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Existence of solutions for a class of Kirchhoff type problems in Orlicz–Sobolev spaces

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Abstract. We consider Kirchhoff type problems of the form

$$\begin{cases} -M(\rho(u))(\operatorname{div}(a(|\nabla u|)\nabla u) - a(|u|)u) = K(x)f(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a smooth bounded domain, ν is the outward unit normal to $\partial\Omega$, $\rho(u) = \int_\Omega (\Phi(|\nabla u|) + \Phi(|u|))\,dx$, $M : [0,\infty) \to \mathbb{R}$ is a continuous function, $K \in L^\infty(\Omega)$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function not satisfying the Ambrosetti–Rabinowitz type condition. Using variational methods, we obtain some existence and multiplicity results.

1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$. Assume that $a : (0,\infty) \to \mathbb{R}$ is a function such that the mapping defined by

$$\varphi(t) := \begin{cases} a(|t|)t & \text{for } t \neq 0, \\ 0 & \text{for } t = 0 \end{cases}$$

is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. For the function φ above, define

$$\Phi(t) = \int_0^t \varphi(s)\,ds \qquad \text{for all } t \in \mathbb{R},$$

on which some suitable conditions will be imposed later.

In this article, we are interested in the existence of weak solutions for the following Kirchhoff type problem:

$$\begin{cases} -M(\rho(u))\big(\operatorname{div}(a(|\nabla u|)\nabla u) - a(|u|)u\big) = K(x)f(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a smooth bounded domain, ν is the outward unit normal to $\partial\Omega$, $\rho(u) = \int_\Omega (\Phi(|\nabla u|) + \Phi(|u|))\,dx$, $M : [0,\infty) \to \mathbb{R}$ is a

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continuous function, $K \in L^\infty(\Omega)$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

It should be noticed that if $\varphi(t) = p|t|^{p-2}t$ then problem (1.1) becomes the well-known p-Kirchhoff type equation

$$\begin{cases} -M\big(\int_\Omega (|\nabla u|^p + |u|^p)\, dx\big)(\Delta_p u - |u|^{p-2}u) = K(x)f(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Since the first equation in (1.2) contains an integral over Ω, it is no longer a pointwise identity, and therefore it is often called a nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density (see [APS, CL]). Problem (1.2) is related to the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.3}$$

presented by Kirchhoff in 1883 (see [K]). This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.3) have the following meaning: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

In recent years, problems involving p-Kirchhoff type operators have been studied in many papers; we refer to [MP, MR1, MR2, CKW, CN, M, R], in which the authors have used different methods to get the existence of solutions for (1.2). In the case when $p(\cdot)$ is a continuous function, problem (1.2) has also been studied by many authors (see for example [CV, C1, CP, DM]). The study of Kirchhoff type problems in Orlicz–Sobolev spaces is a new and interesting topic (see [C2, C3]). Motivated by the ideas introduced in [BMR1, BMR2, CZ, KMR, MR, Y], we study the existence and multiplicity of weak solutions for problem (1.1) without the Ambrosetti–Rabinowitz type condition (see Section 2, condition (F_1)). This condition plays an important role in dealing with problems (1.1) and (1.2) by variational methods (see [CGMS, DM, FT, M]). The situation presented in this paper is different from our recent result [CT].

In order to study problem (1.1), let us introduce the functional spaces where it will be discussed. We will give just a brief review of some basic concepts and facts of the theory of Orlicz and Orlicz–Sobolev spaces, useful for what follows; for more details we refer the readers to the books by Adams [A], Rao and Ren [RR], the papers by Clément et al. [CGMS, CPST], Bonanno et al. [BMR1, BMR2], Mihăilescu et al. [KMR, MR] and Yang [Y].

For $\varphi : \mathbb{R} \to \mathbb{R}$ and Φ introduced at the beginning of the paper, we can see that Φ is a *Young function*, that is, $\Phi(0) = 0$, Φ is convex, and $\lim_{t\to\infty} \Phi(t) = \infty$. Furthermore, Φ is an *N-function*, i.e., Φ is continuous, convex, $\Phi(t) > 0$ for $t > 0$, $\lim_{t\to 0} \Phi(t)/t = 0$, and $\lim_{t\to\infty} \Phi(t)/t = \infty$. The function Φ^* defined by the formula

$$\Phi^*(t) = \int_0^t \varphi^{-1}(s)\, ds \quad \text{for all } t \in \mathbb{R}$$

is called the *complementary function* of Φ and it satisfies the condition

$$\Phi^*(t) = \sup\{st - \Phi(s) : s \geq 0\} \quad \text{for all } t \geq 0.$$

We observe that the function Φ^* is also an N-function, and the following Young inequality holds:

$$st \leq \Phi(s) + \Phi^*(t) \quad \text{for all } s, t \geq 0.$$

Throughout this paper, we assume that

$$1 < \liminf_{t\to\infty} \frac{t\varphi(t)}{\Phi(t)} \leq \varphi^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \infty, \quad t \geq 0, \tag{1.4}$$

$$N < \varphi_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}. \tag{1.5}$$

Further, we also assume that

$$\text{the function } t \mapsto \Phi(\sqrt{t}) \text{ is convex for all } t \in [0, \infty). \tag{1.6}$$

The Orlicz space $L_\Phi(\Omega)$ defined by the N-function Φ is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_\Phi} := \sup\Big\{\Big|\int_\Omega u(x)v(x)\, dx\Big| : \int_\Omega \Phi^*(|v(x)|)\, dx \leq 1\Big\} < \infty.$$

The space $L_\Phi(\Omega)$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_\Phi := \inf\Big\{k > 0 : \int_\Omega \Phi\Big(\frac{u(x)}{k}\Big)\, dx \leq 1\Big\}.$$

For Orlicz spaces, the Hölder inequality reads as follows (see [RR]):

$$\int_\Omega uv\, dx \leq 2\|u\|_{L_\Phi(\Omega)}\|u\|_{L_{\Phi^*}(\Omega)} \quad \text{for all } u \in L_\Phi(\Omega) \text{ and } v \in L_{\Phi^*}(\Omega).$$

We denote by $W^1L_\Phi(\Omega)$ the corresponding Orlicz–Sobolev space for problem (1.1), defined by

$$W^1L_\Phi(\Omega) := \Big\{u \in L_\Phi(\Omega) : \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega),\ i = 1, \ldots, N\Big\}.$$

It is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} := \|u\|_\Phi + \|\,|\nabla u|\,\|_\Phi.$$

Condition (1.5) says that $L_\Phi(\Omega)$ and $W^1L_\Phi(\Omega)$ are separable, reflexive Banach spaces (see [BMR1, BMR2, KMR, Y]). By (1.5), using [CGMS, Lemma D.2], it follows that $W^1L_\Phi(\Omega)$ is continuously embedded into $W^{\varphi_0}(\Omega)$. Moreover, again by (1.5), we deduce that $W^{1,\varphi_0}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Thus, we deduce that $W^1L_\Phi(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Defining $\|u\|_\infty = \sup_{x\in\overline{\Omega}}|u(x)|$, we find a positive constant $c>0$ such that

$$\|u\|_\infty \le c\|u\|_{1,\Phi} \quad \text{for all } u \in W^1L_\Phi(\Omega). \tag{1.7}$$

Proposition 1.1 (see [BMR1, BMR2, KMR, Y]). *On $W^1L_\Phi(\Omega)$ the norms*

$$\|u\|_{1,\Phi} = \|\,|\nabla u|\,\|_\Phi + \|u\|_\Phi,$$
$$\|u\|_{2,\Phi} = \max\{\|\,|\nabla u|\,\|_\Phi, \|u\|_\Phi\},$$
$$\|u\| = \inf\left\{\mu>0 : \int_\Omega \left[\Phi\left(\frac{|u(x)|}{\mu}\right) + \Phi\left(\frac{|\nabla u(x)|}{\mu}\right)\right] dx\right\}$$

are equivalent. More precisely, for every $u \in W^1L_\Phi(\Omega)$ we have

$$\|u\| \le 2\|u\|_{2,\Phi} \le 2\|u\|_{1,\Phi} \le 4\|u\|.$$

Proposition 1.2 (see [BMR1, BMR2, KMR, Y]). *Let $u \in W^1L_\Phi(\Omega)$ and $\rho(u) = \int_\Omega(\Phi(|\nabla u|) + \Phi(|u|))\,dx$. Then*

(i) $\|u\|^{\varphi^0} \le \rho(u) \le \|u\|^{\varphi_0}$ *if* $\|u\|<1$.
(ii) $\|u\|^{\varphi_0} \le \rho(u) \le \|u\|^{\varphi^0}$ *if* $\|u\|>1$.

2. Main result. In this section, we prove the main result of the paper. We shall use C_i to denote general positive constants whose values may change from line to line. We first make the definition of weak solutions for problem (1.1).

Definition 2.1. We say that $u \in W^1L_\Phi(\Omega)$ is a *weak solution* of problem (1.1) if

$$M(\rho(u))\int_\Omega \big(a(|\nabla u|)\nabla u\nabla v + a(|u|)uv\big)\,dx - \int_\Omega K(x)f(u)v\,dx = 0$$

for all $v \in W^1L_\Phi(\Omega)$, where $\rho(u) = \int_\Omega(\Phi(|\nabla u|) + \Phi(|u|))\,dx$.

The main result of the paper can be formulated as follows.

Theorem 2.2. *Assume that M, f satisfy the following conditions:*

(M_1) *$M:[0,\infty)\to\mathbb{R}$ is a continuous function and there exists $m_0>0$ such that $M(t)\ge m_0$ for all $t\in[0,\infty)$.*

(M_2) *$\widehat{M}(t) \geq M(t)t$ for all $t \in [0, \infty)$, where $\widehat{M}(t) = \int_0^t M(s)\,ds$.*

(K_1) *$K \in L^\infty(\Omega)$ and $K(x) \geq k_0 > 0$ for all $x \in \Omega$.*

(F_1) *$f \in C(\mathbb{R}, \mathbb{R})$ and there exist a constant $s_0 \geq 0$ and a decreasing function $\theta(s) \in C(\mathbb{R} \setminus (-s_0, s_0), \mathbb{R})$ such that*
$$0 < (\varphi^0 + \theta(s))F(s) \leq f(s)s, \quad \forall |s| \geq s_0,$$
where $\theta(s) > 0$ and $\lim_{|s|\to\infty} \theta(s)|s| = \infty$, $\lim_{|s|\to\infty} \int_{s_0}^{|s|} \frac{\theta(t)}{t}\,dt = \infty$, $F(s) = \int_0^s f(t)\,dt$.

(F_2) $\lim_{s\to 0}(f(s)/|s|^{\varphi^0-1}) = 0$.

Then problem (1.1) *has a nontrivial weak solution. If further f is odd, then* (1.1) *has infinitely many pairs of weak solutions.*

If $\inf_{|s|\geq s_0} \theta(s) > 0$, then it follows from condition (F_1) that
$$0 < \Big(\varphi^0 + \inf_{|s|\geq s_0} \theta(s)\Big)F(s) \leq f(s)s, \quad \forall |s| \geq s_0,$$
and thus we have the well-known Ambrosetti–Rabinowitz type condition as in [CGMS, FT]. In this paper, we are interested in the case $\inf_{|s|\geq s_0} \theta(s) = 0$. For this reason, we assume throughout this work that $s_0 \geq 1$ and there is a constant $N_0 > 0$ such that $|\theta(s)| \leq N_0$ for all $s \in \mathbb{R} \setminus (-s_0, s_0)$.

Our idea is to prove Theorem 2.2 by using the mountain pass theorem and its $\mathbb{Z}_2$ symmetric version stated in the celebrated paper [AR]. For this purpose, we define the energy functional $J : W^1L_\Phi(\Omega) \to \mathbb{R}$ by
$$J(u) = \widehat{M}(\rho(u)) - \int_\Omega K(x)F(u)\,dx,$$
where $F(s) = \int_0^s f(t)\,dt$ and $\widehat{M}(t) = \int_0^t M(s)\,ds$. Then J is in $C^1(W^1L_\Phi(\Omega), \mathbb{R})$ and its derivative is given by
$$J(u)(v) = M(\rho(u)) \int_\Omega \big(a(|\nabla u|)\nabla u \nabla v + a(|u|)uv\big)\,dx - \int_\Omega K(x)f(u)v\,dx$$
for all $u, v \in W^1L_\Phi(\Omega)$. Hence, the weak solutions of problem (1.1) are exactly the critical points of the functional J.

LEMMA 2.3. *There exist positive constants m_1 and m_2 such that*
$$\widehat{M}(t) \leq m_1 t + m_2, \quad \forall t \in [0, \infty).$$

Proof. Let $t_0 > 0$. By our assumptions (M_1) and (M_2), we have
$$\frac{M(t)}{\widehat{M}(t)} \leq \frac{1}{t}, \quad \forall t \geq t_0.$$
Hence,
$$\int_0^t \frac{M(s)}{\widehat{M}(s)}\,ds = \log \frac{M(t)}{\widehat{M}(t_0)} \leq \log \frac{t}{t_0}, \quad \forall t \geq t_0,$$

which yields

$$\widehat{M}(t) \le \widehat{M}(t_0)\frac{t}{t_0}, \qquad \forall t \ge t_0. \tag{2.1}$$

From (2.1), we conclude that

$$\widehat{M}(t) \le m_1 t + m_2, \quad \forall t \in [0,\infty),$$

where m_1, m_2 can be chosen as $m_1 = \widehat{M}(t_0)/t_0$ and $m_2 = \max_{t\in[0,t_0]} \widehat{M}(t)$. ∎

LEMMA 2.4. *There exist positive constants ρ and α such that $J(u) \ge \alpha$ for all $u \in W^1L_\Phi(\Omega)$ with $\|u\|_{1,\Phi} = \rho$.*

Proof. From (1.7) we have $\|u\|_\infty \to 0$ if $\|u\|_{1,\Phi} \to 0$. By hypothesis (F_2), for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(s)| \le \epsilon |s|^{\varphi^0 - 1}, \qquad \forall |s| < \delta.$$

Hence,

$$|F(s)| \le \frac{\epsilon}{\varphi^0}|s|^{\varphi^0}, \quad \forall |s| < \delta. \tag{2.2}$$

Combining (M_1) with Proposition 1.2, we deduce for $u \in W^1L_\Phi(\Omega)$ with $\|u\|_{1,\Phi} < \min\{1, \delta/c\}$ (where c is given by (1.7)) that

$$\begin{aligned} J(u) &= \widehat{M}(\rho(u)) - \int_\Omega K(x)F(u)\,dx \ge m_0\|u\|_{1,\Phi}^{\varphi^0} - \int_\Omega K(x)|F(u)|\,dx \\ &\ge m_0\|u\|_{1,\Phi}^{\varphi^0} - \frac{\epsilon\|K\|_\infty}{\varphi^0}\int_\Omega |u|^{\varphi^0}\,dx \ge m_0\|u\|_{1,\Phi}^{\varphi^0} - \frac{\epsilon\|K\|_\infty|\Omega|}{\varphi^0}\|u\|_\infty^{\varphi^0} \\ &\ge m_0\|u\|_{1,\Phi}^{\varphi^0} - \frac{\epsilon\|K\|_\infty|\Omega|c^{\varphi^0}}{\varphi^0}\|u\|_{1,\Phi}^{\varphi^0} = \left(m_0 - \frac{\epsilon\|K\|_\infty|\Omega|c^{\varphi^0}}{\varphi^0}\right)\|u\|_{1,\Phi}^{\varphi^0}. \end{aligned} \tag{2.3}$$

From (2.3), there exist positive constants ρ and α such that $J(u) \ge \alpha$ for all $u \in W^1L_\Phi(\Omega)$ with $\|u\|_{1,\Phi} = \rho$. ∎

Let $S = \{w \in W^1L_\Phi(\Omega) : \|w\|_{1,\Phi} = 1\}$. Note that for all $w \in S$ and a.e. $x \in \Omega$ we have $|w(x)| \le L$ for some $L > 0$. There is $s_\lambda \in \{s \in \mathbb{R} : |s| \le |\lambda L|\}$ such that $\theta(s_\lambda) = \min_{s_0 \le |s| \le |\lambda L|} \theta(s)$. Then $|\lambda| \ge |s_\lambda|/L$ and $|s_\lambda| \to \infty$ when $|\lambda| \to \infty$. When $|s| \ge s_0$, we have

$$0 < (\varphi^0 + \theta(s))F(s) \le f(s)s.$$

Hence,

$$F(s) \ge C_1|s|^{\varphi^0}\exp\Big(\int_{s_0}^{|s|}\frac{\theta(t)}{t}\,dt\Big) = C_1|s|^{\varphi^0}G(|s|), \tag{2.4}$$

where $G(|s|) = \exp\big(\int_{s_0}^{|s|}\frac{\theta(t)}{t}\,dt\big)$. Then by ($\mathrm{F}_1$), it follows that $G(|s|)$ increases when $|s|$ increases, and $\lim_{|s|\to\infty} G(|s|) = \infty$.

LEMMA 2.5. *For any $w \in S$ there exist $\delta_w > 0$ and $\lambda_w > 0$ such that, for all $v \in S \cap B(w, \delta_w)$ and for all $|\lambda| \geq \lambda_w$, we have $J(\lambda v) < 0$, where $B(w, \delta_w) = \{v \in W^1L_\Phi(\Omega) : \|v - w\|_{1,\Phi} < \delta_w\}$.*

Proof. Fix $w \in S$. As $\|w\|_{1,\Phi} = 1$, we know that $\mu(\{x \in \Omega : w(x) \neq 0\}) > 0$ and that there exists a $\overline{\lambda}_w > s_0$ such that $\mu(\{x \in \Omega : |\overline{\lambda}_w w(x)| \geq s_0\}) > 0$, where μ is the Lebesgue measure. Let

$$\Omega_w^1 := \{x \in \Omega : |\overline{\lambda}_w w(x)| < s_0\}, \qquad \Omega_w^2 := \{x \in \Omega : |\overline{\lambda}_w w(x)| \geq s_0\}.$$

Then $\mu(\Omega_w^1) > 0$. When $x \in \Omega_w^1$ we have $|w(x)| \geq s_0/\overline{\lambda}_w$. Let $\delta_w = s_0/(2\overline{\lambda}_w)$. Then, for any $v \in S \cap B(w, \delta_w)$,

$$\|v - w\|_\infty \leq L\|v - w\|_{1,\Phi} < \frac{s_0}{2\overline{\lambda}_w}.$$

Hence, when $x \in \Omega_w^1$, we observe that $v(x) \geq s_0/(2\overline{\lambda}_w)$ and

$$|v(x)|^{\varphi^0} \geq \left(\frac{s_0}{2\overline{\lambda}_w}\right)^{\varphi^0} = C_2. \tag{2.5}$$

When $|\lambda| \geq 2\overline{\lambda}_w$, one has $|\lambda v| \geq s_0$ in Ω_w^2. By condition (K$_1$) and (2.4), (2.5) we know that

$$\begin{aligned}|\lambda|^{-\varphi^0} \int_{\Omega_w^2} K(x)F(\lambda v)\,dx &\geq C_1 \int_{\Omega_w^2} K(x)|v|^{\varphi^0} G(|\lambda v|)\,dx \\ &\geq C_1C_2 \int_{\Omega_w^2} K(x)G(|\lambda v|)\,dx \\ &\geq C_1C_2\mu(\Omega_w^2)k_0 G\left(\frac{s_0}{2\overline{\lambda}_w}|\lambda|\right),\end{aligned} \tag{2.6}$$

since $G(|s|)$ increases when $|s|$ increases and $|\lambda v(x)| \geq \frac{s_0}{2\overline{\lambda}_w}|\lambda|$. There exists $C_3 > 0$ such that $F(s) \geq -C_3$ when $|s| \leq s_0$. However, $F(s) > 0$ if $|s| \geq s_0$, so

$$\int_{\Omega_w^1} K(x)F(\lambda v)\,dx \geq \int_{\Omega_w^1 \cap \{x \in \Omega:\, |\lambda v(x)| \leq s_0\}} K(x)F(\lambda v)\,dx \geq -C_3\|K\|_\infty.$$

Hence, by Proposition 1.2, for any $v \in S \cap B(w, \delta_w)$ and $|\lambda| > 1$, we have

$$\begin{aligned}J(\lambda v) &= \widehat{M}(\rho(u)) - \int_\Omega K(x)F(\lambda v)\,dx \\ &\leq m_1\rho(u) + m_2 - \int_\Omega K(x)F(\lambda v)\,dx \\ &\leq m_1|\lambda|^{\varphi^0}\|v\|_{1,\Phi}^{\varphi^0} + m_2 - \int_{\Omega_w^1} K(x)F(\lambda v)\,dx - \int_{\Omega_w^2} K(x)F(\lambda v)\,dx\end{aligned} \tag{2.7}$$

$$= |\lambda|^{\varphi^0}\Big(m_1 - |\lambda|^{-\varphi^0}\int\limits_{\Omega_w^2} K(x)F(\lambda v)\,dx\Big) - \int\limits_{\Omega_w^1} K(x)F(\lambda v)\,dx + m_2$$
$$\le |\lambda|^{\varphi^0}\bigg(m_1 - C_1C_2\mu(\Omega_w^2)k_0G\bigg(\frac{s_0}{2\overline{\lambda}_w}|\lambda|\bigg)\bigg) + C_3\|K\|_\infty + m_2.$$

From (2.7), $J(\lambda v) \to -\infty$ uniformly for $v \in S \cap B(w,\delta_w)$ as $|\lambda| \to \infty$. Therefore, there exists a $\lambda_w > 2\overline{\lambda}_w$ such that $J(\lambda v) < 0$ for any $v \in S \cap B(w,\delta_w)$ and $|\lambda| \ge \lambda_w$. ■

LEMMA 2.6. *The functional J satisfies the* (PS) *condition.*

Proof. Let $\{u_m\}$ be a (PS) *sequence* of the functional J, that is,

$$|J(u_m)| \le c \quad \text{and} \quad |(J'(u_m),h)| \le \epsilon_m\|h\|_{1,\Phi} \tag{2.8}$$

for all $h \in W^1L_\Phi(\Omega)$ with $\epsilon_m \to 0$ as $m \to \infty$. We shall prove that $\{u_m\}$ is bounded in $W^1L_\Phi(\Omega)$. Indeed, if $\{u_m\}$ is not bounded, we may assume that $\|u_m\|_{1,\Phi} \to \infty$ as $m \to \infty$. Let $\{\lambda_m\} \subset \mathbb{R}$ be such that $u_m = \lambda_m w_m$, $w_m \in S$. Then $|\lambda_m| \to \infty$ as $m \to \infty$.

Define

$$\Omega_m^1 = \{x \in \Omega : |\lambda_m w_m(x)| \ge L\} \quad \text{and} \quad \Omega_m^2 = \{x \in \Omega : |\lambda_m w_m(x)| < L\}.$$

Then

$$\begin{aligned}
(2.9)\qquad & -\epsilon_m|\lambda_m| = -\epsilon_m\|u_m\|_{1,\Phi} \le (J'(u_m),u_m)\\
&= M(\rho(u_m))\int\limits_\Omega \big(a(|\nabla u_m|)|\nabla u_m|^2 + a(|u_m|)|u_m|^2\big)\,dx - \int\limits_\Omega K(x)f(u_m)u_m\,dx\\
&= M(\rho(\lambda_m w_m))\int\limits_\Omega \big(a(|\nabla\lambda_m w_m|)|\nabla\lambda_m w_m|^2 + a(|\lambda_m w_m|)|\lambda_m w_m|^2\big)\,dx\\
&\quad - \int\limits_{\Omega_m^1} K(x)f(\lambda_m w_m)\lambda_m w_m\,dx - \int\limits_{\Omega_m^2} K(x)f(\lambda_m w_m)\lambda_m w_m\,dx,
\end{aligned}$$

which implies that

$$\begin{aligned}
(2.10)\qquad & \int\limits_{\Omega_m^1} K(x)f(\lambda w_m)\lambda_m w_m\,dx\\
&\le M(\rho(\lambda_m w_m))\int\limits_\Omega \big(a(|\nabla\lambda_m w_m|)|\nabla\lambda_m w_m|^2 + a(|\lambda_m w_m|)|\lambda_m w_m|^2\big)\,dx\\
&\quad + \epsilon_m|\lambda_m| - \int\limits_{\Omega_m^2} K(x)f(\lambda_m w_m)\lambda_m w_m\,dx\\
&\le \varphi^0 M(\rho(\lambda_m w_m))\int\limits_\Omega \big(\Phi(|\nabla\lambda_m w_m|) + \Phi(|\lambda_m w_m|)\big)\,dx
\end{aligned}$$

$$+ \epsilon_m|\lambda_m| - \int_{\Omega_m^2} K(x)f(\lambda_m w_m)\lambda_m w_m\, dx$$
$$\le \varphi^0 \widehat{M}(\rho(\lambda_m w_m)) + \epsilon_m|\lambda_m| - \int_{\Omega_m^2} K(x)f(\lambda_m w_m)\lambda_m w_m\, dx.$$

We know that

$$0 < (\varphi^0 + \theta(s_{\lambda_m}))F(\lambda_m w_m) \le f(\lambda_m w_m)\lambda_m w_m \quad \text{in } \Omega_m^1.$$

Combining this with (2.10) we then have

$$\begin{aligned}
(2.11)\qquad J(u_m) &= J(\lambda_m w_m)\\
&= \widehat{M}(\rho(\lambda_m w_m)) - \int_{\Omega_m^1} K(x)F(\lambda_m w_m)\, dx - \int_{\Omega_m^2} K(x)F(\lambda_m w_m)\, dx\\
&\ge \widehat{M}(\rho(\lambda_m w_m)) - \frac{1}{\varphi^0 + \theta(s_{\lambda_m})}\int_{\Omega_m^1} K(x)f(\lambda_m w_m)\lambda_m w_m\, dx\\
&\quad - \int_{\Omega_m^2} K(x)F(\lambda_m w_m)\, dx\\
&\ge \widehat{M}(\rho(\lambda_m w_m)) - \frac{\varphi^0}{\varphi^0 + \theta(s_{\lambda_m})}\widehat{M}(\rho(\lambda_m w_m)) - \frac{\epsilon_m|\lambda_m|}{\varphi^0 + \theta(s_{\lambda_m})}\\
&\quad + \frac{1}{\varphi^0 + \theta(s_{\lambda_m})}\int_{\Omega_m^2} K(x)f(\lambda_m w_m)\lambda_m w_m\, dx - \int_{\Omega_m^2} K(x)F(\lambda_m w_m)\, dx\\
&= \frac{\theta(s_{\lambda_m})}{\varphi^0 + \theta(s_{\lambda_m})}\widehat{M}(\rho(\lambda_m w_m)) - \frac{\epsilon_m|\lambda_m|}{\varphi^0 + \theta(s_{\lambda_m})} + \psi(\lambda_m w_m)\\
&\ge m_0\frac{\varphi^0\theta(s_{\lambda_m})}{\varphi^0 + \theta(s_{\lambda_m})}|\lambda_m|^{\varphi^0} - \frac{\epsilon_m|\lambda_m|}{\varphi^0 + \theta(s_{\lambda_m})} + \psi(\lambda_m w_m)\\
&\ge |\lambda_m|\left[m_0\frac{\varphi^0\theta(s_{\lambda_m})}{\varphi^0 + N_0}|\lambda_m|^{\varphi^0 - 1} - \frac{\epsilon_m}{\varphi^0}\right] + \psi(\lambda_m w_m),
\end{aligned}$$

where

$$\psi(\lambda_m w_m) = \int_{\Omega_m^2}\left(\frac{1}{\varphi^0 + \theta(s_{\lambda_m})}K(x)f(\lambda_m w_m)\lambda_m w_m - K(x)F(\lambda_m w_m)\right) dx.$$

By (F_1), the sequence $\{\psi(\lambda_m w_m)\}$ is bounded from below. On the other hand, we know that $|\lambda_m| \to \infty$, and so $|s_{\lambda_m}| \to \infty$ as $m \to \infty$. By (F_1),

$$\lim_{m\to\infty} |\lambda_m|^{\varphi^0 - 1}\theta(s_{\lambda_m}) \ge \lim_{m\to\infty} \frac{|s_{\lambda_m}|\theta(s_{\lambda_m})}{L} = \infty.$$

Hence, $J(u_m) \to \infty$, and we obtain a contradiction. Now, $\{u_m\}$ is bounded in $W^1L_\Phi(\Omega)$. Since $W^1L_\Phi(\Omega)$ is compactly embedded into $L^\infty(\Omega)$, there exist a function $u \in W^1L_\Phi(\Omega)$ and a subsequence of $\{u_m\}$, still denoted by

$\{u_m\}$, which converges strongly to u in $L^\infty(\Omega)$. From this and the continuity of f, we have

$$(2.12)\qquad \Big|\int_\Omega K(x)f(u_m)(u_m-u)\,dx\Big| \le \|K\|_\infty \max_{|s|\le\|u\|_\infty+1}|f(s)|\,\|u_m-u\|_\infty \to 0$$

as $m\to\infty$. Combining (2.8) and (2.12) implies that

$$(2.13)\qquad M(\rho(u_m))\int_\Omega \big(a(|\nabla u_m|)\nabla u_m(\nabla u_m-\nabla u)+a(|u_m|)u_m(u_m-u)\big)\,dx\to 0$$

as $m\to\infty$. By (M_1), it is easy to see from (2.13) that

$$(2.14)\qquad \int_\Omega \big(a(|\nabla u_m|)\nabla u_m(\nabla u_m-\nabla u)+a(|u_m|)u_m(u_m-u)\big)\,dx\to 0$$

as $m\to\infty$. By [MR, Proposition 4.5], the sequence $\{u_m\}$ converges strongly to u in $W^1L_\Phi(\Omega)$ and the functional J satisfies the (PS) condition. ∎

Proof of Theorem 2.2. By Lemmas 2.4–2.6, the functional J satisfies the conditions of the classical mountain pass theorem due to Ambrosetti and Rabinowitz [AR]. Thus, we obtain a nontrivial weak solution of (1.1).

If further f is odd, then J is even. We will use the following $\mathbb{Z}_2$ version of the mountain pass theorem in [AR].

PROPOSITION 2.7. *Let E be an infinite-dimensional Banach space, and let $J\in C^1(E,\mathbb{R})$ be even, satisfy the* (PS) *condition, and have $J(0)=0$. Assume that $E=V\oplus X$, where V is finite-dimensional. Suppose that:*

(i) *there are constants $\rho,\alpha>0$ such that $\inf_{\partial B_\rho\cap X}J\ge\alpha$;*
(ii) *for each finite-dimensional subspace $\widehat{E}\subset E$, there is $R=R(\widehat{E})$ such that $J(u)\le 0$ on $\widehat{E}\setminus B_{R(\widehat{E})}$.*

Then J has an unbounded sequence of critical values.

By Lemma 2.3, the functional J satisfies Proposition 2.7(i) and the (PS) condition. For any finite-dimensional subspace $\widehat{E}\subset E:=W^1L_\Phi(\Omega)$, $S\cap\widehat{E}=\{w\in\widehat{E}:\|w\|_{1,\Phi}=1\}$ is compact. By Lemma 2.5 and the finite covering theorem, it is easy to verify that J satisfies condition (ii) of Proposition 2.7. Therefore, J has a sequence $\{u_m\}$ of critical points. That is, problem (1.1) has infinitely many pairs of solutions. ∎

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