonic functions in networks. Using the Laplace operator Δ in the product network $(X \times Y, t)$, we develop the Δ-potential theory in the usual manner, starting with the definition that a function $f(x, y)$ defined on a subset ω of $X \times Y$ is said to be *superharmonic* in ω if $\Delta f(x, y) \le 0$ at every vertex $(x, y) \in \mathring{\omega}$. In this section, we develop another restricted potential theory in $X \times Y$.

Definition 3.1. A real-valued function $f(x, y)$ defined on a subset ω in $X \times Y$ is said to be *separately superharmonic* on ω if $f_y(x)$ is superharmonic on ω_y for every $y \in Y$ and $f^x(y)$ is superharmonic on ω^x for $x \in X$, whenever ω_y and ω^x are non-empty subsets. Similar definitions hold for separately harmonic functions and separately subharmonic functions.

Thus, a function $f(x, y)$ is separately superharmonic at a vertex (a, b) if $f_y(x)$ is Δ_1-superharmonic at $x = a$ and $f^x(y)$ is Δ_2-superharmonic at $y = b$.

Proposition 3.2. *A separately superharmonic* (*respectively separately harmonic*) *function in* ω *is* Δ*-superharmonic* (*respectively* Δ*-harmonic*) *in* ω.

Proof. Note that if $(x, y) \in \mathring{\omega}$ then x is in the interior of ω_y and y is in the interior of ω^x, and conversely. Hence by using the definition of separately superharmonic functions, we see that $\Delta f(x, y) = \Delta_1 f_y(x) + \Delta_2 f^x(y) \leq 0$. Similarly for separately harmonic functions. ∎

Separately subharmonic functions in $\mathbb{R}^n$. A function f defined on a domain in $\mathbb{C}^n$ that is holomorphic in each variable separately while the others remain fixed is in fact a holomorphic function of several variables (Hartogs). However a separately subharmonic function $u(x, y)$ need not be subharmonic (Wiegerinck [24]). With some additional condition on a separately subharmonic function $u(x, y)$ one can prove that $u(x, y)$ is subharmonic: For $u(x, y)$ locally upper bounded, this was proved by Avanissian [4], see also Lelong [16]; for $u(x, y)$ with L^1-majorant by Arsove [3]; for functions majorised in $L^q (q > 0)$ by Riihentaus [21]; for functions with a local integrability condition by Armitage and Gardiner [2]. An interesting result in this context is that separately harmonic functions are harmonic (Lelong [17]).

Remark. A Δ-superharmonic function need not be separately superharmonic. For example, consider two infinite trees X, Y without terminal vertices. Then, for any $f > 0$ on X and any $g > 0$ on Y, there exist a Δ_1-subharmonic function $u(x)$ on X and a Δ_2-subharmonic function $v(y)$ on Y such that $\Delta_1 u(x) = f(x)$ and $\Delta_2 v(y) = g(y)$ (see [1, p. 114]). Now assuming $f = 1$, $g = 1$, define $s(x, y) = u(x) - 2v(y)$. Then $\Delta s(x, y) = \Delta_1 u(x) - 2\Delta_2 v(y) = 1 - 2 < 0$. Hence, $s(x, y)$ is Δ-superharmonic on $X \times Y$. But, for any fixed $y \in Y$, $\Delta_1 s_y(x) = 1$ so that $s(x, y)$ is not separately superharmonic on $X \times Y$. Similarly, a harmonic function need not be separately harmonic. However, if a harmonic function is separately superharmonic, then it is separately harmonic. A sufficient condition for a superharmonic function to be separately superharmonic is given in the next proposition.

Let $s(x)$ be a real-valued function in $\{X, t_1\}$. For $\rho > 0$, define $t_1^\rho(x, z) = \rho t_1(x, z)$. Write

$$\Delta_1^\rho s(x) = \sum_z t_1^\rho(x,z)[s(z)-s(x)] \quad \text{and} \quad \Delta_1 s(x) = \sum_z t_1(x,z)[s(z)-s(x)].$$

Then $\Delta_1^\rho s(x) = \rho\Delta_1 s(x)$, so that $s(x)$ is superharmonic in $\{X, t_1\}$ if and only if it is superharmonic in $\{X, t_1^\rho\}$.

PROPOSITION 3.3. *Let $\{X, t_1\}, \{Y, t_2\}$ be two infinite networks. Then a real-valued function $u(x,y)$ is separately superhamonic in the product network $\{X, t_1\} \times \{Y, t_2\}$ if and only if $u(x,y)$ is superharmonic in $\{X, t_1^\rho\} \times \{Y, t_2^\sigma\}$ for any $\rho, \sigma > 0$.*

Proof. Suppose $u(x,y)$ is separately superharmonic on $\{X, t_1\} \times \{Y, t_2\}$. Then, for any $\rho, \sigma > 0$, we have $\Delta^{\rho,\sigma} u(x,y) = \rho\Delta_1 u_y(x) + \sigma\Delta_2 u^x(y)$. Note that since $u(x,y)$ is separately superharmonic, it follows that $\Delta_1 u_y(x) \le 0$ and $\Delta_2 u^x(y) \le 0$. Consequently, $u(x,y)$ is superharmonic in $\{X, t_1^\rho\} \times \{Y, t_2^\sigma\}$ for any $\rho, \sigma > 0$.

Conversely, suppose $u(x,y)$ is superharmonic in $\{X, t_1^\rho\} \times \{Y, t_2^\sigma\}$ for any $\rho, \sigma > 0$. Then $\rho\Delta_1 u_y(x) + \sigma\Delta_2 u^x(y) = \Delta^{\rho,\sigma} u(x,y) \le 0$, so if $\Delta_1 u_y(x) > 0$, then by taking σ sufficiently small, we could make $\Delta^{\rho,\sigma} u(x,y) > 0$; this contradiction shows that $\Delta_1 u_y(x) \le 0$. Similarly $\Delta_2 u^x(y) \le 0$. That is, $u(x,y)$ is separately superharmonic in $\{X, t_1\} \times \{Y, t_2\}$. ∎

Kołodziej and Thorbiörnson [15, p. 465] proved that if $v : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, subharmonic in the first variable and harmonic in the other, is a distribution then $v(x,y) = h(x,y) + f(x)$ where $h(x,y)$ is harmonic and $f(x)$ is independent of y and subharmonic in $\mathbb{R}^m$. The following proposition is an analogue of this result in the framework of networks.

PROPOSITION 3.4. *Let X be an infinite tree without terminal vertices and Y be an infinite network in which any positive harmonic function is constant. Then any function $u(x,y)$ in $X \times Y$ that is superharmonic in x for fixed $y \in Y$, and harmonic in y for fixed $x \in X$, is of the form $u(x,y) = h(x,y) + f(x)$, where $h(x,y)$ is separately harmonic (hence Δ-harmonic) in $X \times Y$ and $f(x)$ is Δ_1-superharmonic in X. This decomposition is unique up to an additive harmonic function in X.*

Proof. We have

$$\begin{aligned}\Delta_2\Delta_1 u(x,y) &= \Delta_2\Big\{\sum_z t_1(x,z)[u(z,y)-u(x,y)]\Big\}\\ &= \sum_z t_1(x,z)[\Delta_2 u(z,y) - \Delta_2 u(x,y)] = 0.\end{aligned}$$

That is, $\Delta_1 u(x,y) = \varphi(x,y)$ is a Δ_2-harmonic function in Y for each fixed $x \in X$. Note that for any fixed $y \in Y$, $\Delta_1 u(x,y) \le 0$. Hence $\varphi(x,y)$ is a non-positive harmonic function in Y which by the assumption on Y should be a constant; thus the value of $\varphi(x,y)$ depends only on x. Write $\varphi(x,y) = \vartheta(x)$.

Since X is assumed to be an infinite tree without terminal vertices, there exists [1, p. 114] a function $f(x)$ on X such that $\Delta_1 f(x) = \vartheta(x)$ in X. Since $\vartheta(x) \leq 0$, $f(x)$ is superharmonic in X. Thus, $\Delta_1 u(x,y) = \Delta_1 f(x)$ in X. Write $h(x,y) = u(x,y) - f(x)$. Then, for any fixed $y \in Y$, $\Delta_1 h(x,y) = 0$; and for any fixed $x \in X$, $\Delta_2 h(x,y) = \Delta_2 u(x,y) - 0 = 0$, by the assumption on $u(x,y)$. Thus, $h(x,y)$ is separately harmonic (and hence Δ-harmonic) in $X \times Y$.

Suppose $u(x,y) = h_1(x,y) + f_1(x)$ is another such decomposition. Then $\Delta_1[h(x,y) - h_1(x,y)] = \Delta_1[f(x) - f_1(x)]$, and for any $y \in Y$, the left side of this equality is 0. Hence $f(x) - f_1(x)$ is a harmonic function $v(x)$ in X; and $h(x,y) = h_1(x,y) - v(x)$. ∎

Corollary 3.5. *In Proposition* 3.4*, if* $u(x,y) \geq 0$*, then* $u(x,y) = H(x,y) + p(x)$ *where* $H(x,y)$ *is non-negative and is separately harmonic in* $X \times Y$ *and* $p(x)$ *is a potential in* X*, and the decomposition is unique.*

Proof. Write as in Proposition 3.4, $u(x,y) = h(x,y) + f(x)$. For fixed y, $h_y(x)$ is harmonic in X and $f(x) \geq -h_y(x)$. Since the superharmonic function $f(x)$ has a harmonic minorant, we have $f(x) = p(x) + v(x)$ where $p(x)$ is a potential and $v(x)$ is harmonic in X. Write $H(x,y) = h(x,y) + v(x)$, so $u(x,y) = H(x,y) + p(x)$. Since $p(x) \geq -H_y(x)$, it follows that $-H_y(x) \leq 0$ for any fixed $y \in Y$, and hence $H(x,y) \geq 0$. If $u(x,y) = H_1(x,y) + p_1(x)$ is another such decomposition, then $p(x) - p_1(x)$ is harmonic as shown above, which implies that $p(x) = p_1(x)$, proving the uniqueness of the decomposition. ∎

Some properties of separately superharmonic functions

1. *If* u, v *are separately superharmonic on a subset* ω *of* $X \times Y$*, then for any non-negative numbers* α, β *the function* $\alpha u + \beta v$ *is also separately superharmonic. So is* $s = \inf(u,v)$*; note that* $s_y(x) = \inf(u_y(x), v_y(x))$*.*

2. *Let* $u_n(x,y)$ *be a sequence of separately superharmonic (respectively separately harmonic) functions in* ω *such that* $\lim_{n\to\infty} u_n(x,y) = u(x,y)$ *is finite for each* $(x,y) \in \omega$*. Then* $u(x,y)$ *is separately superharmonic (respectively separately harmonic) in* ω*.*

Proof. For any fixed $x \in X$, $u_n^x(y)$ is superharmonic in ω^x. We know that the limit of superharmonic functions is superharmonic if the limit is finite [1, p. 46]. Hence, the fact that $u_n^x(y) \to u^x(y)$ in ω^x, for every fixed $x \in X$, implies that $u^x(y)$ is superharmonic in ω^x. Similarly, $u_y(x)$ is superharmonic in ω_y. Hence, $u(x,y)$ is separately superharmonic in ω. Similar proof for separately harmonic functions. ∎

3. *Let* $u(x,y)$ *be real-valued and* $v(x,y)$ *be separately subharmonic in a subset* ω *of* $X \times Y$ *such that* $v(x,y) \leq u(x,y)$*. Let* $\mathcal{F}$ *be the family of all*

separately subharmonic functions $s(x,y)$ *on* ω *such that* $s(x,y) \le u(x,y)$. *Then* $\sup\{s(x,y) : s \in \mathcal{F}\}$ *is a separately subharmonic function in* ω.

Proof. Let $\mathcal{F}$ be the family of all separately subharmonic functions $s(x,y)$ in ω such that $s(x,y) \le u(x,y)$ in ω. Since $\mathcal{F}$ is increasingly directed and ω has only a countable number of vertices, we can extract an increasing sequence $s_n(x,y)$ from $\mathcal{F}$ such that $\lim_{n\to\infty} s_n(x,y) = \sup\{s(x,y) : s \in \mathcal{F}\}$. Then, as in property 2, this limit is separately subharmonic in ω. ∎

Consequence: If $\{h_i(x,y)\}$ is an increasingly directed family of separately harmonic functions and $h(x,y) = \sup h_i(x,y)$ is real-valued, then $h(x,y)$ is separately harmonic.

4. *If* $u \ge 0$ *is* Δ_1*-superharmonic in* X, *and* $v \ge 0$ *is* Δ_2*-superharmonic in* Y, *then* $s(x,y) = u(x)v(y)$ *is a separately superharmonic function in* $X \times Y$.

Consequence: If at least one of X, Y is a hyperbolic (transient) network, then $X \times Y$ is Δ-hyperbolic. Indeed, suppose X is hyperbolic. Then there exists a positive Δ_1-superharmonic function $u(x)$ on X that is not Δ_1-harmonic on X. Let $v(y) > 0$ be a Δ_2-superharmonic function in Y. We find that $s(x,y) = u(x)v(y) > 0$ and

$$\Delta s(x,y) = v(y)\Delta_1 u(x) + u(x)\Delta_2 v(y) \le 0.$$

Since $\Delta_1 u(x) < 0$ at least at one vertex by hypothesis, we conclude that $s(x,y)$ is a Δ-superharmonic function that is not Δ-harmonic. Hence, $X \times Y$ is hyperbolic.

Thus, $X \times Y$ being parabolic (recurrent) implies that X and Y are both parabolic networks. The converse is not true: Let Z_n be the lattice network in $\mathbb{R}^n$, $n \ge 1$, with unit conductance between neighbours. Then, Z_1, Z_2 are parabolic while $Z_1 \times Z_2$ is hyperbolic. Indeed, the vertices of Z_n are the integer points in $\mathbb{R}^n$. In Z_1, if a is a vertex then its neighbours are $a \pm 1$ and $t(a, a+1) = t(a, a-1) = 1$. In Z_2, if (a,b) is a vertex then its neighbours are $(a, b\pm 1)$ and $(a\pm 1, b)$ with conductance 1. It is known (see for example Woess [25]) that for the Pólya walk, Z_1, Z_2 are recurrent (parabolic) while $Z_3 = Z_1 \times Z_2$ is transient (hyperbolic).

5 (Minimum Principle for separately superharmonic functions). *Let* A, B *be finite sets in* X, Y *respectively. Suppose* $s(x,y)$ *is a separately superharmonic function on* $A \times B$ *such that* $s(x,y) \ge 0$ *for* $(x,y) \in \partial A \times \partial B$. *Then* $s(x,y) \ge 0$ *for* $(x,y) \in A \times B$.

Proof. For $y \in \partial B$ fixed, $s_y(x)$ is superharmonic on A and $s_y(x) \ge 0$ for $x \in \partial A$. This implies that $s_y(x) \ge 0$ for $x \in A$. Thus, $s(x,y) \ge 0$ for $x \in A$ and $y \in \partial B$. Now for $x \in A$, $s(x,y) = s^x(y) \ge 0$ on ∂B, so that $s^x(y) \ge 0$ for $y \in B$. Thus, for $x \in A$ and $y \in B$, $s(x,y) = s^x(y) \ge 0$. ∎

6. *Let $u_1(x), u_2(y)$ be positive superharmonic functions in X, Y respectively with greatest harmonic minorants $h_1(x), h_2(y)$. Then any non-negative separately harmonic minorant of the Δ-superharmonic function $s(x,y) = u_1(x)u_2(y)$ is smaller than $h_1(x)h_2(y)$. Consequently, $h_1(x)h_2(y)$ is the greatest separately harmonic minorant of $u_1(x)u_2(y)$.*

Proof. Let $h(x,y)$ be a non-negative separately harmonic minorant of $s(x,y)$. Then, for fixed $y \in Y$, $h_y(x) \le s_y(x)$ implies that the Δ_1-harmonic function $h_y(x)/u_2(y) \le u_1(x)$, so that $h_y(x)/u_2(y) \le h_1(x)$. If $h_1(x) = 0$, then there is nothing to prove, hence assume $h_1 > 0$. Thus, for any $y \in Y$, $h(x,y)/h_1(x) \le u_2(y)$, which shows that $h(x,y)/h_1(x) \le h_2(y)$. We therefore conclude that $h(x,y) \le h_1(x)h_2(y)$. Since $h_1(x)h_2(y)$ is already a separately harmonic minorant of $s(x,y)$, we conclude that $h_1(x)h_2(y)$ is the greatest separately harmonic minorant of $u_1(x)u_2(y)$. ■

Definition 3.6. A separately superharmonic function $s(x,y) \ge 0$ in $X \times Y$ is called a *product potential* if whenever $u(x,y)$ is a separately subharmonic function such that $u(x,y) \le s(x,y)$, then $u(x,y) \le 0$. That is, the only non-negative separately subharmonic minorant of $s(x,y)$ is zero.

Example 3.7.

(1) Let $p(x)$ be a potential in X. Suppose $q(y) \ge 0$ is superharmonic (maybe constant) in Y. Then $p(x)q(y)$ is a product potential in $X \times Y$.
(2) More generally, suppose $s(x,y) \ge 0$ is a separately superharmonic function such that for any y fixed in Y, $s_y(x)$ is a potential in X; then $s(x,y)$ is a product potential.
(3) In particular a *separately potential* function $p(x,y)$, that is, $p(x,y)$ is a potential in each variable when the other is fixed, is a product potential. An example of a product potential that is not a separately potential function: The function $p(x)h(y)$ where $p(x)$ is a positive potential in X and $h(y)$ is a positive harmonic function in Y.
(4) If $p_1(x,y), p_2(x,y)$ are two product potentials and if a, b are non-negative constants, then $ap_1(x,y) + bp_2(x,y)$ is also a product potential.
(5) If $p(x,y) = \sum_{n=1}^{\infty} p_n(x,y)$ is the sum of a convergent series of product potentials, then $p(x,y)$ is a product potential.

7. *If there is a positive product potential in $X \times Y$, then there are Δ-potentials in $X \times Y$.*

Proof. If there is a positive product potential in $X \times Y$, then at least one of these networks should be hyperbolic. Indeed, suppose both are parabolic. Let $s(x,y) > 0$ be a separately superharmonic function in $X \times Y$. Then, for fixed $x \in X$, $s^x(y) > 0$ is superharmonic in Y. Since Y is parabolic, $s^x(y) = C(x)$, a constant depending on x. Now, for a fixed y in Y, $C(x) =$

$s^x(y) = s(x, y) = s^y(x)$ is a positive superharmonic function in the parabolic network X and hence a constant. Thus, $C(x)$ is an absolute constant, not depending on x or y. Hence, every positive separately superharmonic function in $X \times Y$ is constant, so that there cannot be any positive product potential in $X \times Y$, contradicting the assumption. Since at least one of the networks is hyperbolic, by the consequence of property 4 above, it follows that $X \times Y$ is hyperbolic. ∎

8. *If there are non-constant positive separately superharmonic functions in $X \times Y$, then there are product potentials in $X \times Y$.*

Proof. As above, the assumption implies that at least one of the networks, say X, is hyperbolic. Let $p(x)$ be a potential in X. Let $q(y)$ be positive superharmonic (maybe constant) in Y. Then, as in Example 3.7(1), $p(x)q(y)$ is a product potential in $X \times Y$. ∎

NOTE. The proofs of properties 7 and 8 show in particular that: Product potentials exist in $X \times Y$ if and only if at least one of X, Y is hyperbolic, whereas Δ-potentials may exist in $X \times Y$ even if both X, Y are parabolic. In fact, if X, Y are parabolic, then any non-negative separately superharmonic function in $X \times Y$ is constant, hence there is no product potential in $X \times Y$. Thus in the lattice network $Z_3 = Z_1 \times Z_2$ there is no product potential, but for the existence and the asymptotic properties of the Green potential in Z_3, see Duffin [10].

In a product network with potentials, there may be no non-constant positive separately harmonic functions.

For example, take Y a hyperbolic network in which every positive harmonic function is constant (an example of such a hyperbolic network is the lattice network Z_3, see [1, p. 10, Example 5]) and take a parabolic network X. Then $X \times Y$ is hyperbolic, but any positive separately harmonic function in $X \times Y$ is constant. For suppose $h(x, y) > 0$ is separately harmonic in $X \times Y$. Then $h_y(x) > 0$ is harmonic in X, for fixed $y \in Y$, hence a constant $C(y)$ since X is parabolic. Now for fixed $x \in X$, $h^x(y) = C(y)$ is a positive harmonic function in Y, hence a constant independent of y, by the assumption. That is, $h(x, y)$ is a constant.

10. *Let X be parabolic and Y be hyperbolic. Then any separately superharmonic function $s(x, y) \geq 0$ is of the form $s(x, y) = v(y)$ where $v(y)$ is superharmonic in Y, and $s(x, y)$ is a product potential if and only if $v(y)$ is a potential in Y.*

Proof. For fixed $y \in Y$, $s_y(x) \geq 0$ is superharmonic in X, hence a constant $v(y)$. Write $s(x, y) = s_y(x) = v(y)$. Since $s^x(y) = s(x, y)$ is superharmonic in Y, we find that $v(y)$ is superharmonic in Y.

Suppose $u(y)$ is the greatest Δ_2-harmonic minorant of $v(y)$ in Y. Then $h(x,y) = u(y)$ is a separately harmonic function such that $0 \le h(x,y) \le s(x,y)$. Hence, if $s(x,y)$ is a product potential, then $h(x,y) = 0$. That implies that $v(y)$ is a potential in Y. Conversely, suppose that $v(y)$ is a potential in Y. Then to show that $s(x,y)$ is a product potential, take a separately subharmonic function $u(x,y)$ such that $u(x,y) \le s(x,y)$. Then the subharmonic function u^x satisfies $u^x(y) \le s^x(y) = v(y)$ in Y. Since $v(y)$ is a potential by assumption, $u^x(y) \le 0$ for every $y \in Y$. Hence $u(x,y) \le 0$, which implies that $s(x,y)$ is a product potential in $X \times Y$. ■

11. *Let $u(x,y)$ be separately superharmonic and $v(x,y)$ be separately subharmonic such that $u(x,y) \ge v(x,y)$ in $X \times Y$. Then $u(x,y) = p(x,y) + q(x,y)$, where $p(x,y)$ is a product potential and $q(x,y)$ is the greatest separately subharmonic minorant of $u(x,y)$.*

Proof. Let $\mathcal{F}$ be the family of separately subharmonic functions $s(x,y)$ on $X \times Y$ such that $s(x,y) \le u(x,y)$. Write $q(x,y) = \sup s(x,y)$, $s \in \mathcal{F}$. Then (property 3) $q(x,y)$ is separately subharmonic. Write $p(x,y) = u(x,y) - q(x,y)$. Then $p(x,y)$ is a non-negative separately superharmonic function that is actually a product potential, hence the stated decomposition. ■

Balayage. In a hyperbolic network X, let $f \ge 0$ be a real-valued function on X. Suppose there exists a Δ_1-superharmonic function $s \ge 0$ on X such that $s \ge f$ on X. Let $\mathcal{F}$ be the family of all Δ_1-superharmonic functions $u \ge 0$ on X such that $u \ge f$ on X. Then $R_f(x) = \inf\{u(x) : u \in \mathcal{F}\}$ is Δ_1-superharmonic on X (since the infimum of a lower directed family of non-negative superharmonic functions is superharmonic) and Δ_1-harmonic at every vertex in X where f is Δ_1-subharmonic. Indeed, if $f(x)$ is Δ_1-subharmonic at $x = a$, then $t_1(a)f(a) \le \sum_y t_1(a,y)f(y)$. Now if u is a Δ_1-superharmonic function with $u \ge f$, define $u_a(x) = u(x)$ if $x \ne a$ and $u_a(a) = \sum_y \frac{t_1(a,y)}{t_1(a)} u(y)$. Then $u_a(x) \ge f(x)$ in X, $u_a(x) \le u(x)$ in X, $u_a(x)$ is Δ_1-superharmonic in X, and $u_a(x)$ is Δ_1-harmonic at $x = a$. Thus $u_a \in \mathcal{F}$ and $R_f(x) = \inf\{u_a(x) : u \in \mathcal{F}\}$ so that $R_f(x)$ is Δ_1-harmonic at $x = a$.

If A is a subset of X, we denote by $R_f^A(x)$ the function $R_{f\chi_A}(x)$, and call it the *balayage* of f on A. Note that $R_f^A(x)$ is Δ_1-superharmonic on X, $R_f^A(x) \ge f(x)$ if $x \in A$, and $R_f^A(x)$ is Δ_1-harmonic at every vertex in $X \setminus A$, since $f\chi_A(x)$ is Δ_1-subharmonic at any vertex z where $f\chi_A(z) = 0$. We remark also that if a Δ_1-potential is in $\mathcal{F}$, then $R_f(x)$ is a Δ_1-potential in X; consequently, if A is a finite subset in the hyperbolic network X, then $R_f^A(x)$ is a Δ_1-potential in X.

In particular, if $s \ge 0$ is Δ_1-superharmonic on X, then $R_s^A(x)$ is Δ_1-superharmonic on X, $R_s^A(x)$ is Δ_1-harmonic at every vertex in $X \setminus A$, $R_s^A(x) \le s(x)$ for each $x \in X$, and $R_s^A(x) = s(x)$ for each $x \in A$. See

[1, Theorem 3.1.10] for another presentation of balayage via the Dirichlet solution.

Definition 3.8. Let $f(x,y) \geq 0$ be a real-valued function on $X \times Y$. Suppose there exists a separately superharmonic function $s(x,y) \geq 0$ on $X \times Y$ such that $s(x,y) \geq f(x,y)$ on a subset E of $X \times Y$. Let $\mathcal{F}$ be the family of all separately superharmonic functions $u(x,y) \geq 0$ on $X \times Y$ such that $u(x,y) \geq f(x,y)$ on E. Then $R_f^E(x,y) = \inf\{u(x,y) : u \in \mathcal{F}\}$ is called the *balayage* of f on E. When $E = X \times Y$, it is simply written $R_f(x,y)$ instead of $R_f^{X\times Y}(x,y)$.

A part of the following theorem is a discrete version of Theorem 4.1 proved by Gowrisankaran [14] in the framework of Brelot axiomatic potential theory.

Theorem 3.9. *Let $u(x) \geq 0$ be Δ_1-superharmonic on X and $v(y) \geq 0$ be Δ_2-superharmonic on Y. Let A, B be subsets of X, Y respectively. Write $f(x,y) = u(x)v(y)$. Then $R_f^{A\times B}(x,y)$ is a separately superharmonic function on $X \times Y$ and $R_{uv}^{A\times B}(x,y) = R_f^{A\times B}(x,y) = R_u^A(x)R_v^B(y)$. Moreover, $R_f^{A\times B}(x,y)$ is a product potential if one of the functions $u(x), v(y)$ is a potential or if at least one of the sets A, B is a non-empty finite set in a hyperbolic network.*

Proof. $f(x,y) \geq 0$ is a separately superharmonic function (property 4). If $\mathcal{F}$ is the family of all non-negative separately superharmonic functions on $X\times Y$ such that $s(x,y) \geq f(x,y)$ on $A\times B$, then $R_f^{A\times B}(x,y) = \inf\{s(x,y) : s \in \mathcal{F}\}$ is a separately superharmonic function on $X \times Y$ (property 3). Note that $R_u^A(x) \geq 0$ is Δ_1-superharmonic in X and $R_v^B(y) \geq 0$ is Δ_2-superharmonic on Y. Hence $R_u^A(x)R_v^B(y)$ is a separately superharmonic function on $X \times Y$ such that if $(x,y) \in A \times B$, then $R_u^A(x)R_v^B(y) = u(x)v(y) = f(x,y)$, so that $R_u^A(x)R_v^B(y) \geq R_f^{A\times B}(x,y)$.

On the other hand, for fixed $y \in B$ and any $s \in \mathcal{F}$, the Δ_1-superharmonic function $s_y(x)$ majorises the Δ_1-superharmonic function $u(x)v(y)$ for any $x \in A$. Hence $s_y(x) \geq R_u^A(x)v(y)$ for fixed $y \in B$ and any $x \in X$. Consequently, for fixed $x \in X$, the Δ_2-superharmonic function $s^x(y)$ on Y satisfies the inequality $s^x(y) \geq R_u^A(x)v(y)$ for all $y \in B$, and hence $s^x(y) \geq R_u^A(x)R_v^B(y)$ for all $y \in Y$. Hence,

$$R_f^{A\times B}(x,y) = \inf\{s(x,y) : s \in \mathcal{F}\} \geq R_u^A(x)R_v^B(y)$$

for any $(x,y) \in X\times Y$, which shows that $R_f^{A\times B}(x,y) = R_u^A(x)R_v^B(y)$ for any $(x,y) \in X \times Y$.

Finally, if one of u, v is a potential (suppose $u(x)$ is a Δ_1-potential on X), then $f(x,y) = u(x)v(y)$ is a product potential (Example 3.7(1) following Definition 3.6). Since the separately superharmonic function $R_f^{A\times B}(x,y) \geq 0$

is majorised by the product potential $f(x,y)$ on $X \times Y$, it follows that $R_f^{A\times B}(x,y)$ is a product potential. On the other hand, suppose one of the sets A, B is non-empty and finite (suppose A is a non-empty finite set) in a hyperbolic network. Then $R_u^A(x)$ is a Δ_1-potential on X, since there is a Δ_1-potential p on X such that $p \geq u$ on A. Hence $R_u^A(x)R_v^B(y)$ is a product potential on $X \times Y$. ■

Dirichlet solution. With the condition that a separately subharmonic function defined on a domain D in $\mathbb{R}^{p+q}$, $p, q \geq 3$, is upper bounded on any compact set in D, Avanissian [4] remarks that the class of separately subharmonic functions is larger than the class of plurisubharmonic functions when p, q are even, and goes on to see what known properties of purisubharmonic functions could be true for separately subharmonic functions as well. J. B. Walsh [23] remarks that separately harmonic functions are useful in the context of some stochastic processes which do not necessarily have the Markov property. He goes on to solve probabilistically a Dirichlet problem for bounded domains; in this case the Dirichlet solution in a domain D in $\mathbb{R}^{p+q}$ is usually a separately subharmonic function and not a separately harmonic function. These Dirichlet solutions are closely connected to Bremermann's study of this problem for pluriharmonic functions, by using the Perron–Wiener–Brelot method. Recall that (Bremermann [6]) also in the case of pluriharmonic functions, for continuous real boundary values prescribed on the boundary of a domain D in $\mathbb{C}^n$, $n > 1$, there may not exist a pluriharmonic (but only a plurisubharmonic) function in D that assumes the given boundary values. However, product domains provide examples of domains in which Dirichlet solutions for continuous functions defined on the distinguished boundary are separately harmonic [23, p. 271]. (The proof of this last result depends on some special properties of separately harmonic functions and consequently is not applicable to the case of pluriharmonic functions.)

So is the case in infinite networks; it turns out that we can always find a separately harmonic function as the Dirichlet solution in $A \times B$ (where A, B are finite subsets of X, Y) for a real-valued function defined on $\partial A \times \partial B$. The following lemma is a discrete analogue of Gowrisankaran's [12, pp. 29–31] result for separately harmonic functions defined on the product of two regular domains in the context of Brelot axiomatic potential theory.

Notation. Let A denote a finite set in X and $\mathring{A}$ the interior of A. Then, for each $a_i \in \partial A = A \setminus \mathring{A}$, there exists a unique function (Dirichlet solution [1, p. 50]) $P_A(x, a_i)$ defined on A such that $\Delta_1 P_A(x, a_i) = 0$ for each $x \in \mathring{A}$ and $P_A(a_k, a_i) = \delta(a_k, a_i)$. In particular, if a is a vertex in X and A is the set consisting of $\{a\}$ and all its neighbours $\{\alpha\}$, then $P_A(a, \alpha) = t_1(a, \alpha)/t_1(a)$. Note that if $\varphi(x)$ is a function defined on the set $\partial A = \{a_i\}$, then $h(x) = \sum_i P_A(x, a_i)\varphi(a_i)$ is defined on A and has $h(a_i) = \varphi(a_i)$ for each a_i and

$\Delta_1 h(x) = 0$ at every $x \in \mathring{A}$. Similarly define $P_B(y, b_j)$ on a finite subset B of Y for which $\partial B = \{b_j\}$.

LEMMA 3.10. *Let A, B be finite sets in X, Y respectively. Let $f(x, y)$ be a real-valued function on $\partial A \times \partial B$. Then there exists a unique function $h(x, y)$ on $A \times B$ which has the following properties:*

(i) *$h(x, y)$ is a separately harmonic function at each vertex (a, b) in $\mathring{A} \times \mathring{B}$.*
(ii) *For fixed $y \in \partial B$, $h_y(x)$ is Δ_1-harmonic in A.*
(iii) *For fixed $x \in \partial A$, $h^x(y)$ is Δ_2-harmonic in B.*
(iv) *$h(x, y) = f(x, y)$ for each vertex $(x, y) \in \partial A \times \partial B$.*

Proof. Define

$$h(x,y) = \begin{cases} \sum_{\alpha\in\partial A,\, \beta\in\partial B} f(\alpha,\beta) P_A(x,\alpha) P_B(y,\beta) & \text{if } (x,y) \in \mathring{A} \times \mathring{B}, \\ \sum_{\alpha\in\partial A} f(\alpha,y) P_A(x,\alpha) & \text{if } x \in \mathring{A},\ y \in \partial B, \\ \sum_{\beta\in\partial B} f(x,\beta) P_B(y,\beta) & \text{if } x \in \partial A,\ y \in \mathring{B}, \\ f(x,y) & \text{if } (x,y) \in \partial A \times \partial B. \end{cases}$$

(i) Let $(a, b) \in \mathring{A} \times \mathring{B}$. Then

$$h_b(x) = \begin{cases} \sum_{\alpha\in\partial A,\, \beta\in\partial B} f(\alpha,\beta) P_A(x,\alpha) P_B(b,\beta) & \text{if } x \in \mathring{A}, \\ \sum_{\beta\in\partial B} f(x,\beta) P_B(b,\beta) & \text{if } x \in \partial A. \end{cases}$$

Hence

$$h_b(a) = \sum_{\alpha\in\partial A} \Big[\sum_{\beta\in\partial B} f(\alpha,\beta) P_B(b,\beta) \Big] P_A(a,\alpha) = \sum_{\alpha\in\partial A} h_b(\alpha) P_A(a,\alpha).$$

Hence, $h_b(x)$ is Δ_1-harmonic at $x = a$.

Similarly, $h^a(y)$ is Δ_2-harmonic at $y = b$.

Thus, $h(x, y)$ is separately harmonic at each vertex $(a, b) \in \mathring{A} \times \mathring{B}$.

(ii) For $y \in \partial B$,

$$h_y(x) = \begin{cases} \sum_{\alpha\in\partial A} f(\alpha,y) P_A(x,\alpha) & \text{if } x \in \mathring{A}, \\ f(x,y) & \text{if } x \in \partial A. \end{cases}$$

Hence, $h_y(x)$ is Δ_1-harmonic at each vertex x in $\mathring{A}$ and for any $y \in \partial B$.

(iii) As above in (ii), it is proved that $h^x(y)$ is Δ_2-harmonic at each vertex $y \in \mathring{B}$ and for any $x \in \partial A$.

(iv) $h(x, y) = f(x, y)$ for $(x, y) \in \partial A \times \partial B$, by definition.

To prove uniqueness, suppose $u(x, y)$ is another function in $A \times B$ with the above four properties. Write $\varphi(x, y) = h(x, y) - u(x, y)$. Then, for fixed $y \in \partial B$, $\varphi_y(x)$ is Δ_1-harmonic on A and for each $\alpha \in \partial A$, $\varphi_y(\alpha) = h(\alpha, y) - u(\alpha, y) = 0$ since $(\alpha, y) \in \partial A \times \partial B$. Consequently, by the Minimum Principle for Δ_1-harmonic functions, $\varphi_y(x) = 0$ for all $x \in A$. Similarly we prove

$\varphi(x,y) = 0$ for $x \in \partial A$ and $y \in B$. This means that $\varphi(x,y) = 0$ on $\partial(A \times B)$. Since $\varphi(x,y)$ is separately harmonic (and hence harmonic) at each vertex in $\mathring{A} \times \mathring{B}$ (which is the interior of $A \times B$) and vanishes on $\partial(A \times B)$, by the Minimum Principle for Δ-harmonic functions, $\varphi(x,y) = 0$ on $A \times B$. ■

THEOREM 3.11. *Let $u(x,y)$ be separately superharmonic and $v(x,y)$ be separately subharmonic in $X \times Y$ such that $u(x,y) \geq v(x,y)$. Then there exists a separately harmonic function $h(x,y)$ such that $u(x,y) \geq h(x,y) \geq v(x,y)$. Moreover, $h(x,y)$ can be chosen so that if $h'(x,y)$ is another separately harmonic function such that $u(x,y) \geq h'(x,y)$, then $h(x,y) \geq h'(x,y)$.*

Proof. Let $\{A_n\}$ be a collection of finite sets in X such that $A_n \subset \mathring{A}_{n+1}$ and $X = \bigcup A_n$. Similarly, let $\{B_n\}$ be a collection of finite sets in Y such that $B_n \subset \mathring{B}_{n+1}$ and $Y = \bigcup B_n$. Then as in Lemma 3.9, let $h_n(x,y)$ be the unique function on $A_n \times B_n$ which has the four properties stated in the lemma, taking $f(x,y) = u(x,y)$.

Since $u(x,y)$ is separately superharmonic on $X \times Y$, we have $u(x,y) \geq h_n(x,y)$ on $\partial(A_n \times B_n)$, so that by the Minimum Principle, $u(x,y) \geq h_n(x,y)$ on $A_n \times B_n$. Also, since $v(x,y)$ is separately subharmonic on $X \times Y$, and $v(x,y) \leq u(x,y)$, we find that $h_n(x,y) \geq v(x,y)$ on $\partial(A_n \times B_n)$ so that $h_n(x,y) \geq v(x,y)$ on $A_n \times B_n$. For a similar reason, $h_{n+1}(x,y) \leq h_n(x,y)$ on $A_n \times B_n$. At any (a,b) in $X \times Y$, $\{h_n(a,b)\}$ is a bounded decreasing sequence for n larger than some m. Hence $\lim_{n\to\infty} h_n(x,y)$ is finite at every vertex in $X \times Y$ and its limit $h(x,y)$ is separately harmonic in $X \times Y$. Moreover, $u(x,y) \geq h(x,y) \geq v(x,y)$. Suppose now $h'(x,y)$ is any separately harmonic function such that $h'(x,y) \leq u(x,y)$ in $X \times Y$. Then by construction, $h' \leq h_n$ on $A_n \times B_n$. Consequently, $h'(x,y) \leq h(x,y)$ on $X \times Y$. ■

REMARK 3.12. We refer to $h(x,y)$ in the above Theorem 3.11 as the greatest separately harmonic minorant of $u(x,y)$. In view of this theorem, we can say that a non-negative separately superharmonic function $p(x,y)$ is a product potential (refer to Definition 3.6) if and only if its greatest separately harmonic minorant is zero. Hence, if $u(x,y)$ is separately superharmonic and $v(x,y)$ is separately subharmonic such that $u(x,y) \geq v(x,y)$ in $X \times Y$, then $u(x,y) = p(x,y) + h(x,y)$, where $p(x,y)$ is a product potential and $h(x,y)$ is the greatest separately harmonic minorant of $u(x,y)$ and this decomposition is unique. Thus any non-negative separately superharmonic function in $X \times Y$ is the unique sum of of a product potential and a non-negative separately harmonic function in $X \times Y$.

In the present context, the Dirichlet problem can be stated as: Let ω be a finite set in a product network $X \times Y$. Let $f(x,y)$ be a real-valued function on $\partial\omega$. Find a function $h(x,y)$ on ω such that $h = f$ on $\partial\omega$ and $h(x,y)$ is separately harmonic at each vertex in the interior of ω. However, in this form a solution may not exist, as suggested by the following proposition.

PROPOSITION 3.13. *Let A, B be two finite sets in X, Y respectively. Let $f(x,y)$ be a real-valued function defined on $\partial(A \times B)$. Then there exists a function $h(x,y)$ on $A \times B$ such that $h = f$ on $\partial(A \times B)$ and separately harmonic at each vertex interior to $A \times B$ if and only if for any $(a,b) \in \mathring{A} \times \mathring{B}$,*

$$\sum_{\alpha \in \partial A} f(\alpha, b) P_A(a, \alpha) = \sum_{\beta \in \partial B} f(a, \beta) P_B(b, \beta).$$

Proof. Suppose that the solution $h(x,y)$ exists on $A \times B$. Then for $(a,b) \in \mathring{A} \times \mathring{B}$, $h_b(x)$ is harmonic at $x = a$ in A, so that

$$h_b(a) = \sum_{\alpha \in \partial A} h_b(\alpha) P_A(a, \alpha) = \sum_{\alpha \in \partial A} f(\alpha, b) P_A(a, \alpha).$$

For similar reasons,

$$h^a(b) = \sum_{\beta \in \partial B} h^a(\beta) P_B(b, \beta) = \sum_{\beta \in \partial B} f(a, \beta) P_B(b, \beta).$$

Since $h_b(a) = h^a(b) = h(a,b)$, we find that

$$\sum_{\alpha \in \partial A} f(\alpha, b) P_A(a, \alpha) = \sum_{\beta \in \partial B} f(a, \beta) P_B(b, \beta).$$

Conversely, suppose the above equality holds for any $(a,b) \in \mathring{A} \times \mathring{B}$. Define

$$h(x,y) = \begin{cases} \sum_{\alpha \in \partial A} f(\alpha, y) P_A(x, \alpha) = \sum_{\beta \in \partial B} f(x, \beta) P_B(y, \beta) & \\ & \text{on } (x,y) \in \mathring{A} \times \mathring{B}, \\ f(x,y) & \text{on } \partial(A \times B). \end{cases}$$

Then $h(x,y)$ satisfies the boundary condition. To show that $h(x,y)$ is separately harmonic at any $(a,b) \in \mathring{A} \times \mathring{B}$, note that

$$h_b(a) = \sum_{\alpha \in \partial A} f(\alpha, b) P_A(a, \alpha) = \sum_{\alpha \in \partial A} h(\alpha, b) P_A(a, \alpha) = \sum_{\alpha \in \partial A} h_b(\alpha) P_A(a, \alpha).$$

Hence $h_b(x)$ is harmonic at $x = a$ in A. Similarly, $h^a(y)$ is harmonic at $y = b$ in B. Thus, $h(x,y)$ is separately harmonic at $(a,b) \in \mathring{A} \times \mathring{B}$. ■

NOTE. If $f(x,y)$ satisfies the above condition, then the separately harmonic solution $h(x,y)$ on $A \times B$ with boundary values $f(x,y)$ is uniquely determined.

THEOREM 3.14 (Dirichlet problem in a product network). *Let $\omega \subseteq X \times Y$. Let f be a real-valued function on $\partial\omega$. Suppose there exist two functions u, v on ω with the properties: $u(x,y)$ is separately superharmonic and $v(x,y)$ is separately subharmonic at each vertex $(x,y) \in \mathring{\omega}$; $v \leq u$ on ω; and $v \leq f \leq u$ on $\partial\omega$. Then there exists $q(x,y)$ on ω such that:*

(i) *$q(x,y)$ is separately subharmonic at each vertex in $\mathring{\omega}$;*
(ii) *$q(x,y) = f(x,y)$ on $\partial\omega$;*

(iii) *if* $q_1(x, y)$ *is a separately subharmonic function in* ω *such that* $q_1 \leq f$ *on* $\partial\omega$, *then* $q_1 \leq q$ *on* ω.

Proof. Let $u_1 = u$ on $\mathring{\omega}$ and $u_1 = f$ on $\partial\omega$; similarly, let $v_1 = v$ on $\mathring{\omega}$ and $v_1 = f$ on $\partial\omega$. Then u_1 is separately superharmonic and v_1 is separately subharmonic at each vertex in $\mathring{\omega}$, $v_1 \leq u_1$ on ω; and $v_1 = u_1 = f$ on $\partial\omega$. Let $\mathcal{F}$ be the family of functions $s(x, y)$ on ω such that $s = f$ on $\partial\omega$; $s \leq u_1$ on ω; and $s(x, y)$ is separately subharmonic at each vertex in $\mathring{\omega}$. Let $q(x, y) = \sup s(x, y)$, $s \in \mathcal{F}$. Then (as in Property 3), $q(x, y)$ is separately subharmonic at each vertex in $\mathring{\omega}$ and $q = f$ on $\partial\omega$. Finally, if $q_1(x, y)$ is a separately subharmonic function in ω such that $q_1 \leq f$ on $\partial\omega$, define $q_2(x, y) = q_1(x, y)$ in $\mathring{\omega}$ and $q_2(x, y) = f(x, y)$ on $\partial\omega$. Then $q_2 \in \mathcal{F}$, and hence $q_2 \leq q$ on ω. In particular, $q_1 \leq q$ on ω. ∎

Corollary 3.15 (Classical Dirichlet problem). *Let* ω *be a finite set in* $X \times Y$, *and* f *be a real-valued function on* $\partial\omega$. *Then there exists a separately subharmonic function* $q(x, y)$ *on* ω *that is maximal in the family of functions* $s(x, y)$ *in* ω *that are separately subharmonic at each vertex in* $\mathring{\omega}$, *and* $s = f$ *on* $\partial\omega$.

Proof. Let $|f| \leq M$ on $\partial\omega$. Take $u = M$ and $v = -M$ in the above theorem to prove the existence and the maximality of a solution q in ω. ∎

Remark 3.16. Let ω be a finite set in $X \times Y$. Let us denote the maximal separately subharmonic function $q(x, y)$ with boundary values f on $\partial\omega$ by $v_f(x, y)$. Analogously, we denote the minimal separately superharmonic function with boundary values f on $\partial\omega$ by $u_f(x, y)$. By the Minimum Principle (since u_f is superharmonic and v_f is subharmonic in the finite subset ω with the same boundary values), $v_f(x, y) \leq u_f(x, y)$ on ω. Since ω is a finite subset, there exists a Δ-Dirichlet solution $h_f(x, y)$ on ω with boundary values f on $\partial\omega$ [1, Theorem 3.1.7] such that $v_f(x, y) \leq h_f(x, y) \leq u_f(x, y)$ on ω.

Integral representation for positive separately superharmonic functions. We now derive a discrete analogue of the integral representation of positive separately superharmonic functions in product Brelot harmonic spaces, given in Drinkwater [9]. Let Σ denote the class of positive separately superharmonic functions in $X \times Y$. Let us define a topology on $\Sigma - \Sigma$ by the seminorms $\|u_1 - u_2\|_{(x,y)} = |u_1(x, y) - u_2(x, y)|$, $(x, y) \in X \times Y$, $u_1, u_2 \in \Sigma$. Since $X \times Y$ is a countable set, these seminorms are countable in number and hence define a locally convex metrisable topology on $\Sigma - \Sigma$. For a fixed $(x_0, y_0) \in X \times Y$, write $B = \{u \in \Sigma : u(x_0, y_0) = 1\}$.

Theorem 3.17 (Harnack property). *Let* $(a, b), (c, d)$ *be two vertices in* $X \times Y$. *Then there exist constants* α, β *such that* $\alpha u(a, b) \leq u(c, d) \leq \beta u(a, b)$ *for any* $u \in \Sigma$.

Proof. Let $\{a, x_1, \ldots, x_n = c\}$ be a path in X connecting a and c. Let $\{b, y_1, \ldots, y_m = d\}$ be a path in Y connecting c and d. Then $\nu = \{(a,b), (x_1,b), \ldots, (x_n,b) = (c,b), (c,y_1), \ldots, (c,y_m) = (c,d)\}$ is a path in $X \times Y$ connecting (a,b) and (c,d). Let $u \in \Sigma$. Since $u(x,y)$ is superharmonic in X for fixed y, we have

$$t_1(a)u(a,b) \geq \sum_{z \sim a} t_1(a,z)u(z,b),$$

so that

$$t_1(a)u(a,b) \geq t_1(a,x_1)u(x_1,b).$$

That is,

$$u(a,b) \geq \frac{t_1(a,x_1)}{t_1(a)} u(x_1,b).$$

Thus proceeding from a vertex to the neighbouring vertex in the path ν, we obtain

$$u(a,b) \geq \frac{t_1(a,x_1)}{t_1(a)} \cdots \frac{t_2(y_{m-1},d)}{t_2(y_{m-1})} u(c,d).$$

This and another analogous inequality show that for fixed constants α, β depending only on the vertices (a,b), (c,d) and the path ν, we have $\alpha u(a,b) \leq u(c,d) \leq \beta u(a,b)$ for any $u \in \Sigma$. ∎

Consequently, since $X \times Y$ has only a countable number of vertices, from any sequence in B we can extract a convergent subsequence. Since the finite limit of a convergent sequence of separately superharmonic functions is separately superharmonic, we conclude that the base B for the cone Σ is compact. Hence if $\mathscr{F}$ denotes the set of extremal elements of B, then for a given positive separately superharmonic function u in Σ, there exists a non-negative Radon measure μ supported by $\mathscr{F}$ (Choquet theorem) such that $u(x,y) = \int v(x,y)\, d\mu(v)$.

This discrete analogue of the integral representation of positive separately superharmonic functions, as its counterpart in product Brelot harmonic spaces given by Drinkwater [9], states only the existence but not the uniqueness of the representing measure μ; nor is it possible to characterise the extremal elements in $\mathscr{F}$, due to the same difficulty as in [9]. However, for a subclass of Σ which we have termed below the doubly supermedian functions, these difficulties will be overcome.

4. Doubly supermedian and doubly median functions in a product network. Recall that if E is a finite set in X, and $\alpha \in \partial E$, then $P_E(x,\alpha)$ denotes the Dirichlet solution in E for $x \in \mathring{E}$ with boundary values $\chi_\alpha(z)$ on ∂E. In particular, if a is a vertex in X and A is the set consisting of $\{a\}$ and all its neighbours $\{\alpha\}$, then $P_A(a,\alpha) = t(a,\alpha)/t(a)$. Let $X \times Y$ be the product network associated with the infinite networks X, Y. Let $u(x,y)$ be a real-valued function defined on $X \times Y$ such that for any fixed $y \in Y$, $\varphi(x) = u_y(x)$ is harmonic in X. Then, for any $(x,y) \in X \times Y$, if A is the set consisting of x and all its neighbours $\{\alpha\}$ and if B is the set consisting of y and all its neighbours $\{\beta\}$, then we have

$$\sum_\alpha u(\alpha, y) P_A(x, \alpha) = u(x, y),$$
$$\sum_\beta u(x, \beta) P_B(y, \beta) = \sum_\beta \Big[\sum_\alpha u(\alpha, \beta) P_A(x, \alpha)\Big] P_B(y, \beta).$$

Hence,

$$\begin{aligned} u(x,y) + \sum_{\alpha,\beta} u(\alpha,\beta) P_A(x,\alpha) P_B(y,\beta) \\ = \sum_\alpha u(\alpha, y) P_A(x, \alpha) + \sum_\beta u(x,\beta) P_B(y,\beta). \end{aligned} \tag{4.1}$$

Among functions with the property (4.1) that are interesting in the case of product networks is the family of separately superharmonic functions that are harmonic in one variable when the other variable is fixed; in particular the family of separately harmonic functions in $X \times Y$. An analogous expression arises when we consider the product of two superharmonic functions, one in X and the other in Y. More precisely, suppose $f(x)$ and $g(y)$ are two superharmonic functions in X and Y respectively. Then, for any $(x,y) \in X \times Y$, if $u(x,y) = f(x) g(y)$ then $f(x) - \sum_{\alpha \sim x} f(\alpha) P_A(x,\alpha) \geq 0$ and $g(y) - \sum_{\beta \sim y} g(\beta) P_B(y, \beta) \geq 0$, so that

$$\Big[f(x) - \sum_{\alpha \sim x} f(\alpha) P_A(x,\alpha)\Big]\Big[g(y) - \sum_{\beta \sim y} g(\beta) P_B(y,\beta)\Big] \geq 0.$$

This expands into

$$\begin{aligned} u(x,y) + \sum_{\alpha,\beta} u(\alpha,\beta) P_A(x,\alpha) P_B(y,\beta) \\ \geq \sum_\alpha u(\alpha, y) P_A(x, \alpha) + \sum_\beta u(x,\beta) P_B(y,\beta). \end{aligned} \tag{4.2}$$

We single out this property in the following definition:

Definition 4.1. A real-valued function $v(x,y)$ defined on $X \times Y$ is said to be *doubly supermedian* if for any $(x,y) \in X \times Y$,

$$\begin{aligned} v(x,y) + \sum_{\alpha,\beta} v(\alpha,\beta) P_A(x,\alpha) P_B(y,\beta) \\ \geq \sum_\alpha v(\alpha, y) P_A(x, \alpha) + \sum_\beta v(x,\beta) P_B(y,\beta). \end{aligned}$$

If the inequality can be replaced by equality, we say that $v(x,y)$ is a *doubly median function.*

PROPOSITION 4.2. *Let $\{v_n(x,y)\}$ be a sequence of doubly supermedian functions such that $v(x,y) = \lim_{n\to\infty} v_n(x,y)$ is finite for every (x,y) in $X \times Y$. Then $v(x,y)$ is a doubly supermedian function.*

Proof. Since for any $(x,y) \in X \times Y$,

$$v_n(x,y) + \sum_{\alpha,\beta} v_n(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) \geq \sum_{\alpha} v_n(\alpha,y)P_A(x,\alpha) + \sum_{\beta} v_n(x,\beta)P_B(y,\beta),$$

letting $n \to \infty$, we conclude that $v(x,y)$ is a doubly supermedian function. ∎

PROPOSITION 4.3. *For any real-valued function $v(x,y)$ in $X \times Y$, the following are equivalent:*

(i) $v(x,y) - \sum_\alpha v(\alpha,y)P_A(x,\alpha) = \psi(y)$ *is superharmonic in* Y.
(ii) $v(x,y) - \sum_\beta v(x,\beta)P_B(y,\beta) = \varphi(x)$ *is superharmonic in* X.
(iii) $v(x,y)$ *is doubly supermedian in* $X \times Y$.

Proof. (ii)⇒(iii): $\sum_\alpha \varphi(\alpha)P_A(x,\alpha) \leq \varphi(x)$ by assumption. That is,

$$\sum_\alpha \Big[v(\alpha,y) - \sum_\beta v(\alpha,\beta)P_B(y,\beta)\Big] P_A(x,\alpha) \leq v(x,y) - \sum_\beta v(x,\beta)P_B(y,\beta),$$

which is the condition for $v(x,y)$ to be doubly supermedian in $X \times Y$.

(iii)⇒(ii): The condition for being doubly supermedian can be written in the form given above in (ii)⇒(iii), so that if we set

$$\varphi(x) = v(x,y) - \sum_\beta v(x,\beta)P_B(y,\beta),$$

then $\varphi(x)$ is superharmonic in X.

A similar argument proves the equivalence (iii)⇔(i). ∎

NOTE. In the above proposition, the statements that $\psi(y)$ is harmonic in Y, $\varphi(x)$ is harmonic in X and $v(x,y)$ is doubly median are equivalent.

Recall that if S^+ stands for the cone of non-negative superharmonic functions in X and if a topology is defined on $S^+ - S^+$ by the seminorms $\|s_1 - s_2\|_x = |s_1(x) - s_2(x)|$, $x \in X$, $s_1, s_2 \in S^+$, then this countable set of seminorms defines a locally convex, metrisable topology on $S^+ - S^+$. For a fixed $x_0 \in X$, if $B = \{s \in S^+ : s(x_0) = 1\}$, then B is a compact base for S^+ (by the Harnack property [1, p. 47]). The extremal elements $\mathfrak{E}$ of B are the minimal harmonic functions $h(x)$ with $h(x_0) = 1$ and the potentials $\{G_e(x)\}_{e\in X}$ where $G_e(x)$ is the unique potential on X which is harmonic outside e and $G_e(x_0) = 1$. (Recall that an element v of B is *extremal* if and only if an expression of the form $v = \lambda v_1 + (1-\lambda)v_2$ where v_1, v_2 are in B and $0 \le \lambda \le 1$, implies that λ is 0 or 1; and a non-negative harmonic function φ is said to be a *minimal harmonic function* if for any harmonic function u such that $0 \le u \le \varphi$, we have $u = \alpha\varphi$ for some α with $0 \le \alpha \le 1$.)

Now to apply the Choquet integral representation theorem [8], all that remains is to show that S^+ is a lattice for its own proper order. (The proper order $\succeq_{S^+}$ of S^+ means: $u \succeq_{S^+} v$ if and only if $u = v + s$ where s is some element of S^+.) Write $S^+ = P + H^+$ where P is the set of potentials and H^+ is the set of non-negative harmonic functions in X.

LEMMA 4.4. *The cone S^+ is a lattice for its own proper order if and only if P and H^+ are lattices for their own proper order. In fact, if $s_1 = p_1 + h_1$, $s_2 = p_2 + h_2$ are functions in S^+, then $s_1 \curlyvee_{S^+} s_2 = p_1 \curlyvee_P p_2 + h_1 \curlyvee_{H^+} h_2$. Similarly for $s_1 \curlywedge_{S^+} s_2$.*

Proof. For H^+, if h_1, h_2 are in H^+, then the usual least harmonic majorant of $\sup(h_1, h_2)$ is $h_1 \curlyvee_{H^+} h_2$ which is the supremum of h_1, h_2 for the order defined by S^+. As for the potential cone P, recall [1, Theorem 3.3.1] that a function $p(x)$ in S^+ is a potential if and only if $p(x) = \sum_{e\in X} c(e)G_e(x)$, where $c(e) \ge 0$ is uniquely determined. Now, if $p_i(x) = \sum_e c_i(x)G_e(x)$, $i = 1, 2$, are two potentials and if $c(e) = \max\{c_1(e), c_2(e)\}$, then $\sum_e c(e)G_e(x)$ is a potential which is the supremum $p_1 \curlyvee_P p_2$ for the order of S^+.

Let us write $u = p_1 \curlyvee_P p_2 + h_1 \curlyvee_{H^+} h_2$. Then $u \succcurlyeq_{S^+} s_i$, $i = 1, 2$, so that $u \succcurlyeq_{S^+} s_1 \curlyvee_{S^+} s_2$.

On the other hand, if we write $s_1 \curlyvee_{S^+} s_2 = p + h$, then $p + h = s_i +$ (a non-negative superharmonic function), $i = 1, 2$. In particular, $h \succcurlyeq_{S^+} h_i$ so that $h \succcurlyeq_{S^+} h_1 \curlyvee_{S^+} h_2$. Similarly, $p \succcurlyeq_{S^+} p_1 \curlyvee_{S^+} p_2$. Hence, $s_1 \curlyvee_{S^+} s_2 = p + h \succcurlyeq_{S^+} u$. Consequently, $s_1 \curlyvee_{S^+} s_2 = p_1 \curlyvee_P p_2 + h_1 \curlyvee_{H^+} h_2$. A similar calculation gives $s_1 \curlywedge_{S^+} s_2 = p_1 \curlywedge_P p_2 + h_1 \curlyvee_{H^+} h_2$.

Consequently, the cone S^+ is a lattice for its own proper order. ∎

Hence by the Choquet integral representation theorem [8], we have: Given any non-negative superharmonic function s in the infinite network X, there exists a unique Radon measure $\mu^s \ge 0$ with support in the extremal set $\mathfrak{E}$ of the base such that $s(x) = \int u(x)\, d\mu^s(u)$ for $x \in X$.

Let us now turn to the class $\mathfrak{I}^+$ of non-negative separately superharmonic functions in a product network $X \times Y$ such that $u \in \mathfrak{I}^+$ if and only if for any fixed y, $u(x, y)$ is a non-negative superharmonic function in X and for fixed x, $u(x, y)$ is a non-negative superharmonic function in Y. Since for fixed y, $u(x, y)$ is a non-negative superharmonic function in X, by the Choquet theorem above there exists a unique measure v^u_y supported by the extremal elements $\mathfrak{G}_1$ and depending on y such that $u(x, y) = \int v(x)\, dv^u_y(v)$ for $x \in X$. We say that the measure v^u_y is *superharmonic* in $y \in Y$ if $v^u_y \ge \sum_\beta P_B(y, \beta) v^u_\beta$.

Proposition 4.5. *Let $u \in \Im^+$. Then $u(x,y)$ is doubly supermedian* (*respectively doubly median*) *if and only if the measure v_y^u is superharmonic* (*respectively harmonic*) *in $y \in Y$.*

Proof. We prove the doubly supermedian case only. Suppose u is a doubly supermedian function. Then by Proposition 4.3,

$$\varphi(x) = u(x,y) - \sum_{\beta} u(x,\beta) P_B(y,\beta)$$

is a superharmonic function in X. Moreover, since for any x, $u(x,y)$ is superharmonic in y, we have

$$u(x,y) \geq \sum_{\beta} u(x,\beta) P_B(y,\beta).$$

That is, $\varphi \geq 0$. Hence there exists a measure $\lambda \geq 0$ supported by $\mathfrak{E}_1$ such that $\varphi(x) = \int v(x)\, d\lambda(v)$. Thus,

$$u(x,y) - \sum_{\beta} u(x,\beta) P_B(y,\beta) = \int v(x)\, d\lambda(v),$$

which gives

$$\int v(x)\, dv_y^u(v) - \sum_{\beta} \Big[\int v(x)\, dv_\beta^u(v)\Big] P_B(y,\beta) = \int v(x)\, d\lambda(v).$$

Hence, by the uniqueness of the representing measure,

$$v_y^u - \sum_{\beta} P_B(y,\beta) v_\beta^u = \lambda \geq 0.$$

In particular, the measure v_y^u is superharmonic in y.

Conversely, suppose v_y^u is superharmonic in y. Then

$$v_y^u - \sum_{\beta} P_B(y,\beta) v_\beta^u = \lambda \geq 0$$

is a measure with support in $\mathfrak{E}_1$. Hence

$$u(x,y) - \sum_{\beta} u(x,\beta) P_\beta(y,\beta) = \int v(x)\, d\lambda(v)$$

is a non-negative superharmonic function in X. This implies by Proposition 4.3 that $u(x,y)$ is doubly supermedian in $X \times Y$. ∎

Remark. (i) By symmetry, we also have the following: If $v \in \Im^+$, then $v(x,y)$ is doubly supermedian if and only if the corresponding measure v_x^v, with support in the extremal elements $\mathfrak{E}_2$ for a base in Y, is superharmonic in $x \in X$.

(ii) Proposition 4.5 shows that the convex cone $\Im_D^+$ of all non-negative separately superharmonic functions that are doubly supermedian is the discrete analogue of separately excessive functions satisfying the condition H in Cairoli's paper [7] and also of the class C in Gowrisankaran's paper [13] in the framework of Brelot axiomatic potential theory.

As before, let us denote by $\Im_D^+$ the set of all non-negative separately superharmonic functions that are doubly supermedian in $X \times Y$. Some subcones of $\Im^+$ are defined as follows:

$\Im_1$: the set of functions $u(x, y)$ that are separately harmonic;
$\Im_2$: the set of functions $u(x, y)$ that are harmonic in x for fixed y, and are potentials in y for fixed x;
$\Im_3$: the set of functions $u(x, y)$ that are potentials in x for fixed y, and harmonic in y for fixed x;
$\Im_4$: the set of functions $u(x, y)$ that are potentials in each variable when the other is fixed.

We emphasise that all the elements in $\Im_i$, $1 \le i \le 4$, are non-negative.

THEOREM 4.6. *Let a doubly supermedian function $u(x, y)$ be in $\Im^+$. Then $u = \sum_{i=1}^4 u_i$ where $u_i \in \Im_i$, and the decomposition is unique.*

Proof. For y fixed, $u_y(x) = u(x, y)$ is a non-negative superharmonic function in X. Hence by using the Choquet representation in X, we obtain

$$\begin{aligned} u(x, y) &= \int_{\mathfrak{E}_1} v(x)\, d\nu_y^u(v) = \int_{\Lambda_1} v(x)\, d\nu_y^u(v) + \int_{\mathfrak{E}_1 \setminus \Lambda_1} v(x)\, d\nu_y^u(v) \\ &= s_1(x, y) + s_2(x, y), \end{aligned}$$

where $\mathfrak{E}_1$ stands for the extremal elements and Λ_1 the minimal (harmonic) elements in a base for the cone of non-negative superharmonic functions in X. Here $s_1(x, y)$ is harmonic in x for fixed y, and superharmonic in y for fixed x (Proposition 4.5). Note that since u is doubly supermedian and s_1 is doubly median, as shown in (4.1) at the beginning of Section 4, we conclude that s_2 is doubly supermedian.

(i) Since for fixed x, $s_1^x(y)$ is a non-negative superharmonic function in Y, write (using notations in Y analogous to the ones given above for X)

$$\begin{aligned} s_1^x(y) &= \int_{\mathfrak{E}_2} v'(y)\, d\nu_x^{s_1}(v') = \int_{\Lambda_2} v'(y)\, d\nu_x^{s_1}(v') + \int_{\mathfrak{E}_2 \setminus \Lambda_2} v'(y)\, d\nu_x^{s_1}(v') \\ &= u_1^x(y) + u_2^x(y), \end{aligned}$$

where $u_1^x(y)$ is the greatest harmonic minorant of $s_1^x(y)$ in Y, and $u_2^x(y)$ is a potential in $y \in Y$.

Now $\nu_x^{s_1}$ is superharmonic in X. Hence for fixed y, we conclude that $u_1^x(y)$, $u_2^x(y)$ are superharmonic in $x \in X$. Thus $u_1(x, y), u_2(x, y)$ are separately superharmonic in $X \times Y$, and $s_1(x,y) = u_1(x,y) + u_2(x,y)$. Here for fixed y, $s_1(x,y)$ is harmonic in x and is the sum of two superharmonic functions $u_1(x,y), u_2(x,y)$, so that $u_1(x,y), u_2(x,y)$ are harmonic in x for fixed y. Consequently, $s_1(x,y) = u_1(x,y) + u_2(x,y)$, where $u_1 \in \Im_1$ and $u_2 \in \Im_2$.

(ii) Similarly taking up the function $s_2(x,y)$ which is doubly supermedian, we have

$$\begin{aligned} s_2^x(y) &= \int_{\mathfrak{E}_2} v'(y)\, dv_x^{s_2}(v') \qquad \text{for fixed } x \\ &= \int_{\Lambda_2} v'(y)\, dv_x^{s_2}(v') + \int_{\mathfrak{E}_2 \setminus \Lambda_2} v'(y)\, dv_x^{s_2}(v') = u_3^x(y) + u_4^x(y), \end{aligned}$$

where $u_3^x(y) = u_3(x,y)$ and $u_4^x = u_4(x,y)$ are separately superharmonic functions in $X \times Y$.

For fixed x, $u_3(x,y)$ is harmonic in y and $u_4(x,y)$ is a potential in y; for fixed y, $u_3(x,y)$ and $u_4(x,y)$ are non-negative superharmonic functions in X whose sum is $s_2(x,y)$, which is a potential in X. Hence for fixed y, $u_3(x,y)$ and $u_4(x,y)$ are potentials in X. Consequently, $s_2(x,y) = u_3(x,y) + u_4(x,y)$, where $u_3 \in \Im_3$ and $u_4 \in \Im_4$.

Finally, $u = \sum_{i=1}^4 u_i$, where $u_i \in \Im_i$. Here for the claim that $u_4 \in \Im_4$, it remains to check that u_4 is doubly supermedian; but this is easy to see, since u_1, u_2, u_3 are all doubly median and u is doubly supermedian, hence u_4 should be doubly supermedian.

(iii) For the uniqueness of decomposition, suppose $u(x,y) = \sum_{i=1}^4 u_i'(x,y)$ is another such decomposition. Then, for fixed y, $u_1 + u_2$ and $u_1' + u_2'$ are harmonic in X and $u_3 + u_4, u_3' + u_4'$ are potentials in X. Hence by the uniqueness of Riesz decomposition, for any (x,y) in $X \times Y$,

$$\begin{aligned} u_1(x,y) + u_2(x,y) &= u_1'(x,y) + u_2'(x,y), \\ u_3(x,y) + u_4(x,y) &= u_3'(x,y) + u_4'(x,y). \end{aligned}$$

Now for fixed x, the two sides of both the above equations are the sum of a potential and a harmonic function in Y. Hence, again by the uniqueness of Riesz decomposition, $u_i(x,y) = u_i'(x,y)$ for any $(x,y) \in X \times Y$ and $1 \le i \le 4$. ∎

5. Integral representation of positive separately superharmonic functions that are doubly supermedian. In this section, by using Theorem 4.6, an integral representation is given for any $u \in \Im_D^+$. This method brings forth some precision on the representing measure and the extremal elements. Let us denote by B_i the base determined in $\Im_i$ by $B_i = \{u \in \Im_i : u(x_0,y_0) = 1\}$ for a fixed vertex $(x_0,y_0) \in X \times Y$; also $B = \{u \in \Im_D^+ : u(x_0,y_0) = 1\}$. Using the countably many seminorms for the topology as before, we see that B_i, B are all metrisable. Since each function in B_i, B

is separately superharmonic and hence superharmonic, these functions have the Harnack property. Lemmas 5.1–5.4 below show that $\Im_D^+$ is a lattice for its own proper order. Let us use the symbols $\curlyvee, \curlywedge, \preccurlyeq, \succcurlyeq$, with respect to the order determined by $\Im_D^+$ and $\succcurlyeq_i$, etc. with respect to the order determined by the cone $\Im_i$.

LEMMA 5.1. *If $u, v \in \Im_1$, then $u \curlyvee_1 v$ exists in $\Im_1$.*

Proof. Let $s(x, y) = \sup\{u(x, y), v(x, y)\}$ for any $(x, y) \in X \times Y$. Then $s(x, y)$ is a non-negative separately subharmonic function in $X \times Y$. Let $h(x, y)$ be the least separately harmonic majorant of $s(x, y)$. It is easy to conclude that $h(x, y) = u(x, y) \curlyvee_1 v(x, y)$. ∎

LEMMA 5.2. *If $u, v \in \Im_2$, then $u \curlyvee_2 v$ exists in $\Im_2$; similarly for $\Im_3$.*

Proof. We know that $u(x, y), v(x, y)$ are harmonic in x for fixed y and potentials in y for fixed x. Hence for fixed x,

$$u^x(y) = u(x, y) = \sum_z k_x^u(z) G_z(y), \qquad v^x(y) = v(x, y) = \sum_z k_x^v(z) G_z(y),$$

where $k_x^u(z) = (-\Delta_2)u^x(z)$ and $k_x^v(z) = (-\Delta_2)v^x(z)$.

For fixed z, we have $(-\Delta_1)k_x^u(z) = (-\Delta_2)(-\Delta_1)u^x(z) = 0$. Hence $k_x^u(z)$ (and similarly $k_x^v(z)$) is harmonic in x for fixed z. Let $\mu_x(z)$ be the harmonic majorant (for the order determined by the cone of non-negative harmonic functions in X) of $k_x^u(z)$ and $k_x^v(z)$. Note that $\mu_x(z) \le k_x^u(z) + k_x^v(z)$ so that $Q(x, y) = \sum_z \mu_x(z) G_z(y)$ is a potential in y for fixed x. On the other hand, $Q(x, y)$ is harmonic in x, for fixed y, for

$$\begin{aligned}
\sum_\alpha Q(\alpha, y) P_A(x, \alpha) &= \sum_\alpha \Big[\sum_z \mu_\alpha(z) G_z(y)\Big] P_A(x, \alpha) \\
&= \sum_z \Big[\sum_\alpha \mu_\alpha(z) P_A(x, \alpha)\Big] G_z(y) \\
&= \sum_z \mu_x(z) G_z(y) = Q(x, y).
\end{aligned}$$

Hence, $Q \in \Im_2$. Clearly, $Q \succcurlyeq_2 u$ and $Q \succcurlyeq_2 v$. In fact, $Q = u \curlyvee_2 v$. Indeed, let $g \in \Im_2$ be such that $g \succcurlyeq_2 u$ and $g \succcurlyeq_2 v$. Since $g(x, y)$ is a potential in y for fixed x, write $g(x, y) = \sum_z \lambda_x(z) G_z(y)$. Since $g = u+$(a function $\varphi \in \Im_2$), we find that for fixed z, $\lambda_x(z)$ majorizes $k_x^u(z)$ (and similarly $k_x^v(z)$) in the order determined by the cone of non-negative harmonic functions in X. Hence $\lambda_x(z)$ majorizes $\mu_x(z)$ in that order, that is, $\lambda_x(z) = \mu_x(z) + \xi_x(z)$, where $\xi_x(z)$ is a non-negative harmonic function in x for fixed z. Consequently,

$$\begin{aligned}
g(x, y) = \sum_z \lambda_x(z) G_z(y) &= \sum_z \mu_x(z) G_z(y) + \sum_z \xi_x(z) G_z(y) \\
&= Q(x, y) + (\text{a function in } \Im_2).
\end{aligned}$$

Hence, $g \succcurlyeq_2 Q$, proving our claim $Q(x,y) = u(x,y) \curlyvee_2 v(x,y)$. ∎

Lemma 5.3. *Let* $p(x) = \sum_z a(z)G_z(x)$ *and* $q(x) = \sum_z b(z)G_z(x)$ *be two potentials in* X. *Let* $c(z) = \max\{a(z), b(z)\}$. *Then the potential* $Q(x) = \sum_z c(z)G_z(x)$ *is* $\sup(p(x), q(x))$ *for the order determined by the cone* $S^+(X)$ *of non-negative superharmonic functions in* X.

Proof. Suppose that $s(x)$ is a non-negative superharmonic function majorizing $p(x)$ and $q(x)$ for the order determined by $S^+(X)$. Then $s = p + u$, where $u \in S^+(X)$. Write $s(x) = \sum_z m(z)G_z(x) +$ (a non-negative harmonic function) in the Riesz representation, and note that $m(z) \ge a(z)$; similarly $m(z) \ge b(z)$. Hence, $m(z) \ge c(z)$. This shows that $s(x)$ majorizes $Q(x)$ for the order determined by $S^+(X)$. Hence $Q(x) = p(x) \curlyvee_{S^+(X)} q(x)$ and the lemma is proved. ∎

Lemma 5.4. *Let* $u, v \in \Im_4$. *Then* $u \curlyvee_4 v$ *exists in* $\Im_4$.

Proof. Since $u(x,y)$ is doubly supermedian, the measure $v_x^u(\xi)$ is superharmonic in x in X for fixed ξ in $\mathfrak{E}_2$ (Proposition 4.5). Now,

$$u(x,y) = u^x(y) = \int_{\mathfrak{E}_2} \xi(y)\, dv_x^u(\xi) = \int_{\Lambda_2} \xi(y)\, dv_x^u(\xi) + \int_{\mathfrak{E}_2 \setminus \Lambda_2} \xi(y)\, dv_x^u(\xi).$$

Since $u(x,y)$ is a potential in y for fixed x, observe that $v_x^u(\xi) = 0$ if ξ is in Λ_2. Hence,

$$u(x,y) = \int_{\mathfrak{E}_2 \setminus \Lambda_2} \xi(y)\, dv_x^u(\xi).$$

Now $v_x^u(\xi)$ being a superharmonic function in x for fixed ξ, write $v_x^u(\xi) = h_x^u(\xi) + p_x^u(\xi)$ by using the Riesz decomposition. Then identifying $\mathfrak{E}_2 \setminus \Lambda_2$ with Y and noting that $\xi(y) = G_\xi(y)$ when $\xi \in \mathfrak{E}_2 \setminus \Lambda_2$, we have

$$u(x,y) = \sum_{z \in Y} h_x^u(z)G_z(y) + \sum_{z \in Y} p_x^u(z)G_z(y).$$

But $u(x,y)$ is a potential in x for fixed y, so that $h_x^u(z) = 0$ for $z \in Y$. Consequently, $v_x^u(z) = p_x^u(z)$ is a potential in x for any fixed z in Y.

Let $\mu_x(z) = v_x^u(z) \curlyvee_{S^+(X)} v_x^v(z)$. Note that by Lemma 5.3, $\mu_x(z)$ is a potential in x for fixed z. Also, $\mu_x(z) \le v_x^u(z) + v_x^v(z)$, so that $Q(x,y) = \sum_{z \in Y} \mu_x(z)G_z(y)$ is well-defined as a potential in x for fixed y and a potential in y for fixed x. For fixed z, $\mu_x(z)$ is a potential in X and $G_z(y)$ is a potential in Y, hence $\mu_x(z)G_z(y) \in \Im_4$; consequently, as a convergent sum in $\Im_4$, $Q(x,y) \in \Im_4$. Then as in the proof of Lemma 5.2, we show that $Q(x,y) = u(x,y) \curlyvee_4 v(x,y)$. ∎

Proposition 5.5. *Let* $u, v \in \Im_D^+$. *If* $u = \sum_{i=1}^4 u_i$ *and* $v = \sum_{i=1}^4 v_i$, *then* $u \curlyvee v = \sum_{i=1}^4 u_i \curlyvee_i v_i$.

Proof. Suppose $s \in \Im_D^+$ and $s = \sum_{i=1}^4 s_i \succcurlyeq u$. That is, $s = u + \varphi$ for some $\varphi = \sum_{i=1}^4 \varphi_i \in \Im_D^+$. Then $s_i = u_i + \varphi_i$ so that $s_i \succcurlyeq_i u_i$. Consequently, if $s \succcurlyeq u$ and $s \succcurlyeq v$, then $s_i \succcurlyeq_i u_i \curlyvee_i v_i$ so that $s \succcurlyeq \sum_{i=1}^4 u_i \curlyvee_i v_i$. This implies that $u \curlyvee v \succcurlyeq \sum_{i=1}^4 u_i \curlyvee_i v_i$. (Here the uniqueness of decomposition given in Theorem 4.6 has been useful.)

On the other hand, $u_i \curlyvee_i v_i = u_i + f_i$ where $f_i \in \Im_i$, and hence $\sum_{i=1}^4 u_i \curlyvee_i v_i = u + f$ where $f = \sum_{i=1}^4 f_i \in \Im_D^+$. Hence, $\sum_{i=1}^4 u_i \curlyvee_i v_i \succcurlyeq u$; and similarly with respect to v. Hence $\sum_{i=1}^4 u_i \curlyvee_i v_i \succcurlyeq u \curlyvee v$. ■

PROPOSITION 5.6. *Let $u, v \in \Im_D^+$. Then $u \curlywedge v = u + v - (u \curlyvee v)$.*

Proof. Since $u + v \succcurlyeq u \curlyvee v$, we have $\rho = u + v - (u \curlyvee v) \in \Im_D^+$. Since $(u \curlyvee v) - v \in \Im_D^+$, it follows that $u \succcurlyeq \rho$; similarly, $v \succcurlyeq \rho$. Hence $u \curlywedge v \succcurlyeq \rho$.

On the other hand, suppose $\psi \in \Im_D^+$ is such that $\psi \preccurlyeq u$ and $\psi \preccurlyeq v$. Then $u + (v - \psi) \succcurlyeq u$ and $v + (u - \psi) \succcurlyeq v$. Consequently, $u + v - \psi \succcurlyeq u \curlyvee v$. That is, $\rho = u + v - (u \curlyvee v) \succcurlyeq \psi$. This implies $\rho \succcurlyeq u \curlywedge v$. ■

Extremal elements in $B \subset \Im_D^+$. In $\Im_D^+$, let us consider the base $B = \{u \in \Im_D^+ : u(x_0, y_0) = 1\}$ where (x_0, y_0) is a fixed vertex in $X \times Y$. We shall now determine the form of the extremal elements in $\Im_D^+$.

PROPOSITION 5.7. *A function $s \in B$ is extremal if and only if $s \in B_i$ for some i, $1 \le i \le 4$, and s is extremal in B_i.*

Proof. Let $s = \sum_{i=1}^4 s_i$. If s is extremal in B, then the non-zero $\{s_i\}$'s should be proportional. This implies, since $\{\Im_i\}$'s are mutually disjoint, that there can be only one non-zero s_i. That is, $s_i = s \in B_i$ for some i. ■

Thus, to get the form of the extremal elements in $B \subset F_D^+$, we should look for the extremal elements in each $B_i \subset \Im_i$.

LEMMA 5.8. *An extremal element $u(x,y)$ in B_1 is of the form $u(x,y) = \varphi(x)\psi(y)$ where $\varphi(x), \psi(y)$ are minimal harmonic functions in X, Y respectively, and $\varphi(x_0) = \psi(y_0) = 1$.*

Proof. (i) Suppose $u(x,y) = \varphi(x)\psi(y)$ where $\varphi(x)$ and $\psi(y)$ are minimal. If $h(x,y) \in B_1$ and $h(x,y) \le \varphi(x)\psi(y)$, then for fixed y, $h_y(x)/\psi(y) \le \varphi(x)$, which implies that $h_y(x)/\psi(y) = \alpha(y)\varphi(x)$ where $\alpha(y)$ is a constant depending only on y and $0 < \alpha(y) \le 1$. Similarly, $h^x(y)/\varphi(x) = \beta(x)\psi(y)$, $0 < \beta(x) \le 1$. Hence, $\alpha(y) = \beta(x)$ should be an absolute constant λ. Thus $h(x,y) = \lambda\varphi(x)\psi(y)$, which shows that $u(x,y)$ is extremal in B_1.

Conversely, let $u(x,y)$ be extremal in B_1. Then, for fixed y, there exists a unique measure v_y^u on the minimal boundary Λ_1 of X such that $u(x,y) = \int_{\Lambda_1} \varphi(x)\, dv_y^u(\varphi)$. Suppose the support of v_y^u contains two distinct elements φ', φ''. Then we can find two continuous functions f', f'' on the minimal boundary such that $f' = 1$ in a neighbourhood of φ' and $f'' = 1$ in a neighbourhood of φ'' and $f' + f'' = 1$. Then

$$u(x,y) = \int\limits_{\Lambda_1} f'(\varphi)\varphi(x)\,dv_y^u(\varphi) + \int\limits_{\Lambda_1} f''(\varphi)\varphi(x)\,dv_y^u(\varphi).$$

Since u is doubly median, v_y^u is harmonic in y (Proposition 4.5). Thus each one of the integrals on the right side represents a positive separately harmonic function in $X \times Y$, and hence should be proportional to $u(x,y)$, which is not possible. So we conclude that v_y^u has point support.

Suppose v_y^u has different point supports φ_1, φ_2 for two different vertices y_1, y_2 in Y. Take a continuous function g on φ_1 such that $g = 1$ in a neighbourhood of φ_1 and $g = 0$ in a neighbourhood of φ_2. Then $\int_{\Lambda_1} g(\varphi)\varphi(x)\,dv_y^u(\varphi)$ is a non-negative harmonic function on Y for fixed x, which is positive at the vertex y_1 and 0 at the vertex y_2, which is impossible. Hence v_y^u has the same point support for all y in Y.

This shows that $u(x,y) = \varphi(x)\psi(y)$ where $\varphi(x)$ is a positive minimal harmonic function in X and $\psi(y)$ is a positive function in Y. Now a separately harmonic function is harmonic so that $\Delta u = 0$; also $\Delta^x(\varphi) = 0$. Since $\Delta u = \psi(\Delta^x\varphi) + \varphi(\Delta_y\psi)$, we conclude that $\Delta_y\psi = 0$ in Y, so that $\psi(y)$ is harmonic in Y. In fact, $\psi(y)$ is a minimal harmonic function in Y. For suppose $w(y) > 0$ is harmonic in Y and $w(y) \le \psi(y)$. Then $\varphi(x)w(y)$ is separately harmonic and $\varphi(x)w(y) \le u(x,y)$. Hence $\varphi(x)w(y) = \lambda u(x,y)$ since $u(x,y)$ is a minimal separately harmonic function. Consequently, $w(y) = \lambda\psi(y)$ so that $\psi(y)$ is a minimal harmonic function in Y. ∎

Remark. (1) Suppose we say that a separately harmonic function $u(x,y) \ge 0$ is a minimal separately harmonic function if for any separately harmonic function $h(x,y)$ such that $0 \le h(x,y) \le u(x,y)$, we have $h(x,y) = \lambda u(x,y)$. Note that a minimal separately harmonic function need not be a minimal harmonic function. Thus what the above lemma says is that the extremal elements in B_1 are the same as the minimal separately harmonic functions in B_1.

(2) The above proof is modeled after a similar one given for minimal multiply harmonic functions in Brelot axiomatic potential theory by Gowrisankaran [12]. Here we have considered only minimal harmonic functions that are separately harmonic. Recall that a harmonic function in $X \times Y$ need not be separately harmonic. The following results in product Markov chains and product Riemannian manifolds are indicative of the modifications that are needed in the above lemma if we consider minimal harmonic functions in $X \times Y$.

Let $\{X, P = p(x,y)\}$ be a Markov chain where P is a stochastic transition operator on a time-homogeneous irreducible Markov chain with countable discrete state space X. For a real number t and a real-valued function $h(x)$

on X, h is said to be *t-harmonic* for P if $th(x) = \sum_y p(x,y)h(y)$ (that is, h is an eigenfunction on $\{X, P\}$ corresponding to the eigenvalue t). Let now $\{X, P_1\}$ and $\{X, P_2\}$ be two such Markov chains. In the Cartesian product $X \times Y$, define the transition Q_a, where $0 < a < 1$, by $Q_a = aP_1 + (1-a)P_2$. Then Picardello and Woess [20] prove that a positive function on $X \times Y$ is minimal t-harmonic for Q_a if and only if it can be written as $h_1(x)h_2(y)$ where $h_1(x)$ is minimal t_1-harmonic in $H^+_{t_1}(X, P_1)$ and $h_2(y)$ is minimal t_2-harmonic in $H^+_{t_2}(Y, P_2)$ with $at_1 + (1-a)t_2 = t$. Actually they go further to show that Cartesian products serve to construct examples where the Martin boundary M contains non-minimal elements even when the minimal boundary is closed in M.

Earlier, Molchanov [19] showed that minimal harmonic functions for a Markov chain which is the product of two Markov chains (with some restrictions) are products of minimal eigenfunctions for the corresponding chains. Related results in Freire [11] and Taylor [22] in the continuous setting of potential theory on Riemannian manifolds show the following: Let X, Y be two complete Riemannian manifolds. Let u, v be minimal eigenfunctions for Brownian motions on X, Y. Then uv is minimal for the Brownian motion for the product manifold $X \times Y$.

Lemma 5.9. *An extremal element in B_2 is of the form $h(x)G_z(y)$ where $h(x)$ is minimal harmonic in X, $h(x_0) = 1$, and $G_z(y)$ is the potential in Y with vertex harmonic support z in Y and $G_z(y_0) = 1$.*

Proof. (i) Let $u(x,y)$ be an extremal element in B_2. Since for fixed x, $u(x,y)$ is a potential in y, we have $u(x,y) = \sum_z v^u_x(z)G_z(y)$ and $G_z(y_0) = 1$. As already remarked, since $u \in \Im_2$, $v^u_x(z)$ is harmonic in x for fixed z. If in this expression, $v^u_x(z) \neq 0$ for $z = z_1$ and $z = z_2 \neq z_1$, then

$$u(x,y) = v^u_x(z_1)G_{z_1}(y) + \sum_{z \neq z_1} v^u_x(z)G_z(y).$$

Since the two terms on the right side are non-proportional and $u(x,y)$ is extremal, we conclude that $v^u_x(z) \neq 0$ for only one z which at the outset depends on x; actually, it is independent of x. For take two values x_1, x_2 of x such that $v^u_{x_1}(z_1) \neq 0$ and $v^u_{x_2}(z_2) \neq 0$. Suppose $z_1 \neq z_2$. Take a function $\varphi(z)$, $0 \leq \varphi(z) \leq 1$, such that $\varphi(z_1) = 1$ and $\varphi(z_2) = 0$, and consider $v(x,y) = \sum_z \varphi(z)v^u_x(z)G_z(y)$ which belongs to $\Im_2$. Hence for a fixed y, $v_y(x) = v(x,y)$ is a non-negative harmonic function in X. Now,

$$v_y(x_1) = \varphi(z_1)v^u_{x_1}(z_1)G_{z_1}(y) > 0, \qquad v_y(x_2) = \varphi(z_2)v^u_{x_2}(z_2)G_{z_2}(y) = 0.$$

This is not possible since $v_y(x)$ is a non-negative harmonic function in X. In conclusion, $u(x,y)$ should be of the form $u(x,y) = v^u_x(z)G_z(y)$ for a fixed z in Y, with $G_z(y_0) = 1$. Here, for fixed $z \in Y$, $G_z(y)$ is a potential in Y and $v^u_x(z)$ is harmonic in X. In fact, $v^u_x(z)$ is a minimal harmonic function in X. For suppose $0 < f(x) \le v_x^u(z)$, where $f(x)$ is harmonic in X. Then $f(x)G_z(y) \in \Im_2$ and

$$u(x,y) = f(x)G_z(y) + [v_x^u(z) - f(x)]G_z(y).$$

Now the two terms on the right side belong to $\Im_2$ and $u(x,y)$ is extremal in $\Im_2$. Hence, for some λ with $0 < \lambda \le 1$, $\lambda u(x,y) = f(x)G_z(y)$ so that $f(x) = \lambda v_x^u(z)$, which means that $v_x^u(z)$ is a minimal harmonic function in X.

(ii) Conversely, suppose $u \in B_2$ is of the form $u(x,y) = h(x)G_z(y)$ for some $z \in Y$, $h(x)$ being minimal harmonic in X, $h(x_0) = 1 = G_z(y_0)$. To show that $u(x,y)$ is extremal in B_2, let $v, w \in \Im_2$ be such that $u = v + w$. Since $v(x,y) \le u(x,y) = h(x)G_z(y)$, for a fixed y the function $v_y(x)/G_z(y)$ is harmonic in X, majorized by $h(x)$. Since $h(x)$ is minimal, we should have $v(x,y)/G_z(y) = \alpha(y)h(x)$ where $\alpha(y)$ is a constant $0 < \alpha(y) \le 1$, dependent on y. Hence $v(x,y) = \alpha(y)h(x)G_z(y)$.

Again, for fixed x, we have $v^x(y)/h(x) + w^x(y)/h(x) = G_z(y)$. Here, the two terms on the left side are potentials in Y, while the right side is a potential with harmonic support at the vertex z. Hence, $v^x(y)/h(x), w^x(y)/h(x)$ are also potentials with harmonic support at the same vertex z, so they are proportional to $G_z(y)$, the constant of proportionality depending on x only. That is, $v(x,y) = \beta(x)h(x)G_z(y)$. From the two expressions for $v(x,y)$, we conclude that $\alpha(y) = \beta(x) = \sigma$, an absolute constant. This shows that $u(x,y)$ is extremal in B_2. ∎

Finally, on the lines of Lemmas 5.8 and 5.9, we can prove the following two lemmas:

Lemma 5.10. *An extremal element u in B_3 is of the form $u(x,y) = G_\tau(x)v(y)$ where $G_\tau(x)$ is a potential in X with harmonic support at τ and with $G_\tau(x_0) = 1$, and $v(y)$ is minimal harmonic in Y with $v(y_0) = 1$.*

Lemma 5.11. *An extremal element u in B_4 is of the form $u(x,y) = G_\tau(x)G'_\rho(y)$ where $G_\tau(x)$ is a potential in X with harmonic support at τ and with $G_\tau(x_0) = 1$, and $G'_\rho(y)$ is a potential in Y with harmonic support at ρ and with $G'_\rho(y_0) = 1$.*

Actually Lemma 5.11 can be completed as follows:

Proposition 5.12. *A non-negative function u is in $\Im_4$ if and only if it is of the form $u(x,y) = \sum_{\xi\in X,\,\eta\in Y}\lambda(\xi,\eta)G_\xi(x)G'_\eta(y)$, where $\lambda(\xi,\eta)$'s are non-negative constants and $G_\xi(x), G'_\eta(y)$ are Green potentials in X, Y respectively, that is, $(-\Delta_1)G_\xi(x) = \delta_\xi(x)$ and $(-\Delta_2)G'_\eta(y) = \delta'_\eta(y)$.*

Proof. Suppose

$$u(x,y) = \sum_{\xi,\eta}\lambda(\xi,\eta)G_\xi(x)G'_\eta(y).$$

We know that $G_\xi(x)G'_\eta(y)$ is in $\Im_4$. Thus in the sum on the right side, each term is in $\Im_4$ and so is the convergent sum.

Conversely, suppose $u \in \Im_4$. Hence, for a fixed y, $u(x,y)$ is a potential in X, hence $u_y(x) = \sum_\xi v_y(\xi)G_\xi(x)$, and moreover $\varphi(x) = u(x,y) - \sum_\beta u(x,\beta)P_B(y,\beta)$ is a superharmonic function in X (Proposition 4.3). Now for any β, $u(x,\beta)$ is a potential in X. Hence the superharmonic function $\varphi(x)$ is the difference of two potentials in X, so $\varphi(x)$ itself is a potential in X. Write $\varphi(x) = \sum_{\xi\in X} k(\xi)G_\xi(x)$. Then

$$\begin{aligned}\sum_\xi k(\xi)G_\xi(x) &= \sum_\xi v_y(\xi)G_\xi(x) - \sum_\beta\Big[\sum_\xi v_\beta(\xi)G_\xi(x)\Big]P_B(y,\beta)\\ &= \sum_\xi\Big[v_y(\xi) - \sum_\beta v_\beta(\xi)P_B(y,\beta)\Big]G_\xi(x).\end{aligned}$$

This implies, for every $\xi \in X$, $v_y(\xi) - \sum_\beta v_\beta(\xi)P_B(y,\beta) = k(\xi) \geq 0$. That is, for any fixed $\xi \in X$, $v_y(\xi)$ is a non-negative superharmonic function in Y.

Hence, for fixed x, $\sum_\xi v_y(\xi)G_\xi(x)$ is the sum of a series of non-negative superharmonic functions in Y, the sum being $u(x,y)$ which is a potential in Y. The implication is that for fixed x,ξ, $v_y(\xi)G_\xi(x)$ is a potential in Y and in particular $v_y(\xi)$ is a potential in y, for fixed ξ. Write

$$v_y(\xi) = \sum_{\eta\in Y} \lambda(\xi,\eta)G'_\eta(y),$$

where $\lambda(\xi,\eta)$ is a non-negative constant and $G'_\eta(y)$ is the Green potential in Y with harmonic support η. Thus, we finally have the expansion

$$u(x,y) = \sum_{\xi\in X}\Big[\sum_{\eta\in Y}\lambda(\xi,\eta)G'_\eta(y)\Big]G_\xi(x) = \sum_{\xi,\eta}\lambda(\xi,\eta)G_\xi(x)G'_\eta(y). \ \blacksquare$$

Remark. (i) When $u(x,y)$ has the above series expansion,

$$\lambda(\xi,\eta) = \mathit{\Delta}_1\mathit{\Delta}_2 u(\xi,\eta) \quad \text{since} \quad (-\mathit{\Delta}_1)(-\mathit{\Delta}_2)G_\xi(x)G'_\eta(y) = \delta_\xi(x)\delta'_\eta(y).$$

(ii) A little modification at the end of the above proof gives the following representation for functions in $\Im_3$: A real-valued function u is in $\Im_3$ if and only if it is of the form $u(x,y) = \sum_{\xi\in X} v_y(\xi)G_\xi(x)$, where for $\xi \in X$, $v_y(\xi)$ is a non-negative harmonic function in Y.

Integral representation. We have seen so far that the cone $\Im_D^+$ is a lattice for its own order and is provided with a topology for which the base $B = \{u \in \Im_D^+ : u(x_0,y_0) = 1\}$ is compact and metrisable. Hence, a recourse to the Choquet theorem gives an integral representation for functions in $\Im_D^+$. Some details about the extremal elements and the unique representing measure are given in the following:

THEOREM 5.13. *Let $u(x,y) \in \Im_D^+$. Then there exist four uniquely determined measures, μ_1 with support in $\Lambda_1 \times \Lambda_2$, μ_2 with support in $\Lambda_1 \times (\mathfrak{E}_2 \setminus \Lambda_2)$, μ_3 with support in $(\mathfrak{E}_1 \setminus \Lambda_1) \times \Lambda_2$, μ_4 with support in $(\mathfrak{E}_1 \setminus \Lambda_1) \times (\mathfrak{E}_2 \setminus \Lambda_2)$ such that if $\mu = \sum_{i=1}^4 \mu_i$, then for any $(x,y) \in X \times Y$,*

$$u(x,y) = \int_{\mathfrak{E}_1 \times \mathfrak{E}_2} v(x)v'(y)\, d\mu(v,v').$$

As a converse to the above theorem, we prove the following:

THEOREM 5.14. *Suppose for some measure λ on $\mathfrak{E}_1 \times \mathfrak{E}_2$,*

$$u(x,y) = \int_{\mathfrak{E}_1 \times \mathfrak{E}_2} f(x)g(y)\, d\lambda(f,g)$$

is finite for some $(\xi,\eta) \in X \times Y$. Then $u(x,y) \in \Im_D^+$.

Proof. Let (x_0,y_0) be an arbitrary vertex in $X \times Y$. Since $f(x)$ is a positive superharmonic function on X, there is a constant C_1 such that $f(x_0) \le C_1 f(\xi)$ for all $f \in \mathfrak{E}_1$ (Harnack property). Similarly, there is a constant C_2 such that $g(y_0) \le C_2 g(\eta)$ for all $g \in \mathfrak{E}_2$. Consequently,

$$u(x_0,y_0) = \int_{\mathfrak{E}_1 \times \mathfrak{E}_2} f(x_0)g(y_0)\, d\lambda(f,g) \le C_1C_2 \int_{\mathfrak{E}_1 \times \mathfrak{E}_2} f(\xi)g(\eta)\, d\lambda(f,g) < \infty.$$

Thus, $u(x,y)$ is real-valued for all $(x,y) \in X \times Y$. Moreover, $u(x,y)$ is separately superharmonic in $X \times Y$, for if $y \in Y$ is fixed, then

$$\begin{aligned} u_y(x) &= \int_{\mathfrak{E}_1 \times \mathfrak{E}_2} f(x)g(y)\, d\lambda(f,g) \\ &\ge \int_{\mathfrak{E}_1 \times \mathfrak{E}_2} \Big[\sum_\alpha f(\alpha)P_A(x,\alpha)\Big] g(y)\, d\lambda(f,g) \\ &= \sum_\alpha \Big[\int_{\mathfrak{E}_1 \times \mathfrak{E}_2} f(\alpha)g(y)\, d\lambda(f,g)\Big] P_A(x,\alpha) \\ &= \sum_\alpha u_y(\alpha)P_A(x,\alpha). \end{aligned}$$

Hence, for fixed y in Y, $u_y(x)$ is superharmonic in X, and similarly for fixed x, $u^x(y)$ is superharmonic in Y. We now show that in fact $u \in \Im_D^+$.

Indeed, $f(x) - \sum_\alpha f(\alpha)P_A(x,\alpha) \ge 0$ and $g(y) - \sum_\beta g(\beta)P_B(y,\beta) \ge 0$ imply that

$$\begin{aligned} f(x)g(y) &+ \sum_{\alpha,\beta} f(\alpha)g(\beta)P_A(x,\alpha)P_B(y,\beta) \\ &\ge \sum_\alpha f(\alpha)g(y)P_A(x,\alpha) + \sum_\beta f(x)g(\beta)P_B(y,\beta). \end{aligned}$$

Integrating both sides with respect to the measure λ, we get

$$u(x,y)+\sum_{\alpha,\beta}u(\alpha,\beta)P_A(x,\alpha)P_B(y,\beta) \geq \sum_{\alpha}u(\alpha,y)P_A(x,\alpha)+\sum_{\beta}u(x,\beta)P_B(y,\beta).$$

That is, $u \in \Im_D^+$. ∎

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