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Inequalities for radial Blaschke–Minkowski homomorphisms

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Abstract. We establish L_p Brunn–Minkowski type inequalities for radial Blaschke–Minkowski homomorphisms, which in special cases yield some new results for intersection bodies. Moreover, we obtain two monotonicity inequalities for radial Blaschke–Minkowski homomorphisms.

1. Introduction. The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 3$). Let $\mathcal{S}^n$ denote the set of *star bodies* (i.e., compact sets, star shaped with respect to the origin) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and B for the unit ball centered at the origin. The surface of B is S^{n-1}. We shall use $V(K)$ for the n-dimensional volume of the body K.

In 1988, the notion of intersection body was explicitly defined by Lutwak in the important paper [Lu2]. It is a central notion in dual Brunn–Minkowski theory. During the past 30 years, significant progress has been made in the understanding of the intersection body operator and the class of intersection bodies by Gardner, Grinberg, Goodey, Koldobsky, Ludwig, Lutwak, Zhang and others (see e.g. [G1, G2, GKS, GLW, H, K1, K2, LZ, L3, Lu1, Zh]).

In recent years real valued valuations have received much attention in a series of articles by Ludwig [L1, L2, L3]. She proved that the intersection operator is the only nontrivial $\mathrm{GL}(n)$ (the group of general linear transformations) contravariant radial valuation. Radial Blaschke–Minkowski homomorphisms were introduced by Schuster [S1], and further investigated in [ABS, Liu, S2, W, WLH, Z1, Z2, Z3, ZC]. The intersection operator is an example of a radial Blaschke–Minkowski homomorphism.

A map $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ is called a *radial Blaschke–Minkowski homomorphism* if it satisfies the following conditions:

(1) Ψ is continuous.

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(2) $\Psi(K)$ is radial Blaschke–Minkowski additive, i.e., $\Psi(K \check{+} L) = \Psi K \tilde{+} \Psi L$ for all $K, L \in \mathcal{S}^n$.
(3) Ψ intertwines rotations, i.e., $\Psi(\vartheta K) = \vartheta \Psi K$ for all $K \in \mathcal{S}^n$ and all $\vartheta \in \mathrm{SO}(n)$.

Here, $\check{+}$ denotes the radial Blaschke sum, and $\tilde{+}$ denotes the radial Minkowski sum. $\mathrm{SO}(n)$ is the group of rotations in n-dimensional space.

In [S1], Schuster established the following Brunn–Minkowski inequality for radial Blaschke–Minkowski homomorphisms of star bodies.

Theorem 1.A. *If $K, L \in \mathcal{S}^n$, then*

$$V(\Psi(K \tilde{+} L))^{\frac{1}{n(n-1)}} \le V(\Psi K)^{\frac{1}{n(n-1)}} + V(\Psi L)^{\frac{1}{n(n-1)}}, \tag{1.1}$$

with equality if and only if K and L are dilates of each other.

In fact, associated with the dual quermassintegrals $\widetilde{W}_i(K)$ of $K \in \mathcal{S}^n$, Schuster established a more general version of the Brunn–Minkowski inequality for radial Blaschke–Minkowski homomorphisms: If $K, L \in \mathcal{S}^n$ and integers i, j satisfy $0 \le i \le n-1$, $0 \le j \le n-2$, then

$$\widetilde{W}_i(\Psi_j(K \tilde{+} L))^{\frac{1}{(n-i)(n-j-1)}} \le \widetilde{W}_i(\Psi_j K)^{\frac{1}{(n-i)(n-j-1)}} + \widetilde{W}_i(\Psi_j L)^{\frac{1}{(n-i)(n-j-1)}}, \tag{1.2}$$

with equality if and only if K and L are dilates of each other. Here Ψ_j are the mixed radial Blaschke–Minkowski homomorphisms which are defined as follows [S1]:

Theorem 1.B. *There is a continuous operator*

$$\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$$

symmetric in its arguments such that, for $K_1, \ldots, K_m \in \mathcal{S}^n$ and $\lambda_1, \ldots, \lambda_m \ge 0$,

$$\Psi(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_m K_m) = \sum_{i_1, \ldots, i_{n-1}}^{\sim} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Psi(K_{i_1}, \ldots, K_{i_{n-1}}), \tag{1.3}$$

where the sum is with respect to radial Minkowski addition.

Clearly, Theorem 1.B generalizes the notion of radial Blaschke–Minkowski homomorphisms and we will call the map $\Psi : \mathcal{S}^n \times \cdots \times \mathcal{S}^n \to \mathcal{S}^n$ the *mixed radial Blaschke–Minkowski homomorphism* induced by Ψ. For $K, L \in \mathcal{S}^n$, let $\Psi_i(K, L)$ denote the mixed radial Blaschke–Minkowski homomorphism $\Psi_i(K, \ldots, K, L, \ldots, L)$, with i copies of L and $n-i-1$ copies of K. We write $\Psi_i(K)$ for $\Psi_i(K, \ldots, K, B, \ldots, B)$ and call $\Psi_i(K)$ the *radial Blaschke–Minkowski homomorphism of order i.*

The main goal of this paper is to prove L_p Brunn–Minkowski type inequalities for radial Blaschke–Minkowski homomorphisms. Associated with L_p harmonic radial linear combination, we obtain

THEOREM 1.1. *If $K, L \in \mathcal{S}^n$, and $p \geq 1$, then*

$$V(\Psi(K \hat{+}_p L))^{-\frac{p}{n(n-1)}} \geq V(\Psi K)^{-\frac{p}{n(n-1)}} + V(\Psi L)^{-\frac{p}{n(n-1)}}. \tag{1.4}$$

with equality if and only if K and L are dilates of each other, where $\hat{+}_p$ is the L_p harmonic radial sum.

The next theorem shows that radial Blaschke–Minkowski homomorphisms satisfy the L_p Brunn–Minkowski inequality for the L_p radial Blaschke linear combination.

THEOREM 1.2. *If $K, L \in \mathcal{S}^n$ and $1 \leq p < n$, then*

$$V(\Psi(K \check{+}_p L))^{\frac{n-p}{n(n-1)}} \leq V(\Psi K)^{\frac{n-p}{n(n-1)}} + V(\Psi L)^{\frac{n-p}{n(n-1)}}. \tag{1.5}$$

with equality if and only if K and L are dilates of each other, where $\check{+}_p$ is the L_p radial Blaschke sum.

Since the intersection operator I is an example of a radial Blaschke–Minkowski homomorphism, Theorems 1.1 and 1.2 provide the following new dual L_p Brunn–Minkowski inequalities for the volume of intersection bodies:

COROLLARY 1.3. *If $K, L \in \mathcal{S}^n$ and $p \geq 1$, then*

$$V(\mathrm{I}(K \hat{+}_p L))^{-\frac{p}{n(n-1)}} \geq V(\mathrm{I}K)^{-\frac{p}{n(n-1)}} + V(\mathrm{I}L)^{-\frac{p}{n(n-1)}}, \tag{1.6}$$

with equality if and only if K and L are dilates of each other.

COROLLARY 1.4. *If $K, L \in \mathcal{S}^n$ and $1 \leq p < n$, then*

$$V(\mathrm{I}(K \check{+}_p L))^{\frac{n-p}{n(n-1)}} \leq V(\mathrm{I}K)^{\frac{n-p}{n(n-1)}} + V(\mathrm{I}L)^{\frac{n-p}{n(n-1)}}, \tag{1.7}$$

with equality if and only if K and L are dilates of each other.

In fact, more general L_p Brunn–Minkowski type inequalities than Theorems 1.1 and 1.2 are established in Section 3. In Section 4, we establish two monotonicity inequalities for mixed radial Blaschke–Minkowski homomorphisms.

2. Notation and background material

2.1. Basic notions. We recall some preliminary properties of star bodies and convex bodies; for general references consult the books of Gardner [G3] and Schneider [Sc].

For a compact subset K in $\mathbb{R}^n$ which is star shaped with respect to the origin, its *radial function* $\rho(K, \cdot) : S^{n-1} \to \mathbb{R}$ is defined by

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}. \tag{2.1}$$

If $\rho(K,\cdot)$ is positive and continuous, then K will be called a *star body*. Let δ denote the *radial Hausdorff metric*, i.e., if $K,L\in\mathcal{S}^n$, then $\delta(K,L)=|\rho(K,\cdot)-\rho(L,\cdot)|_\infty$.

For $K,L\in\mathcal{S}^n$ and $\lambda_1,\lambda_2\ge 0$ (not both 0), the radial Minkowski linear combination $\lambda_1K\,\tilde{+}\,\lambda_2L$ is the star body defined by

$$\rho(\lambda_1K\,\tilde{+}\,\lambda_2L,u)=\lambda_1\rho(K,u)+\lambda_2\rho(L,u). \tag{2.2}$$

For real $p\ge 1$, $K,L\in\mathcal{S}^n$ and $\lambda_1,\lambda_2\ge 0$ (not both 0), Firey [F] defined the *L_p harmonic radial linear combination*, $\lambda_1\cdot K\,\hat{+}_p\,\lambda_2\cdot L$, of K and L by

$$\rho(\lambda_1\cdot K\,\hat{+}_p\,\lambda_2\cdot L,u)^{-p}=\lambda_1\rho(K,u)^{-p}+\lambda_2\rho(L,u)^{-p}. \tag{2.3}$$

Note that the restriction to $p\ge 1$ is necessary.

For $K,L\in\mathcal{S}^n$ and $\lambda_1,\lambda_2\ge 0$ (not both 0), Lutwak [Lu2] defined the *radial Blaschke linear combination*, $\lambda_1 * K\,\check{+}\,\lambda_2 * L$, of K and L by

$$\rho(\lambda_1 * K\,\check{+}\,\lambda_2 * L,u)^{n-1}=\lambda_1\rho(K,u)^{n-1}+\lambda_2\rho(L,u)^{n-1}. \tag{2.4}$$

Here, we extend the radial Blaschke linear combination to the L_p case: If $K,L\in\mathcal{S}^n$, $\lambda_1,\lambda_2\ge 0$ (not both 0) and $p\ge 1$, we introduce the *L_p radial Blaschke linear combination*, $\lambda_1\circ K\,\check{+}_p\,\lambda_2\circ L$, as the star body whose radial function is given by

$$\rho(\lambda_1\circ K\,\check{+}_p\,\lambda_2\circ L,u)^{n-p}=\lambda_1\rho(K,u)^{n-p}+\lambda_2\rho(L,u)^{n-p}. \tag{2.5}$$

In the special case $p=1$, (2.5) is just the classical radial Blaschke linear combination.

2.2. Dual mixed volumes. If $K_1,\dots,K_m\in\mathcal{S}^n$ and $\lambda_1,\dots,\lambda_m\ge 0$ (not all 0), the volume of $\lambda_1K_1\,\tilde{+}\cdots\tilde{+}\,\lambda_mK_m$ is a homogeneous polynomial of degree n:

$$V(\lambda_1K_1\,\tilde{+}\cdots\tilde{+}\,\lambda_mK_m)=\sum_{i_1,\dots,i_n}\widetilde{V}(K_{i_1},\dots,K_{i_n})\lambda_{i_1}\cdots\lambda_{i_n}. \tag{2.6}$$

The coefficients $\widetilde{V}(K_{i_1},\dots,K_{i_n})$ are called the *dual mixed volumes* of $K_{i_1},\dots,K_{i_n}$. They are continuous, non-negative, symmetric and monotone (with respect to set inclusion). For simplicity, let $dS(u)=dS^{n-1}$ be the spherical Lebesgue measure of S^{n-1}. We have the following integral representation of the dual mixed volumes:

$$\widetilde{V}(K_1,\dots,K_n)=\frac{1}{n}\int_{S^{n-1}}\rho(K_1,u)\cdots\rho(K_n,u)\,dS(u). \tag{2.7}$$

For $K,L\in\mathcal{S}^n$ and $i=0,1,\dots,n-1$, we write $\widetilde{W}_i(K,L)$ for $\widetilde{V}(K,\dots,K,B,\dots,B,L)$, where K appears $n-i-1$ times, B appears i times and L appears once. The dual mixed volume $\widetilde{W}_i(K,K)$ will be written as $\widetilde{W}_i(K)$

and it is called the *dual quermassintegral* of K. From (2.6), it follows that

$$\widetilde{W}_i(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n-i-1}\rho(L,u)\,dS(u). \tag{2.8}$$

We will use the following dual Minkowski inequality [Lu1]: If integers i satisfy $0 \le i \le n-2$, then

$$\widetilde{W}_i(K,L)^{n-i} \le \widetilde{W}_i(K)^{n-i-1}\widetilde{W}_i(L), \tag{2.9}$$

with equality if and only if K and L are dilates of each other.

2.3. Intersection body. For $K \in \mathcal{S}^n$, there is a unique star body $\mathrm{I}K$ whose radial function satisfies, for $u \in S^{n-1}$,

$$\rho(\mathrm{I}K,u) = v(K \cap u^{\perp}), \tag{2.10}$$

where v is the $(n-1)$-dimensional volume and $u^{\perp}$ is the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to u. It is called the *intersection body* of K (see [Lu2]).

By using the polar coordinate formula for volume, it is trivial to verify that

$$\rho(\mathrm{I}K,u) = \frac{1}{n-1} \int_{S^{n-1}\cap u^{\perp}} \rho(K,u)^{n-1}\,dS(u). \tag{2.11}$$

An important generalization of this notion is the mixed intersection body (see [LZ]). The *mixed intersection body* of the bodies $K_1,\dots,K_{n-1} \in \mathcal{S}^n$, $\mathrm{I}(K_1,\dots,K_{n-1})$, is defined by

$$\rho(\mathrm{I}(K_1,\dots,K_{n-1}),u) = \widetilde{v}(K_1 \cap u^{\perp},\dots,K_{n-1} \cap u^{\perp}), \tag{2.12}$$

where $\widetilde{v}$ denotes the $(n-1)$-dimensional dual mixed volume. If $K_1 = \dots = K_{n-i-1} = K, K_{n-i} = \dots = K_{n-1} = L$, $i = 0,1,\dots,n-1$, then

$$\mathrm{I}(\underbrace{K,\dots,K}_{n-i-1},\underbrace{L,\dots,L}_{i})$$

is usually written as $\mathrm{I}_i(K,L)$.

3. L_p Brunn–Minkowski type inequalities for radial Blaschke–Minkowski homomorphisms. In this section we will prove Theorems 1.1 and 1.2. The following lemmas are needed.

LEMMA 3.1 ([S1]). *Let $\varPsi : \underbrace{\mathcal{S}^n \times \dots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ be a mixed radial Blaschke–Minkowski homomorphism. If $K,L \in \mathcal{S}^n$ integers i,j satisfy $0 \le i \le n-2$ and $0 \le j \le n-2$, then*

$$\widetilde{W}_i(K,\varPsi_j L) = \widetilde{W}_j(L,\varPsi_i K). \tag{3.1}$$

Note that the image of the unit ball under a radial Blaschke–Minkowski homomorphism Ψ is again a ball. Let r_Ψ denote the radius of this ball, $j = n-2$ and $L = B$. From (3.1), we get

$$\widetilde{W}_{n-1}(\Psi_i(K)) = r_\Psi \widetilde{W}_{i+1}(K). \tag{3.2}$$

Lemma 3.2. *Let $K, L, Q \in \mathcal{S}^n$ and i is an integer with $0 \le i \le n-2$. If $p \ge 1$, then*

$$\widetilde{W}_i(K \hat{+}_p L, Q)^{-\frac{p}{n-i-1}} \ge \widetilde{W}_i(K, Q)^{-\frac{p}{n-i-1}} + \widetilde{W}_i(L, Q)^{-\frac{p}{n-i-1}}, \tag{3.3}$$

with equality if and only if K and L are dilates of each other.

Proof. From (2.8), (2.3) and the Minkowski integral inequality with the condition $-\frac{p}{n-i-1} < 0$, it follows that

$$\begin{aligned}
\widetilde{W}_i(K \hat{+}_p L, Q)^{-\frac{p}{n-i-1}} &= \Bigg[\frac{1}{n}\int\limits_{S^{n-1}} \rho(K \hat{+}_p L, u)^{n-i-1}\rho(Q, u)\,du\Bigg]^{-\frac{p}{n-i-1}} \\
&= \Bigg\{\frac{1}{n}\int\limits_{S^{n-1}} [\rho(K, u)^{-p} + \rho(L, u)^{-p}]^{-\frac{n-i-1}{p}}\rho(Q, u)\,du\Bigg\}^{-\frac{p}{n-i-1}} \\
&\ge \Bigg[\frac{1}{n}\int\limits_{S^{n-1}} \rho(K, u)^{n-i-1}\rho(Q, u)\,du\Bigg]^{-\frac{p}{n-i-1}} \\
&\quad + \Bigg[\frac{1}{n}\int\limits_{S^{n-1}} \rho(L, u)^{n-i-1}\rho(Q, u)\,du\Bigg]^{-\frac{p}{n-i-1}} \\
&= \widetilde{W}_i(K, Q)^{-\frac{p}{n-i-1}} + \widetilde{W}_i(L, Q)^{-\frac{p}{n-i-1}}.
\end{aligned}$$

Equality holds in (3.3) if and only if the functions $\rho(K, \cdot)$ and $\rho(L, \cdot)$ are positively proportional, $\rho(K, \cdot) = \lambda\rho(L, \cdot)$. Hence, equality holds if and only if K and L are dilates of each other. ∎

A generalization of Theorem 1.1 is

Theorem 3.3. *Let $\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ be a mixed radial Blaschke–Minkowski homomorphism. If $K, L \in \mathcal{S}^n$ and integers i, j satisfy $0 \le i, j \le n-2$ and $p \ge 1$, then*

$$\begin{aligned}
&\widetilde{W}_i(\Psi_j(K \hat{+}_p L))^{-\frac{p}{(n-i)(n-j-1)}} \\
&\qquad \ge \widetilde{W}_i(\Psi_j K)^{-\frac{p}{(n-i)(n-j-1)}} + \widetilde{W}_i(\Psi_j L)^{-\frac{p}{(n-i)(n-j-1)}},
\end{aligned} \tag{3.4}$$

with equality if and only if K and L are dilates of each other.

Proof. Suppose N, K, L are star bodies. Let $p \ge 1$. By (3.1), (3.3) and (3.1) again and (2.9), we have

$$\begin{aligned}\widetilde{W}_i(N,\Psi_j(K \hat{+}_p L))^{-\frac{p}{n-j-1}} &= \widetilde{W}_j(K \hat{+}_p L,\Psi_i N)^{-\frac{p}{n-j-1}}\\ &\geq \widetilde{W}_j(K,\Psi_i N)^{-\frac{p}{n-j-1}} + \widetilde{W}_j(L,\Psi_i N)^{-\frac{p}{n-j-1}} \quad (3.5)\\ &= \widetilde{W}_i(N,\Psi_j K)^{-\frac{p}{n-j-1}} + \widetilde{W}_i(N,\Psi_j L)^{-\frac{p}{n-j-1}}\\ &\geq \widetilde{W}_i(N)^{-\frac{(n-i-1)p}{(n-i)(n-j-1)}}[\widetilde{W}_i(\Psi_j K)^{-\frac{p}{(n-i)(n-j-1)}} \quad (3.6)\\ &\quad + \widetilde{W}_i(\Psi_j L)^{-\frac{p}{(n-i)(n-j-1)}}].\end{aligned}$$

Set $N = \Psi_j(K \hat{+}_p L)$, note that $\widetilde{W}_i(N,N) = \widetilde{W}_i(N)$ and rearrange the above inequality to obtain (3.4).

By the equality conditions of (3.5) and (3.6), equality in (3.4) holds if and only if K and L are dilates of each other. ∎

Obviously, taking $i, j = 0$ in Theorem 3.3, we obtain Theorem 1.1.

LEMMA 3.4. *Let $K, L, Q \in \mathcal{S}^n$ and $0 \leq i \leq n-2$. If $i+1 \leq p < n$, then*

$$\widetilde{W}_i(K \check{+}_p L, Q)^{\frac{n-p}{n-i-1}} \leq \widetilde{W}_i(K,Q)^{\frac{n-p}{n-i-1}} + \widetilde{W}_i(L,Q)^{\frac{n-p}{n-i-1}}, \tag{3.7}$$

with equality if and only if K and L are dilates of each other.

Proof. From (2.8), (2.5) and the Minkowski integral inequality with $\frac{n-i-1}{n-p} \geq 1$ (i.e., $i+1 \leq p < n$), it follows that

$$\begin{aligned}\widetilde{W}_i(K \check{+}_p L, Q)^{\frac{n-p}{n-i-1}} &= \Big[\frac{1}{n}\int_{S^{n-1}} \rho(K \check{+}_{n-p} L, u)^{n-i-1}\rho(Q,u)du\Big]^{\frac{n-p}{n-i-1}}\\ &= \Big\{\frac{1}{n}\int_{S^{n-1}} [\rho(K,u)^{n-p} + \rho(L,u)^{n-p}]^{\frac{n-i-1}{n-p}}\rho(Q,u)du\Big\}^{\frac{n-p}{n-i-1}}\\ &\leq \Big[\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i-1}\rho(Q,u)du\Big]^{\frac{n-p}{n-i-1}}\\ &\quad + \Big[\frac{1}{n}\int_{S^{n-1}} \rho(L,u)^{n-i-1}\rho(Q,u)du\Big]^{\frac{n-p}{n-i-1}}\\ &= \widetilde{W}_i(K,Q)^{\frac{n-p}{n-i-1}} + \widetilde{W}_i(L,Q)^{\frac{n-p}{n-i-1}},\end{aligned}$$

with equality if and only if K and L are dilates of each other. The above inequality gives (3.7). ∎

A generalization of Theorem 1.2 is

THEOREM 3.5. *Let $\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n$ be a mixed radial Blaschke–Minkowski homomorphism. If $K, L \in \mathcal{S}^n$ and integers i, j satisfy $0 \leq i, j \leq$*

$n-2$ and $j+1 \le p < n$, then

$$\widetilde{W}_i(\Psi_j(K \check{+}_p L))^{\frac{n-p}{(n-i)(n-j-1)}} \le \widetilde{W}_i(\Psi_j K)^{\frac{n-p}{(n-i)(n-j-1)}} + \widetilde{W}_i(\Psi_j L)^{\frac{n-p}{(n-i)(n-j-1)}}, \tag{3.8}$$

with equality if and only if K and L are dilates of each other.

Proof. Suppose N, K, L are star bodies. Let $j+1 \le p < n$. Using (3.1), (3.7), (3.1) again and (2.9), we have

$$\begin{aligned}
\widetilde{W}_i(N, \Psi_j(K \check{+}_{n-p} L))^{\frac{n-p}{n-j-1}} &= \widetilde{W}_j(K \check{+}_p L, \Psi_i N)^{\frac{n-p}{n-j-1}} \\
&\le \widetilde{W}_j(K, \Psi_i N)^{\frac{n-p}{n-j-1}} + \widetilde{W}_j(L, \Psi_i N)^{\frac{n-p}{n-j-1}} \\
&= \widetilde{W}_i(N, \Psi_j K)^{\frac{n-p}{n-j-1}} + \widetilde{W}_i(N, \Psi_j L)^{\frac{n-p}{n-j-1}} \\
&\le \widetilde{W}_i(N)^{\frac{(n-i-1)(n-p)}{(n-i)(n-j-1)}} [\widetilde{W}_i(\Psi_j K)^{\frac{n-p}{(n-i)(n-j-1)}} \\
&\quad + \widetilde{W}_i(\Psi_j L)^{\frac{n-p}{(n-i)(n-j-1)}}].
\end{aligned}$$

Set $N = \Psi_j(K \check{+}_p L)$, note that $\widetilde{W}_i(N, N) = \widetilde{W}_i(N)$ and rearrange to obtain

$$\widetilde{W}_i(\Psi_j(K \check{+}_p L))^{\frac{n-p}{(n-i)(n-j-1)}} \le \widetilde{W}_i(\Psi_j K)^{\frac{n-p}{(n-i)(n-j-1)}} + \widetilde{W}_i(\Psi_j L)^{\frac{n-p}{(n-i)(n-j-1)}},$$

with equality if and only if K and L are dilates of each other. ■

Obviously, taking $i, j = 0$ in Theorem 3.3, we obtain Theorem 1.2.

The following lemma is the representation for radial Blaschke–Minkowski homomorphisms:

LEMMA 3.6 ([S1]). *A map $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ is a radial Blaschke–Minkowski homomorphism if and only if there is a measure $\mu \in \mathcal{M}_+(S^{n-1}, \hat{e})$ such that*

$$\rho(\Psi K, \cdot) = \rho^{n-1}(K, \cdot) * \mu, \tag{3.9}$$

where $\mathcal{M}_+(S^{n-1}, \hat{e})$ denotes the set of nonnegative zonal measures on S^{n-1}.

From Lemma 3.6 and (2.11), we can see that the intersection body operator is a radial Blaschke–Minkowski homomorphism. The generating measure of the intersection body operator I is the invariant measure $\mu_{S_o^{n-2}}$ which is concentrated on $S_o^{n-2} := S^{n-1} \cap \hat{e}^{\perp}$, i.e.,

$$\rho(\mathrm{I}K, \cdot) = \rho^{n-1}(K, \cdot) * \mu_{S_o^{n-2}}. \tag{3.10}$$

Thus, taking $i, j = 0$ and changing Ψ to the intersection body operator I in Theorems 3.3 and 3.5 we obtain the dual L_p Brunn–Minkowski inequalities for the volume of intersection bodies, presented as Corollaries 1.3 and 1.4 in the introduction.

4. Monotonicity inequalities. In the following we establish the monotonicity inequalities for mixed radial Blaschke–Minkowski homomorphisms of star bodies.

THEOREM 4.1. *If $K, L \in \mathcal{S}^n$ are such that $\widetilde{W}_i(K,Q) \le \widetilde{W}_i(L,Q)$ for $i = 0, 1, \ldots, n-2$, then*

$$\widetilde{W}_i(\Psi_i K) \le \widetilde{W}_i(\Psi_i L), \tag{4.1}$$

with equality if and only if $\Psi_i K = \Psi_i L$.

Proof. Since $K, L \in \mathcal{S}^n$ and $\widetilde{W}_i(K,Q) \le \widetilde{W}_i(L,Q)$, taking $Q = \Psi_i(\Psi_i K)$, and using (3.1), we get

$$\widetilde{W}_i(K, \Psi_i(\Psi_i K)) = \widetilde{W}_i(\Psi_i K) \le \widetilde{W}_i(\Psi_i K, \Psi_i L). \tag{4.2}$$

Now by using the dual Minkowski inequality (2.9) with the equality condition, we immediately get the desired result. ∎

THEOREM 4.2. *If $K \in \mathcal{S}^n$ and $i = 0, 1, \ldots, n-2$, then*

$$\frac{\widetilde{W}_{n-i-1}(\Psi_i K)}{\widetilde{W}_{n-i-1}(K)^i} \le \frac{\widetilde{W}_{n-i-1}(\Psi_i^2 K)}{\widetilde{W}_{n-i-1}(\Psi_i K)^i}, \tag{4.3}$$

with equality if and only if K and $\Psi_i^2 K$ are dilates of each other.

Proof. Let $Q, K \in \mathcal{S}^n$. By (3.1) and the dual Minkowski inequality (2.9), it follows that

$$\begin{aligned} \widetilde{W}_{n-i-1}(Q, \Psi_i K)^{i+1} &= \widetilde{W}_{n-i-1}(K, \Psi_i Q)^{i+1} \\ &\le \widetilde{W}_{n-i-1}(K)^i \widetilde{W}_{n-i-1}(\Psi_i Q), \end{aligned} \tag{4.4}$$

with equality if and only if K and $\Psi_i Q$ are dilates of each other. Taking $Q = \Psi_i K$, we obtain

$$\widetilde{W}_{n-i-1}(\Psi_i K)^{i+1} \le \widetilde{W}_{n-i-1}(K)^i \widetilde{W}_{n-i-1}(\Psi_i^2 K), \tag{4.5}$$

i.e.,

$$\frac{\widetilde{W}_{n-i-1}(\Psi_i K)}{\widetilde{W}_{n-i-1}(K)^i} \le \frac{\widetilde{W}_{n-i-1}(\Psi_i^2 K)}{\widetilde{W}_{n-i-1}(\Psi_i K)^i},$$

with equality if and only if K and $\Psi_i^2 K$ are dilates of each other. ∎

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