ANNALES
POLONICI MATHEMATICI
113.3 (2015)

Rotation surfaces with L_1-pointwise 1-type Gauss map in pseudo-Galilean space

by Dae Won Yoon (Jinju), Young Ho Kim (Daegu)
and Jae Seong Jung (Jinju)

Abstract. We study rotation surfaces in the three-dimensional pseudo-Galilean space G_3^1 such that the Gauss map G satisfies the condition $L_1 G = f(G + C)$ for a smooth function f and a constant vector C, where L_1 is the Cheng–Yau operator.

1. Introduction. The Gauss map remains an interesting object in Euclidean space and pseudo-Euclidean space and it has been investigated from various viewpoints by many differential geometers [1], [6], [7], [9], [11], [16], etc.

If the Gauss map G of a surface M satisfies

$$\Delta G = \lambda(G + C) \tag{1.1}$$

for a constant λ and a constant vector C, where Δ denotes the Laplacian operator on M, then M is said to have 1-*type Gauss map*; it is a special case of a finite type Gauss map introduced by Chen [3]. A plane, a circular cylinder and a sphere are surfaces with 1-type Gauss map. However, the Laplacian operator of the Gauss map of some well-known surfaces such as a helicoid, a catenoid and a right cone in the three-dimensional Euclidean space $\mathbb{E}^3$ take a somewhat different form:

$$\Delta G = f(G + C) \tag{1.2}$$

for a smooth function f and a constant vector C. If the Gauss map G of a surface M satisfies condition (1.2), M is said to have *pointwise* 1-*type Gauss map* (cf. [11], [4]). Many results on submanifolds with pointwise 1-type Gauss map were obtained in [1], [4], [8], [11], [17], etc. when the ambient spaces are the Euclidean space, Minkowski space and Galilean space.

2010 *Mathematics Subject Classification*: Primary 53A35; Secondary 53B25.
Key words and phrases: Cheng–Yau operator, pointwise 1-type Gauss map, pseudo-Galilean space, rotation surface.

DOI: 10.4064/ap113-3-3

The Laplacian operator of a hypersurface M immersed in $\mathbb{E}^{n+1}$ is a second-order linear differential operator which arises naturally as the linearized operator of the first variation of the mean curvature for normal variations of hypersurfaces. From this point of view, the Laplacian operator $\varDelta$ can be seen as the first one of a sequence of n operators $L_0, L_1, \dots, L_{n-1}$, where L_k stands for the linearized operator of the first variation of the $(k+1)$th mean curvature arising from normal variations of hypersurfaces (cf. [2]). These operators are given by $L_k(f) = \operatorname{trace}(P_k \circ \nabla^2 f)$ for a smooth function f on M (see Section 2). When $k = 0$, $L_0 = -\varDelta$ is nothing but the Laplacian operator; when $k = 1$ the operator L_1 is the operator $\square$ introduced by Cheng and Yau [5] and called the *Cheng–Yau operator*. Hypersurfaces in terms of the linearized operator L_k have been studied in [9], [12] and [13].

Mimicking the condition (1.2), we can consider the following condition in terms of the Gauss map and the Cheng–Yau operator:

$$L_1 G = f(G + C) \tag{1.3}$$

for a smooth function f and a constant vector C.

A surface M is said to have *L_1-pointwise 1-type Gauss map* if its Gauss map G satisfies condition (1.3). In particular, a L_1-pointwise 1-type Gauss map is said to be of the *first kind* if (1.3) is satisfied for $C = 0$; otherwise, it is said to be of the *second kind* [10].

Recently, in [9] and [10] the authors studied constant curvature surfaces and helicoidal surfaces with L_1-pointwise 1-type Gauss map.

In this paper, we classify rotation surfaces in the three-dimensional pseudo-Galilean space G_3^1 satisfying condition (1.3).

2. Preliminaries. Let $\mathbf{x} : M \to \widetilde{M}$ be an isometric immersion of a connected oriented hypersurface into an $(n+1)$-dimensional Riemannian manifold $\widetilde{M}$. Let $\widetilde{\nabla}$ and ∇ be the Levi-Civita connections on $\widetilde{M}$ and M, respectively. Then the Gauss and Weingarten formulas are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle SX, Y\rangle N \quad \text{and} \quad \widetilde{\nabla}_X N = -SX$$

for all tangent vector fields $X, Y \in \mathcal{X}(M)$, where S and N are the shape operator and the unit normal vector field of M, respectively. It is well-known that S defines a self-adjoint linear operator on each tangent space and its eigenvalues $\kappa_1(p), \dots, \kappa_n(p)$ are the principal curvatures of M at p. The functions $s_k(p)$ defined by

$$s_k(p) = \sigma_k(\kappa_1(p), \dots, \kappa_n(p)), \quad 1 \le k \le n,$$

are called the *algebraic invariants* of the shape operator S of M, where σ_k

is the kth symmetric function in $\mathbb{R}^n$ given by

$$\sigma_k(t_1,\dots,t_n)=\sum_{1\le i_1<\dots<i_k\le n} t_{i_1}\cdots t_{i_k}.$$

The classical Newton transformations $P_k:\mathcal{X}(M)\to\mathcal{X}(M)$ are defined from the shape operator S by

$$P_k=s_kP_0-S\circ P_{k-1},\quad k=1,\dots,n,$$

where $P_0=I$ denotes the identity operator acting on $\mathcal{X}(M)$. We consider the second-order linear differential operator $L_k:\mathcal{C}^\infty(M)\to\mathcal{C}^\infty(M)$ given by

$$L_k(f)=\operatorname{trace}(P_k\circ\nabla^2 f),$$

where $\nabla^2 f$ is the Hessian of f. It is a consequence of the Codazzi equation that

$$L_k(f)=\operatorname{div}(P_k(\nabla f)). \tag{2.1}$$

Here ∇f stands for the gradient of f and div for the divergence operator (see [14]).

Now, let M be a surface and e_1,e_2 be the principal directions corresponding to the principal curvatures κ_1,κ_2 of M. By (2.1), for $f\in\mathcal{C}^\infty(M)$ the Cheng–Yau operator L_1f of f can be expressed as

$$\begin{aligned}L_1f&=\operatorname{div}(P_1(\nabla f))\\&=e_1(\kappa_2)e_1f+e_2(\kappa_1)e_2f+\kappa_2(e_1e_1-\nabla_{e_2}e_2)f+\kappa_1(e_2e_2-\nabla_{e_1}e_1)f.\end{aligned}$$

Thus, the Cheng–Yau operator L_1 is given by [9]

$$L_1=e_1(\kappa_2)\widetilde{\nabla}_{e_1}+e_2(\kappa_1)\widetilde{\nabla}_{e_2}+\kappa_2(\widetilde{\nabla}_{e_1}\widetilde{\nabla}_{e_1}-\widetilde{\nabla}_{\nabla_{e_2}e_2})+\kappa_1(\widetilde{\nabla}_{e_2}\widetilde{\nabla}_{e_2}-\widetilde{\nabla}_{\nabla_{e_1}e_1}). \tag{2.2}$$

LEMMA 2.1 ([9]). *Let M be an oriented surface in $\mathbb{E}^3$ with Gaussian curvature K and mean curvature H. Then the Gauss map G of M satisfies*

$$L_1G=-\nabla K-2KHG. \tag{2.3}$$

3. Pseudo-Galilean space. The pseudo-Galilean space G_3^1 is a Cayley–Klein space with the absolute figure consisting of an ordered triple $\{\omega,f,I\}$, where ω is the ideal (absolute) plane in the three-dimensional real projective space $\mathbb{R}P_3$, f the line (the absolute line) in ω and I the fixed hyperbolic involution of points of f.

Homogeneous coordinates in G_3^1 are introduced in such a way that the absolute plane ω is given by $x_0=0$, the absolute line f by $x_0=x_1=0$ and the hyperbolic involution η by $\eta:(x_0:x_1:x_2:x_3)\mapsto(0:0:x_3:x_2)$. The last condition is equivalent to the requirement that the conic $x_2^2-x_3^2=0$ is the absolute conic. Metric relations are introduced with respect to the absolute figure. In affine coordinates defined by $(x_0:x_1:x_2:x_3)=(1:x:$

$y : z$), the distance between the points $P_i = (x_i, y_i, z_i)$ $(i = 1, 2)$ is defined by (cf. [15])

$$d(P_1, P_2) = \begin{cases} |x_2 - x_1| & \text{if } \ x_1 \neq x_2, \\ \sqrt{|(y_2 - y_1)^2 - (z_2 - z_1)^2|} & \text{if } \ x_1 = x_2. \end{cases}$$

The group motion of G_3^1 is a six-parameter group given (in affine coordinates) by

$$\begin{aligned} \bar{x} &= a + x, \\ \bar{y} &= b + cx + y \cosh \varphi + z \sinh \varphi, \\ \bar{z} &= d + ex + y \sinh \varphi + z \cosh \varphi. \end{aligned}$$

Let $\mathbf{x} = (x_1, y_1, z_1)$ and $\mathbf{y} = (x_2, y_2, z_2)$ be vectors in G_3^1. A vector $\mathbf{x}$ is called *isotropic* if $x_1 = 0$, otherwise it is *non-isotropic*. The pseudo-Galilean scalar product of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \begin{cases} x_1 x_2 & \text{if } x_1 \neq 0 \text{ or } x_2 \neq 0, \\ y_1 y_2 - z_1 z_2 & \text{if } x_1 = 0 \text{ and } x_2 = 0. \end{cases}$$

From this, the pseudo-Galilean norm of a vector $\mathbf{x}$ in G_3^1 is given by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$ and all unit non-isotropic vectors are the form $(1, y_1, z_1)$. There are four types of isotropic vectors: spacelike $(y_1^2 - z_1^2 > 0)$, timelike $(y_1^2 - z_1^2 < 0)$, and two types of lightlike $(y_1 = \pm z_1)$ vectors. A non-lightlike isotropic vector is a unit vector if $y_1^2 - z_1^2 = \pm 1$.

The pseudo-Galilean cross product of $\mathbf{x}$ and $\mathbf{y}$ on G_3^1 is defined by

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} 0 & -e_2 & e_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix},$$

where $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Consider a C^r-surface M, $r \geq 1$, in G_3^1 parameterized by

$$\mathbf{x}(u_1, u_2) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)).$$

Let us denote

$$g_i = \frac{\partial x}{\partial u_i}, \qquad h_{ij} = \left\langle \frac{\partial \tilde{\mathbf{x}}}{\partial u_i}, \frac{\partial \tilde{\mathbf{x}}}{\partial u_j} \right\rangle \qquad (i, j = 1, 2),$$

where $\tilde{}$ stands for the projection of a vector on the pseudo-Euclidean yz-plane. A surface M is called *admissible* if it does not have Euclidean tangent planes. Therefore a surface M is admissible if and only if $x_{,i} \neq 0$ for some $i = 1, 2$.

Let M be an admissible surface. Then the unit normal vector field U of a surface M is defined by

$$U = \frac{1}{W}(0, x_{,1} z_{,2} - x_{,2} z_{,1}, x_{,1} y_{,2} - x_{,2} y_{,1}),$$

where

$$W=\sqrt{|(x_{,1}y_{,2}-x_{,2}y_{,1})^2-(x_{,1}z_{,2}-x_{,2}z_{,1})^2|}.$$

Moreover, the matrix of the first fundamental form ds^2 of M in G_3 is given by (cf. [15])

$$ds^2=\begin{pmatrix} ds_1^2 & 0\\ 0 & ds_2^2\end{pmatrix},$$

where $ds_1^2=(g_1du_1+g_2du_2)^2$ and $ds_2^2=h_{11}du_1^2+2h_{12}du_1du_2+h_{22}du_2^2$. Here $g_i=x_{,i}$ and $h_{ij}=\langle\tilde{\mathbf{x}}_{,i},\tilde{\mathbf{x}}_{,j}\rangle$ $(i,j=1,2)$.

The Gaussian curvature K of M is defined by means of the coefficients L_{ij} $(i,j=1,2)$ of the second fundamental form, which are the normal components of $\mathbf{x}_{,i,j}$ $(i,j=1,2)$, that is,

$$L_{ij}=\frac{1}{g_1}\langle g_1\tilde{\mathbf{x}}_{,i,j}-g_{i,j}\tilde{\mathbf{x}}_{,1},U\rangle=\frac{1}{g_2}\langle g_2\tilde{\mathbf{x}}_{,i,j}-g_{i,j}\tilde{\mathbf{x}}_{,2},U\rangle.$$

Thus, the Gaussian curvature K of M is defined by

$$K=-\epsilon\frac{L_{11}L_{22}-L_{12}^2}{W^2} \tag{3.1}$$

and the mean curvature H is given by

$$H=-\frac{\epsilon}{2W^2}(g_2^2L_{11}-2g_1g_2L_{12}+g_1^2L_{22}), \tag{3.2}$$

where ϵ $(=\pm1)$ is the sign of the unit normal vector field.

In the pseudo-Galilean space G_3^1, there are two types of rotations: pseudo-Euclidean rotations given by the normal form

$$\begin{aligned}\bar{x}&=x,\\ \bar{y}&=y\cosh t+z\sinh t,\\ \bar{z}&=y\sinh t+z\cosh t,\end{aligned}$$

and isotropic rotations with the normal form

$$\begin{aligned}\bar{x}&=x+bt,\\ \bar{y}&=y+xt+bt^2/2,\\ \bar{z}&=z,\end{aligned}$$

where $t\in\mathbb{R}$ and $b=\text{constant}>0$.

The trajectory of a single point under a pseudo-Euclidean rotation is a pseudo-Euclidean circle (i.e., a rectangular hyperbola)

$$x=\text{constant},\qquad y^2-z^2=r^2,\qquad r\in\mathbb{R}.$$

The invariant r is the radius of the circle. Pseudo-Euclidean circles intersect the absolute line f in the fixed points of the hyperbolic involution (F_1,F_2). There are three kinds of pseudo-Euclidean circles: circles of real radius, of imaginary radius, and of radius zero. Circles of real radius are timelike curves (having timelike tangent vectors) and imaginary radius spacelike curves (having spacelike tangent vectors).

The trajectory of a point under isotropic rotation is an isotropic circle whose normal form is

$$z = \text{constant}, \qquad y = \frac{x^2}{2b}.$$

The invariant b is the radius of the circle. The fixed line of the isotropic rotation (3.2) is the absolute line f.

First of all, we rotate a non-isotropic curve α parameterized by

$$\alpha(u) = (h(u), g(u), 0) \quad \text{or} \quad \alpha(u) = (h(u), 0, g(u))$$

around the x-axis by pseudo-Euclidean rotation (3.1), where g is a positive function and h is a smooth function on an open interval I. Then the surface M of revolution can be written as

$$\mathbf{x}(u, v) = (h(u), g(u)\cosh v, g(u)\sinh v), \tag{3.3}$$

or

$$\mathbf{x}(u, v) = (h(u), g(u)\sinh v, g(u)\cosh v), \tag{3.4}$$

for any $v \in \mathbb{R}$.

Next, we consider the isotropic rotations. By rotating the isotropic curve $\alpha(u) = (0, h(u), g(u))$ about the z-axis by isotropic rotation (3.2), we obtain a surface

$$\mathbf{x}(u, v) = \left(v, h(u) + \frac{v^2}{2b}, g(u)\right), \tag{3.5}$$

where h and g are smooth functions and $b \neq 0$ [15].

4. Rotation surface generated by a non-isotropic curve. Let M be a rotation surface generated by a non-isotropic curve $\alpha(u) = (u, g(u), 0)$. Then M is parameterized by

$$\mathbf{x}(u, v) = (u, g(u)\cosh v, g(u)\sinh v), \tag{4.1}$$

where $g(u)$ is a positive function. By using the natural frame $\{\mathbf{x}_u, \mathbf{x}_v\}$ of M we define an orthonormal frame $\{e_1, e_2\}$ by

$$\begin{aligned} e_1 &= \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|} = (1, g'(u)\cosh v, g'(u)\sinh v), \\ e_2 &= \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|} = (0, \sinh v, \cosh v); \end{aligned} \tag{4.2}$$

from this the Gauss map G of M is given by

$$G = (0, \cosh v, \sinh v). \tag{4.3}$$

On the other hand, the Gaussian curvature K and the mean curvature H are given by

$$K = -\epsilon \frac{g''(u)}{g(u)}, \quad H = \frac{1}{2g(u)}. \tag{4.4}$$

Thus from (2.3), (4.3) and (4.4) the operator $L_1 G$ of the Gauss map G can be expressed as

$$L_1 G = \left(\epsilon \frac{g(u)g'''(u) - g'(u)g''(u)}{g(u)^2} \right) e_1 + \epsilon \frac{g''(u)}{g(u)^2} G. \tag{4.5}$$

4.1. Rotation surface with L_1-harmonic Gauss map. First of all, we consider a rotation surface M with L_1-harmonic Gauss map, that is, $L_1 G = 0$. From (4.5) we have

$$g(u)g'''(u) - g'(u)g''(u) = 0, \quad g''(u) = 0,$$

and it follows that $g(u) = au + b$ with $a, b \in \mathbb{R}$. In this case, M is a flat surface. If $a = 0$, M is a Lorentzian hyperbolic cylinder $y^2 - z^2 = b^2$. If $a \neq 0$, M is a Lorentzian cone $(ax + b)^2 = y^2 - z^2$.

THEOREM 4.1. *Let M be a rotation surface defined by* (4.1) *in the three-dimensional pseudo-Galilean space G_3^1. Then M has L_1-harmonic Gauss map if and only if it is an open part of a Lorentzian hyperbolic cylinder or a Lorentzian cone.*

4.2. Rotation surface with L_1-pointwise 1-type Gauss map of the first kind. In this subsection, we study rotation surfaces with L_1-pointwise 1-type Gauss map of the first kind. From (4.5) we can obtain the equations

$$g(u)g'''(u) - g'(u)g''(u) = 0, \quad \epsilon \frac{g''(u)}{g(u)^2} = f.$$

The first equation implies $g''(u) = cg(u)$, where $c \in \mathbb{R}$. So, from (4.4) the Gaussian curvature K is constant. On the other hand, rotation surfaces with constant Gaussian curvature were obtained in [15].

Thus, we have the following theorem.

THEOREM 4.2. *Let M be a rotation surface defined by* (4.1) *in the three-dimensional pseudo-Galilean space G_3^1. Then M has L_1-pointwise* 1*-Gauss map of the first kind if and only if M is an open part of one of the following surfaces:*

1. $\mathbf{x}(u, v) = (u, a\cos(ku + b)\cosh v, a\cos(ku + b)\sinh v), \quad c = -k^2,$
2. $\mathbf{x}(u, v) = (u, a\cosh(ku + b)\cosh v, a\cosh(ku + b)\sinh v), \quad c = k^2,$

where $a, b, k \in \mathbb{R}$.

4.3. Rotation surface with L_1-pointwise 1-type Gauss map of the second kind. Let M be a rotation surface with L_1-pointwise 1-type Gauss map of the second kind. Then equation (1.3) is satisfied for a non-zero constant vector $C = (c_1, c_2, c_3)$ and a smooth function f and we have

$$\begin{aligned} &\epsilon\frac{g(u)g'''(u) - g'(u)g''(u)}{g(u)^2} = f\langle C, e_1\rangle, \\ &\epsilon\frac{g''(u)}{g(u)^2} = f(1 + \langle C, G\rangle), \qquad 0 = \langle C, e_2\rangle. \end{aligned} \tag{4.6}$$

Let us distinguish the following cases:

1. If $c_1 = 0$, then from $\langle C, e_2\rangle = 0$ we can obtain

$$c_2 g(u)\cosh v - c_3 g(u)\sinh v = 0.$$

Since $\{\sinh v, \cosh v\}$ forms a set of linearly independent functions, we get

$$c_2 = 0, \qquad c_3 = 0,$$

because $g(u)$ is a positive function. In this case, the constant vector C vanishes identically. This is a contradiction.

2. If $c_1 \neq 0$, then from $\langle C, e_1\rangle = c_1$, $\langle C, e_2\rangle = 0$ and $\langle C, G\rangle = 0$ the constant vector C becomes $C = c_1 e_1$, which is impossible because e_1 is a non-constant vector except for $g'(u) = 0$. If $g'(u) = 0$, from (4.6) the smooth function f is identically zero.

THEOREM 4.3. *There do not exist rotation surfaces defined by* (4.1) *in* G_3^1 *with* L_1*-pointwise* 1*-Gauss map of the second kind.*

5. Rotation surface generated by isotropic curve. In this section, we consider isotropic rotations. By rotating an isotropic curve $\alpha(u) = (0, h(u), g(u))$ about the z-axis by an isotropic rotation, we obtain a rotation surface parameterized by

$$\mathbf{x}(u, v) = \left(v, h(u) + \frac{v^2}{2b}, g(u)\right), \tag{5.1}$$

where b is a non-zero constant. We assume that the isotropic curve is parameterized by arc length, that is,

$$h'(u)^2 - g'(u)^2 = -\epsilon. \tag{5.2}$$

Then the orthonormal frame $\{e_1, e_2\}$ of the tangent space of M is given by

$$\begin{aligned} e_1 &= (0, h'(u), g'(u)), \\ e_2 &= (1, v/b, 0). \end{aligned} \tag{5.3}$$

On the other hand, the Gauss map G of M is

$$G = (0, -g'(u), -h'(u)). \tag{5.4}$$

From (3.1) and (3.2), the Gaussian curvature K and the mean curvature H are given by

$$K = -\frac{h''(u)}{b}, \qquad H = -\frac{\epsilon h''(u)}{2g'(u)}. \tag{5.5}$$

Thus the operator $L_1 G$ of the Gauss map G can be expressed as

$$L_1 G = \frac{h'''(u)}{b} e_1 - \frac{\epsilon h''(u)^2}{bg'(u)} G. \tag{5.6}$$

5.1. Rotation surface with L_1-harmonic Gauss map. Suppose that a rotation surface M satisfies $L_1 G = 0$. Then, from (5.6), $h(u) = au + b$ and $g(u) = \pm\sqrt{a^2+\epsilon}u + c$ with $a, b, c \in \mathbb{R}$.

THEOREM 5.1. *Let M be a rotation surface defined by* (5.1) *in the three-dimensional pseudo-Galilean space G_3^1. Then M has L_1-harmonic Gauss map if and only if M is parameterized by*

$$\mathbf{x}(u, v) = \left(v, c_1 u + c_2 + \frac{v^2}{2b}, c_3 u + c_4\right),$$

where c_i $(i = 1, \ldots, 4)$ are constants.

5.2. Rotation surface with L_1-pointwise 1-type Gauss map of the first kind. Let M be a rotation surface with L_1-pointwise 1-type Gauss map of the first kind. Then from (5.6) we have

$$\frac{h'''(u)}{b} e_1 - \frac{\epsilon h''(u)^2}{bg'(u)} G = fG, \tag{5.7}$$

which implies $h'''(u) = 0$, and it follows that the Gaussian curvature K is a constant K_0. Combining this with the result in [15] we have the following theorem:

THEOREM 5.2. *Let M be a rotation surface generated by an isotropic curve in the three-dimensional pseudo-Galilean space G_3^1. Then M has L_1-pointwise* 1*-type Gauss map of the first kind if and only if M is parameterized as*

$$\mathbf{x}(u, v) = \left(v, h(u) + \frac{v^2}{2b}, g(u)\right),$$

where either

$$h(u) = bK_0 u^2/2 + c_1 u + c_2,$$

$$g(u) = -\frac{1}{2bK_0}\big((c_1 - bK_0 u)\sqrt{(c_1 - bK_0)^2 - 1} - \cosh^{-1}(c_1 - bK_0 u) + c_2\big),$$

or

$$h(u) = bK_0u^2/2 + c_1u + c_2,$$
$$g(u) = -\frac{1}{2bK_0}\big((c_1 - bK_0u)\sqrt{(c_1 - bK_0)^2 + 1} + \sinh^{-1}(c_1 - bK_0u) + c_2\big).$$

5.3. Rotation surface with L_1-pointwise 1-type Gauss map of the second kind. We suppose that a rotation surface M satisfies the condition $L_1G = f(G + C)$ for some smooth function f and non-zero constant vector $C = (c_1, c_2, c_3)$. Then from (5.6) we have

$$-\epsilon\frac{h'''(u)}{b} = f\langle C, e_1\rangle, \tag{5.8}$$
$$-\frac{h''(u)^2}{bg'(u)} = f(\epsilon + \langle C, G\rangle, \tag{5.9}$$
$$\langle C, e_2\rangle = 0. \tag{5.10}$$

From the scalar product of C and e_2 of (5.10), we find $c_1 = 0$. By taking the covariant derivative of (5.10) with respect to e_2 we have

$$0 = \widetilde{\nabla}_{e_2}\langle C, e_2\rangle = \left\langle C, -\epsilon\frac{h'(u)}{b}e_1 - \epsilon\frac{g'(u)}{b}G\right\rangle,$$

which implies that $c_2(h'(u)^2 - g'(u)^2) = 0$. Thus $c_2 = 0$. Combining (5.8) and (5.9) we get

$$\epsilon h'''(u)(\epsilon + \langle C, G\rangle) - \frac{h''(u)^2}{g'(u)}\langle C, e_1\rangle = 0;$$

from this equation, we have the following ODE:

$$(1 + \epsilon c_3h'(u))h'''(u) + c_3h''(u)^2 = 0. \tag{5.11}$$

To solve (5.11), we set $h'(u) = y(u)$; then

$$(1 + \epsilon c_3y(u))y''(u) + c_3y'(u)^2 = 0.$$

Again, we set $y'(u) = p(u)$ then the above equation becomes

$$(1 + \epsilon c_3y)\frac{dp}{dy} + c_3p = 0,$$

and its general solution is

$$p(u) = d_1(1 + \epsilon c_3y(u))^{-\epsilon}, \tag{5.12}$$

where $d_1 \in \mathbb{R}$.

If $\epsilon = 1$, then from (5.12) we find

$$y(u)^2 + \frac{2}{c_3} y(u) - \frac{2}{c_3}(d_1 u + d_2) = 0,$$

that is,

$$y(u) = -\frac{1}{c_3} \pm \frac{1}{c_3}\sqrt{1 + 2c_3(d_1 u + d_2)},$$

where $d_2 \in \mathbb{R}$. Thus the general solution of (5.11) is

$$h(u) = -\frac{1}{c_3} u \pm \frac{1}{3c_3^2 d_1}(1 + 2c_3(d_1 u + d_2))^{3/2} + d_3, \tag{5.13}$$

where $d_3 \in \mathbb{R}$. On the other hand, from (5.2) the function $g(u)$ is given by

$$g(u) = \pm\int \left(1 + \left(-\frac{1}{c_3} \pm \frac{1}{c_3}\sqrt{1 + 2c_3(d_1 u + d_2)}\right)^2\right)^{1/2} du. \tag{5.14}$$

If $\epsilon = -1$, then from (5.12) we get

$$y(u) = \frac{dh}{du} = \frac{1}{c_3}(1 - e^{-c_3(d_1 u + d_2)}),$$

it follows that we have the general solution of (5.11) as

$$h(u) = \frac{1}{c_3}\left(u + \frac{1}{c_3 d_1} e^{-c_3(d_1 u + d_2)}\right) + d_3 \tag{5.15}$$

with $d_3 \in \mathbb{R}$, and from (5.2) the function $g(u)$ is given by

$$g(u) = \pm\int \left(\frac{1}{c_3^2}(1 - e^{-c_3(d_1 u + d_2)})^2 - 1\right)^{1/2} du. \tag{5.16}$$

Consequently, we have the following theorem:

THEOREM 5.3. *Let M be a rotation surface generated by an isotropic curve in the three-dimensional pseudo-Galilean space G_3^1. Then M has L_1-pointwise 1-type Gauss map of the second kind if and only if M is parameterized as*

$$\mathbf{x}(u, v) = \left(v, h(u) + \frac{v^2}{2b}, g(u)\right),$$

where either

$$h(u) = -\frac{1}{c_3} u \pm \frac{1}{3c_3^2 d_1}(1 + 2c_3(d_1 u + d_2))^{3/2} + d_3,$$

$$g(u) = \pm\int \left(1 + \left(-\frac{1}{c_3} \pm \frac{1}{c_3}\sqrt{1 + 2c_3(d_1 u + d_2)}\right)^2\right)^{1/2} du$$

or

$$h(u) = \frac{1}{c_3}\left(u + \frac{1}{c_3 d_1} e^{-c_3(d_1 u + d_2)}\right) + d_3,$$
$$g(u) = \pm \int \left(\frac{1}{c_3^2}(1 - e^{-c_3(d_1 u + d_2)})^2 - 1\right)^{1/2} du,$$

with $d_1, d_2, d_3 \in \mathbb{R}$.

Acknowledgements. The first author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012R1A1A2003994).

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Dae Won Yoon
Department of Mathematics Education
and RINS
Gyeongsang National University
Jinju 660-701, South Korea
E-mail: dwyoon@gnu.ac.kr

Young Ho Kim
Department of Mathematics Education
Kyungpook National University
Daegu 702-701, South Korea
E-mail: yhkim@knu.ac.kr

Jae Seong Jung
Department of Mathematics Education
Gyeongsang National University
Jinju 660-701, South Korea
E-mail: jungzzang@naver.com

Received 27.2.2014
and in final form 6.8.2014 (3337)