ANNALES
POLONICI MATHEMATICI
89.2 (2006)

Weierstrass division theorem in definable C^∞ germs in a polynomially bounded o-minimal structure

by Abdelhafed Elkhadiri and Hassan Sfouli (Kénitra)

Abstract. We give some examples of polynomially bounded o-minimal expansions of the ordered field of real numbers where the Weierstrass division theorem does not hold in the ring of germs, at the origin of $\mathbb{R}^n$, of definable C^∞ functions.

Introduction. In this paper we consider the problem of extending the Weierstrass division theorem to some quasianalytic local rings of germs of C^∞ functions of n real variables, $n \geq 2$. C. L. Childress [4] proved that, in the ring of germs of C^∞ functions in some fixed quasianalytic Denjoy–Carleman class, the Weierstrass division theorem does not hold unless the class is analytic. Another way to yield quasianalytic C^∞ functions is to consider the germs of C^∞ functions definable in a polynomially bounded o-minimal expansion of the ordered field of real numbers. The ring of germs of these functions shares some good properties with the analytic germs ring, for instance, it is Henselian and closed under differentiations [9]. It is unknown whether it is Noetherian.

We give an example of a polynomially bounded o-minimal structure, $\mathcal{R}$, where the Weierstrass division theorem does not hold in the ring of germs, at the origin, of C^∞ functions definable in $\mathcal{R}$. In this connection the following question asked by Lou van den Dries in [6] arises. Does the Weierstrass division theorem hold for the ring of germs of real analytic functions definable in an o-minimal structure (not necessarily polynomially bounded), extending the structure of real numbers? We also give a negative answer to this question.

2000 *Mathematics Subject Classification*: Primary 03C50, 26E10; Secondary 26E05, 03C65.

Key words and phrases: o-minimal structures, power series, Weierstrass division theorem.

Partially supported by PARS MI 33.

Finally, we give a condition under which an element in the ring of germs, at the origin, of C^∞ functions, definable in a polynomially bounded o-minimal structure, has the Weierstrass division property. As an application, we give examples of closed ideals in these rings.

1. Definitions and recalls. Throughout this paper, $\mathcal{R}$ denotes a fixed (but arbitrary) expansion of the structure $\overline{\mathbb{R}} = (\mathbb{R}, <, 0, 1, +, -, \cdot)$ in a first-order language extending $\{<, 0, 1, +, -, \cdot\}$. *Definable* means first-order definable in $\mathcal{R}$ with parameters from $\mathbb{R}$. A function $f : X \to \mathbb{R}$, $X \subseteq \mathbb{R}$, is said to be *definable* if its graph is definable. We say that $\mathcal{R}$ is *polynomially bounded* if for every definable function $f : \mathbb{R} \to \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $|f(t)| \leq t^N$ for all sufficiently large positive t. We say that $\mathcal{R}$ is *o-minimal* if the definable subsets of $\mathbb{R}$ are just finite unions of intervals of all kinds, including singletons.

(A) The structure $\overline{\mathbb{R}} = (\mathbb{R}, <, 0, 1, +, -, \cdot)$ is polynomially bounded and o-minimal (by Tarski–Seidenberg); the sets definable in this structure are precisely the semialgebraic sets (see [3] for a thorough treatment of semialgebraic sets).

(B) A polynomially bounded o-minimal structure in which non-semialgebraic sets are definable, due to Denef and van den Dries [5], is the ordered field of real numbers with restricted analytic functions:

$$\mathbb{R}_{\mathrm{an}} = (\mathbb{R}, <, 0, 1, \cdot, +, -, (\widetilde{f})_{f \in \mathbb{R}\{X,m\}, m \in \mathbb{N}})$$

where $\mathbb{R}\{X, m\}$ denotes the ring of all power series in $X_1, \ldots, X_n$ over $\mathbb{R}$ that converge in a neighborhood of $[0,1]^m$, and where for each $f \in \mathbb{R}\{X, m\}$, we define $\widetilde{f} : \mathbb{R}^m \to \mathbb{R}$ by

$$\widetilde{f}(\overline{x}) := \begin{cases} f(\overline{x}), & \overline{x} \in [0,1]^m, \\ 0, & \overline{x} \in \mathbb{R}^m \setminus [0,1]^m. \end{cases}$$

The sets definable in $\mathbb{R}_{\mathrm{an}}$ are the finitely subanalytic sets (see [2] for general facts about subanalytic sets).

(C) The structure $\mathbb{R}_{\mathrm{an}}^{\mathbb{R}} := (\mathbb{R}_{\mathrm{an}}, (x \mapsto x^r)_{r \in \mathbb{R}})$, where we set $x^r = 0$ for $x \leq 0$, is a polynomially bounded o-minimal expansion of $(\mathbb{R}, +, \cdot)$ [10]. The class of sets definable in this structure properly contains the class of finitely subanalytic sets. By [10], the function $x^r : (0, \infty) \to \mathbb{R}$ is definable in $\mathbb{R}_{\mathrm{an}}$ if and only if r is rational.

(D) The structure $\mathbb{R}_{\mathcal{G}}$ is an expansion of $\mathbb{R}_{\mathrm{an}}$ by adding functions given by multisummable real series. Among the basic operations of $\mathbb{R}_{\mathcal{G}}$ are the C^∞ functions $f : [0,1] \to \mathbb{R}$, whose restrictions to $(0,1]$ extend to holomorphic functions on a sector

$$S(R,\phi) := \{z \in \mathbb{C} : 0 < |z| < R,\ |\arg z| < \phi\}$$

for some $R > 1$ and $\phi > \pi/2$, such that there exist positive constants A, B with $|f^{(n)}(z)| \le AB^n(n!)^2$ for all $z \in S(R,\phi)$, and

$$\lim_{S(R,\phi)\ni z\to 0} f^{(n)}(z) = f^{(n)}(0).$$

An example of such a function is

$$f(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\,dt \quad \text{for } 0 \le x \le 1. \tag{1.1}$$

Its Taylor expansion at 0 is the *divergent* series $\sum_{n=0}^{\infty}(-1)^n n! x^n$. The structure $\mathbb{R}_{\mathcal{G}}$ is polynomially bounded and o-minimal (see [7]).

Let U be an open subset of $\mathbb{R}^n$ ($n \ge 1$), and let $f : U \to \mathbb{R}$ be given. We say that f is C^N, $N \in \mathbb{N} \cup \{\infty\}$, at $a \in U$ if f is C^N on some open neighborhood of a, and that f is analytic at a if f is real analytic on some open neighborhood of a.

Given $a \in \mathbb{R}^n$ we define an equivalence relation $\sim$ on the set of real-valued functions whose domain contains a neighborhood of a by $f \sim g$ if there is a neighborhood V of a, $V \subseteq \mathrm{dom}(f) \cap \mathrm{dom}(g)$, such that $f\lceil V = g\lceil V$. The equivalence classes are called *germs* at a. The equivalence classes of definable functions that are C^∞ at a are called *definable C^∞ germs* at a. These germs can be added and multiplied in the usual way and are easily seen to form a ring, denoted by $\mathcal{D}_a^\infty$. We also let $\mathcal{D}_a^w$ denote the ring of definable real analytic germs at a. For $a = 0 \in \mathbb{R}^n$, we write $\mathcal{D}_n^\infty$ and $\mathcal{D}_n^w$ as appropriate. Clearly, $\mathcal{D}_a^w \subseteq \mathcal{D}_a^\infty$ for all $a \in \mathbb{R}^n$. By [9, Prop. 2], $\mathcal{D}_a^w$ and $\mathcal{D}_a^\infty$ are local rings, and the maximal ideals of $\mathcal{D}_n^\infty$ and $\mathcal{D}_n^w$ are each generated by the germs at 0 of the coordinate functions $x \mapsto x_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$.

If $\mathcal{R}$ is polynomially bounded and o-minimal, then the function

$$T : \mathcal{D}_n^\infty \to \mathbb{R}[[X_1, \ldots, X_n]]$$

sending the germ at 0 of a definable function $f : \mathbb{R}^n \to \mathbb{R}$, C^∞ at 0, to its formal Taylor expansion at 0, is an injective ring homomorphism [9]; thus, if $\mathcal{R}$ is polynomially bounded and o-minimal, we have $\mathcal{D}_n^\infty \simeq T[\mathcal{D}_n^\infty]$. In the following, when $\mathcal{R}$ is polynomially bounded and o-minimal, we will not distinguish notationally between a germ and its image under T, i.e., its Taylor expansion at the origin.

From [10], if $\mathcal{R}$ is a polynomially bounded o-minimal expansion of $\mathbb{R}_{\mathrm{an}}$ such that $\mathcal{D}_1^\infty = \mathcal{D}_1^w$, then $\mathcal{D}_n^\infty = \mathcal{D}_n^w$ for all $n \ge 1$. In [9], it is shown that $\mathcal{D}_1^\infty = \mathcal{D}_1^w$ for $\mathbb{R}_{\mathrm{an}}^{\mathbb{R}}$. For $\mathcal{R} = \mathbb{R}_{\mathcal{G}}$ we can see that $\mathcal{D}_1^w$ is strictly contained in $\mathcal{D}_1^\infty$.

A local ring R with maximal ideal $\underline{m}$ is called *Henselian* if given $P \in R[T]$ and $a \in R$ with $P(a) \in \underline{m}$ and $P'(a)$ invertible, there exists $b \in R$ with $P(b) = 0$ and $a \equiv b \bmod \underline{m}$. It is easy to see that the implicit function theorems (C^∞ and analytic versions) yield definable functions when the data are definable; thus $\mathcal{D}_n^\infty$ and $\mathcal{D}_n^w$ are Henselian rings.

2. Weierstrass systems. Let $\mathbb{K}$ be a field of characteristic 0 and $\mathbb{K}[[X_1, \ldots, X_n]]$ the formal power series ring in $X = (X_1, \ldots, X_n)$ over $\mathbb{K}$. (In this paper we need only the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.)

DEFINITION 2.1 ([6]). We define a *Weierstrass system* over $\mathbb{K}$ to be a family $(\mathcal{A}_n)$ of rings such that for all n the following conditions hold:

- (w_1) $\mathbb{K}[X_1, \ldots, X_n] \subset \mathcal{A}_n \subset \mathbb{K}[[X_1, \ldots, X_n]]$, and if σ is a permutation of $\{1, \ldots, n\}$ and $f(X_1, \ldots, X_n) \in \mathcal{A}_n$, then $f(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) \in \mathcal{A}_n$. Moreover, for each $m > 0$, $\mathcal{A}_{n+m} \cap \mathbb{K}[[X_1, \ldots, X_n]] = \mathcal{A}_n$.
- (w_2) If $f \in \mathcal{A}_n$ is a unit in $\mathbb{K}[[X_1, \ldots, X_n]]$, then f is a unit in $\mathcal{A}_n$.
- (w_3) (Weierstrass division) If $f \in \mathcal{A}_{n+1}$ and $f(0, X_{n+1}) \in \mathbb{K}[[X_{n+1}]]$ is non-zero of order d, then for every $g \in \mathcal{A}_{n+1}$ there are $Q \in \mathcal{A}_{n+1}$ and $R_i \in \mathcal{A}_n$, $i = 1, \ldots, d-1$, such that
$$g = Qf + (R_{d-1} X_{m+1}^{d-1} + \cdots + R_0).$$

We deduce that, for each $n \in \mathbb{N}$, $\mathcal{A}_n$ is a Noetherian local ring with maximal ideal $X\mathcal{A}_n$ and completion $\mathbb{K}[[X_1, \ldots, X_n]]$.

EXAMPLES 2.2.

1. (1) If $\mathcal{R} = \overline{\mathbb{R}}$, the system $(\mathcal{D}_n^w)_{n \in \mathbb{N}}$ is a Weierstrass system ($\mathbb{K} = \mathbb{R}$) (algebraic germs, see [3]).
2. (2) The system $(\mathbb{R}\langle X_1, \ldots, X_n \rangle)_{n \in \mathbb{N}}$ is a Weierstrass system. Here the ring $\mathbb{R}\langle X_1, \ldots, X_n \rangle$ consists of all power series in $\mathbb{R}[[X_1, \ldots, X_n]]$ that converge in some neighborhood of the origin.

3. Some counterexamples. Recall that the structure $\mathbb{R}_\mathcal{G}$ is model complete, polynomially bounded and o-minimal [7].

We first give an example which shows that the Weierstrass division theorem does not generalize to the germs of C^∞ functions definable in the structure $\mathbb{R}_\mathcal{G}$.

We suppose $\mathcal{R} = \mathbb{R}_\mathcal{G}$ and we put $\Phi := \sum_{n=0}^\infty (-1)^n n! Y^{2n}$. We can see that Φ is the Taylor expansion at 0 of the function
$$g(x) := \int_0^\infty \frac{e^{-t}}{1 + tx^2}\, dt.$$
Note that $g(x) = f(x^2)$, where f is the function given by the equation (1.1) of Section 1.

Let $P := Y^2 + X^2$. If the Weierstrass division holds in the system $(\mathcal{D}_n^\infty)_{n\in\mathbb{N}}$, then there are $R_0, R_1 \in \mathcal{D}_1^\infty$ and $Q \in \mathcal{D}_2^\infty$ such that

$$\Phi = (X^2 + Y^2)Q + R_1Y + R_0. \tag{3.1}$$

By substitution $Y = iX$ and $Y = -iX$ in the last equation, we find that

$$R_0 = \sum_{n=0}^{\infty} n!X^{2n}, \qquad R_1 = 0.$$

Since R_0 is definable in $\mathbb{R}_{\mathcal{G}}$ and its Taylor series, at the origin, has all it coefficients non-negative, by [7, Corollary 8.6] this series must converge, which is a contradiction.

As indicated in the introduction, this leads us to ask if the Weierstrass division theorem holds for the ring of germs of real analytic functions definable in an o-minimal structure, a question asked by Lou van den Dries in [6]. We give here a negative answer.

Recall that the structure $\mathbb{R}_{\mathrm{e}} = (\overline{\mathbb{R}}, \exp_{|[0,1]})$ is model complete, polynomially bounded and o-minimal [12].

Example 3.1. If $\mathcal{R} = \mathbb{R}_{\mathrm{e}}$, then the Weierstrass division theorem does not hold in the system $(\mathcal{D}_n^w)_{n\in\mathbb{N}}$; in particular, $(\mathcal{D}_n^w)_{n\in\mathbb{N}}$ is not a Weierstrass system.

Indeed, let $P := X^2 + Y^2$ and put $\exp(Y) := \sum_{n=0}^{\infty} Y^n/n! \in \mathbb{R}[[X, Y]]$. If the Weierstrass division theorem holds in the system $(\mathcal{D}_n^w)_{n\in\mathbb{N}}$, then there are $R_0, R_1 \in \mathcal{D}_1^w$ and $Q \in \mathcal{D}_2^w$ such that

$$\exp(Y) = PQ + R_1Y + R_0. \tag{3.2}$$

By substitution $Y = iX$ and $Y = -iX$ in the last equation, we have

$$R_0 + iXR_1 = \exp(iX), \qquad R_0 - iXR_1 = \exp(-iX),$$

hence

$$R_0 = \cos(X), \qquad R_1 = \sin(X)/X.$$

But we know, by a result of R. Bianconi [1], that the restriction of the sine function to any interval is not definable in $\mathbb{R}_{\mathrm{e}}$, hence the contradiction.

Remark 3.2. By using the result of R. Bianconi, we can also see that if $\mathcal{R} = (\overline{\mathbb{R}}, \sin\lceil[0, \pi])$, then the Weierstrass division theorem does not hold in the system of germs, at the origin, of C^∞ functions definable in $\mathcal{R}$.

4. Weierstrass division property. We fix a polynomially bounded o-minimal structure $\mathcal{R}$ extending $\overline{\mathbb{R}} = (\mathbb{R}, <, 0, 1, +, -, \cdot)$. Let U be an open set in $\mathbb{R}^n$. We denote by $\mathcal{E}_n(U)$ the ring of C^∞ functions on $U \subset \mathbb{R}^n$ and by $\mathcal{D}_n^\infty(U)$ the subring of $\mathcal{E}_n(U)$ of definable functions on U. We recall that $\mathcal{D}_n^\infty$ is the ring of germs of C^∞ functions, in a neighborhood of the origin in $\mathbb{R}^n$, definable in $\mathcal{R}$.

Let $f \in \mathcal{D}_n^\infty$. We say that f is *regular* of order p with respect to x_n if the Taylor series of f at the origin, Tf, is regular of order p with respect to x_n.

DEFINITION 4.1. Let $f \in \mathcal{D}_n^\infty$ be regular of order p with respect to x_n. Then f is said to have the *Weierstrass division property* if given $g \in \mathcal{D}_n^\infty$ there exist unique elements $q \in \mathcal{D}_n^\infty$ and $r_j(x_1, \dots, x_{n-1}) \in \mathcal{D}_{n-1}^\infty$, $0 \le j \le p-1$, such that

$$g = fq + \sum_{j=0}^{p-1} r_j x_n^j.$$

We remark that the uniqueness assumption is superfluous, since the rings $(\mathcal{D}_n^\infty)_n$ are quasianalytic.

As in the analytic case, any $f \in \mathcal{D}_n^\infty$ having the Weierstrass division property can be factorized uniquely as QP, where $Q \in \mathcal{D}_n^\infty$ is unit and $P \in \mathcal{D}_{n-1}^\infty[x_n]$ is a monic polynomial of degree p.

Let U be a subset of $\mathbb{R}^n$ and let $a_1, \dots, a_p$ be real functions on U. We consider the polynomial $P(x,t) = t^p + a_1(x)t^{p-1} + \dots + a_p(x)$ as a function on $U \times \mathbb{C}$.

DEFINITION 4.2. We say that the polynomial $P(x,t)$ is *hyperbolic* if, for each $x \in U$, all the roots of $t \mapsto P(x,t)$ are real.

We suppose that U is a definable bounded connected open subset of $\mathbb{R}^n$ and the functions $a_1, \dots, a_p$ are C^∞ on the closure of U. Then there exists $\eta > 0$ such that, for each $x \in U$, all the roots of $t \mapsto P(x,t)$ are in $I =]-\eta, \eta[$.

THEOREM 4.3. *Suppose that the polynomial $P(x,t)$ is hyperbolic and $a_1, \dots, a_p$ are in $\mathcal{D}_n^\infty(U)$. Then for each $g \in \mathcal{E}_{n+1}(U \times \mathbb{R})$ whose restriction to $U \times I$ is in $\mathcal{D}_{n+1}^\infty(U \times I)$, there exist $Q \in \mathcal{D}_{n+1}^\infty(U \times I)$ and $r_j \in \mathcal{D}_n^\infty(U)$, $0 \le j \le p-1$, such that*

$$g(x,t) = P(x,t)Q(x,t) + \sum_{j=0}^{p-1} r_j(x)t^j, \quad \forall (x,t) \in U \times I.$$

Furthermore, Q and $r_1, \dots, r_{p-1}$ are uniquely determined.

Proof. Let $g \in \mathcal{E}_{n+1}(U \times \mathbb{R})$. By the differentiable preparation theorem [11, IX.2.7], there exist $Q \in \mathcal{E}_{n+1}(U \times \mathbb{R})$ and $r_j \in \mathcal{E}_n(U)$, $j = 1, \dots, p-1$, such that

$$g(x,t) = Q(x,t)P(x,t) + \sum_{j=0}^{p-1} r_j(x)t^j, \quad \forall (x,t) \in U \times \mathbb{R}, \tag{4.1}$$

Let us first prove that $r_j \in \mathcal{D}_n^\infty(U)$ for each $0 \le j \le p-1$. For each $l = 1, \dots, p$, we put

$$B_l = \{x \in U : \exists t_1, \dots, t_l,\ t_i \ne t_j \text{ if } i \ne j, \text{ and } P(x,t_i) = 0,\ \forall i = 1, \dots, l\}.$$

With $B_{p+1}=\emptyset$, we have the sequence of inclusions

$$\emptyset = B_{p+1} \subset B_p \subset B_{p-1} \subset \cdots \subset B_1 = U.$$

We put $A_q = B_q - B_{q+1},\ 1 \le q \le p$; then $U = \bigcup_{q=1}^p A_q$. We remark that A_q is definable for each $q = 1, \ldots, p$.

We will show that the restriction of r_j, $0 \le j \le p-1$, to each A_q is definable. Suppose first that $q = p$, hence $A_p = B_p$. For each $x \in A_p$, the function $t \mapsto P(x,t)$ has p distinct roots. By the proof of Lemma 2 in [8, §20], there exist $\alpha_i : A_p \to \mathbb{R}$ continuous, $1 \le i \le p$, such that, for each $x \in A_p$, $\alpha_1(x) < \cdots < \alpha_p(x)$ and $\alpha_1(x), \ldots, \alpha_p(x)$ are the roots of $P(x,t)$. We remark that the functions $\alpha_i, 1 \le i \le p$, are definable and $\alpha_i(A_p) \subset I$ for all $i = 1, \ldots, p$. For each $x \in A_p$, the functions $r_1(x), \ldots, r_{p-1}(x)$ are determined, from (4.1), by the relation

$$\begin{pmatrix} 1 & \alpha_1(x) & \ldots & \alpha_1(x)^{p-1} \\ 1 & \alpha_2(x) & \ldots & \alpha_2(x)^{p-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_p(x) & \ldots & \alpha_p(x)^{p-1} \end{pmatrix} \begin{pmatrix} r_0(x) \\ \vdots \\ r_{p-1}(x) \end{pmatrix} = \begin{pmatrix} g(x,\alpha_1(x)) \\ \vdots \\ g(x,\alpha_p(x)) \end{pmatrix}$$

Since the determinant, say $\Delta(x)$, of the matrix is not null, $\Delta(x) = \prod_{i<j}(\alpha_i(x) - \alpha_j(x))^2$, and the functions $x \mapsto g(x,\alpha_i(x))$, $1 \le i \le p$, are definable on A_p, we see that $r_{0|A_p}, \ldots, r_{p-1|A_p}$ are definable.

Consider now $A_{p-1} = B_{p-1} - B_p$. As for A_p, for each $x \in A_{p-1}$, the function $t \mapsto P(x,t)$ has $p-1$ distinct roots, one of them double. Again by the proof of Lemma 2 in [8, §20], there exist $\alpha_i : A_{p-1} \to \mathbb{R}$ continuous and definable, $1 \le i \le p-1$, such that, for each $x \in A_{p-1}$, $\alpha_1(x) < \cdots < \alpha_{p-1}(x)$ and $\alpha_1(x), \ldots, \alpha_{p-1}(x)$ are the roots of $P(x,t)$. For $i = 1, \ldots, p-1$, we put

$$\Lambda_i = \left\{ x \in A_{p-1} : \frac{\partial P(x,\alpha_i(x))}{\partial t} = 0 \right\};$$

then $A_{p-1} = \bigcup_{i=1}^{p-1} \Lambda_i$.

We want to show that, for each $j = 1, \ldots, p-1$, the restriction of r_j to each Λ_i, $1 \le i \le p-1$, is definable. We prove it for Λ_1, for the others the proof is the same.

For each $x \in \Lambda_1$, the roots of $P(x,t)$ are $\alpha_1(x) < \cdots < \alpha_{p-1}(x)$ and $\alpha_1(x)$ is a double root. We have the following identity, obtained from (4.1):

$$(4.2) \quad \frac{\partial g(x,t)}{\partial t} = Q(x,t)\,\frac{\partial P(x,t)}{\partial t} + \frac{\partial Q(x,t)}{\partial t}\,P(x,t) + \sum_{j=1}^{p-1} r_j(x) t^{j-1},$$

hence the following relation:

$$\begin{pmatrix} 1 & \alpha_1(x) & \alpha_1(x)^2 & \dots & \alpha_1(x)^{p-1} \\ 0 & 1 & 2\alpha_1(x) & \dots & (p-1)\alpha_1(x)^{p-2} \\ 1 & \alpha_3(x) & \alpha_3(x)^2 & \dots & \alpha_3(x)^{p-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_p(x) & \alpha_p(x)^2 & \dots & \alpha_p(x)^{p-1} \end{pmatrix} \begin{pmatrix} r_0(x) \\ \vdots \\ r_{p-1}(x) \end{pmatrix} = \begin{pmatrix} g(x,\alpha_1(x)) \\ \frac{\partial g}{\partial t}(x,\alpha_1(x)) \\ \vdots \\ g(x,\alpha_p(x)) \end{pmatrix},$$

where the second line in the matrix is obtained from (4.2) and the other lines from (4.1). Now the matrix

$$M := \begin{pmatrix} 1 & \alpha_1(x) & \alpha_1(x)^2 & \dots & \alpha_1(x)^{p-1} \\ 0 & 1 & 2\alpha_1(x) & \dots & (p-1)\alpha_1(x)^{p-2} \\ 1 & \alpha_3(x) & \alpha_3(x)^2 & \dots & \alpha_3(x)^{p-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_p(x) & \alpha_p(x)^2 & \dots & \alpha_p(x)^{p-1} \end{pmatrix}$$

is again non-singular for each $x \in \Lambda_1$. We can see this by taking the homogeneous system with this matrix, i.e.,

$$M \begin{pmatrix} X_1 \\ \vdots \\ X_p \end{pmatrix} = 0. \tag{4.3}$$

Indeed, if $\lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{R}^p$ is a solution of (4.3), then the polynomial $\lambda_1 + \lambda_2 t + \dots + \lambda_p t^{p-1}$ has p roots, hence $\lambda_1 = \dots = \lambda_p = 0$. We deduce that the restriction of each r_j, $1 \le j \le p-1$, to Λ_1 is definable.

Suppose now that $p > 1$ and $q < p-1$, and consider $A_q = B_q - B_{q+1}$. There exist definable continuous functions $\alpha_i : A_q \to \mathbb{R}$, $1 \le i \le q$, such that, for each $x \in A_q$, $\alpha_1(x) < \dots < \alpha_q(x)$ and $\alpha_1(x), \dots, \alpha_q(x)$ are the roots of $P(x,t)$. For each finite sequence of positive integers $n_1, \dots, n_q$ with $n_1 + \dots + n_q = p$, let

$$\Lambda_{n_1,\dots,n_q} = \{x \in A_q : \alpha_j(x) \text{ is a root of order } n_j,\ 1 \le j \le q\}.$$

Then $A_q = \bigcup \Lambda_{n_1,\dots,n_q}$, where the union is over all $n_1, \dots, n_q$ such that $n_1 + \dots + n_q = p$. We have to show that the restriction of each r_j to $\Lambda_{n_1,\dots,n_q}$ is definable. By taking the partial derivatives of (4.1) with respect to t of orders $n_1, \dots, n_q$, we get p equations and a non-singular matrix as above. We deduce that the restrictions of $r_0, \dots, r_{p-1}$ to $\Lambda_{n_1,\dots,n_q}$ are definable. Hence the restriction of each r_j, $1 \le j \le p-1$, to A_q is definable.

By (4.1), the function $(x,t) \mapsto Q(x,t)P(x,t)$ is in $\mathcal{D}^\infty_{n+1}(U \times I)$, and the assertion about the restriction of Q to $U \times I$ follows from Lemma 4.4.

To prove uniqueness in (4.1), suppose that

$$g(x,t) = P(x,t)Q(x,t) + \sum_{j=0}^{p-1} r_j(x)t^j = P(x,t)Q'(x,t) + \sum_{j=0}^{p-1} r'_j(x)t^j.$$

Then

$$(Q - Q')(x,t)P(x,t) = \sum_{j=0}^{p-1} (r_j - r'_j)(x)t^j.$$

This implies that, for each $x \in U$, the polynomial $\sum_{j=0}^{p-1}(r_j - r'_j)(x)t^j$ has p roots, hence $r_j(x) = r'_j(x)$ for all $j = 1, \dots, p-1$. We then have $(Q - Q')(x,t)P(x,t) = 0$ for all $(x,t) \in U \times I$. Since $\mathcal{D}^\infty_{n+1}(U \times I)$ is a domain, it follows that $Q(x,t) = Q'(x,t)$ for all $(x,t) \in U \times I$.

Lemma 4.4. *Let $f, g \in \mathcal{D}^\infty_n(U) - \{0\}$ and suppose that $g = hf$, where $h \in \mathcal{E}_n(U)$. Then $h \in \mathcal{D}^\infty_n(U)$.*

Proof. We have to prove that the graph of h, Γ_h, is definable. Let $Y = \{x \in U : f(x) = 0\}$. Since U is connected, the interior of Y is empty, hence $\overline{U - Y} \cap U = U$. The set $\Gamma_h \cap (U - Y) \times \mathbb{R}$ is definable. Hence so is $\overline{\Gamma_h \cap (U - Y) \times \mathbb{R}} \cap U \times \mathbb{R}$. Since h is continuous, we have $\Gamma_h = \overline{\Gamma_h \cap (U - Y) \times \mathbb{R}} \cap U \times \mathbb{R}$, hence the result.

Definition 4.5. Let $P \in \mathcal{D}^\infty_n[t]$. We say that P is a *hyperbolic polynomial* if P is monic and there exists a neighborhood U of $0 \in \mathbb{R}^n$ such that all the coefficients of P are C^∞, definable on U, and $P(x,t)$ is hyperbolic on $U \times \mathbb{C}$.

As a consequence of the theorem, we have

Corollary 4.6. *Let $P \in \mathcal{D}^\infty_n[t]$ be a hyperbolic polynomial. Suppose that P is regular of order p with respect t. Then P has the Weierstrass division property.*

Proof. Let $g \in \mathcal{D}^\infty_{n+1}$. There exists a definable connected open bounded neighborhood U of 0 in $\mathbb{R}^n$ and a nonempty interval $I =]-\eta, \eta[\subset \mathbb{R}$ such that:

- all the coefficients of P are in $\mathcal{D}^\infty_n(U)$ and they are C^∞ on the closure of U;
- for each $x \in U$, all the roots of $t \mapsto P(x,t)$ are in I and $g \in \mathcal{D}^\infty_{n+1}(U \times I) \cap \mathcal{E}_{n+1}(\overline{U \times I})$.

By Whitney's extension theorem, there exists $\widetilde{g} \in \mathcal{E}_{n+1}(\mathbb{R}^n \times \mathbb{R})$ such that $\widetilde{g}_{|U\times I} = g$. We put $\widetilde{G} = \widetilde{g}_{|U\times\mathbb{R}}$, and apply Theorem 4.3 to $\widetilde{G}$ and P.

5. Closed ideals. Let $U \subset \mathbb{R}^n$ be an open set and $\mathcal{E}(U)$ the ring of C^∞ functions on U. For each non-negative integer k and each compact subset $K \subset U$, we define the seminorm

$$p_{k,K}(\varphi) = \sup\{|D^\alpha\varphi(x)| : |\alpha| \le k,\ x \in K\},$$

where $\varphi \in \mathcal{E}(U)$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_1}$. The family of all seminorms $p_{k,K}$ induces a locally convex Hausdorff topology on $\mathcal{E}(U)$.

If $\varphi \in \mathcal{E}(U)$ and $a \in U$, let $T_a\varphi$ denote the Taylor series of φ at a. Let $J \subset \mathcal{E}(U)$ be an ideal. By Malgrange's theorem [11, V.1.5], the closure of J, $\overline{J}$, is the set of all $\varphi \in \mathcal{E}(U)$ such that $T_a\varphi \in T_aJ$ for each $a \in U$, where

$$T_aJ = \{T_a\psi : \psi \in J\}.$$

We fix a polynomially bounded o-minimal structure $\mathcal{R}$ extending $\overline{\mathbb{R}} = (\mathbb{R}, <, 0, 1, +, -, \cdot)$. Let U be as in Theorem 4.3 and $P \in \mathcal{D}_n^\infty(U)[t]$ be a hyperbolic polynomial of degree p. We also suppose that P satisfies the hypothesis of Theorem 4.3. We have:

Proposition 5.1. *Let P be as above. Then*

$$\overline{P\mathcal{D}_{n+1}^\infty(U \times \mathbb{R})} \cap \mathcal{D}_{n+1}^\infty(U \times \mathbb{R}) = P\mathcal{D}_{n+1}^\infty(U \times \mathbb{R}).$$

Proof. Let $g \in \mathcal{D}_{n+1}^\infty(U \times \mathbb{R})$. Since the polynomial P is hyperbolic, we have, by Theorem 4.3,

$$g(x,t) = q(x,t)P(x,t) + \sum_{j=0}^{p-1} r_j(x)t^j, \tag{5.1}$$

where $q \in \mathcal{D}_{n+1}^\infty(U \times \mathbb{R})$, and $r_j \in \mathcal{D}_n^\infty(U)$ for all $j = 0, \ldots, p-1$. If $g \in \overline{P\mathcal{D}_{n+1}^\infty(U \times \mathbb{R})}$, then $R := \sum_{j=0}^{p-1} r_j(x)t^j \in \overline{P\mathcal{D}_{n+1}^\infty(U \times \mathbb{R})}$. Hence for each $x \in U$, the polynomial $\sum_{j=0}^{p-1} r_j(x)t^j$ has p roots, so we have $r_j = 0$ for all $j = 0, \ldots, p-1$, which proves the proposition.

Acknowledgements. The authors want to thank the referee for his remarks.

References

[1] R. Bianconi, *Nondefinability results for expansions of the field of real numbers by the exponential function and by the restricted sine function*, J. Symbolic Logic 62 (1997), 1173–1178.

[2] E. Bierstone and P. Milman, *Semianalytic and subanalytic sets*, Inst. Hautes Études Sci. Publ. Math. 67 (1988), 27–33.

[3] J. Bochnak, M. Coste et M. F. Roy, *Géométrie algébrique réelle*, Springer, Berlin, 1987.

[4] L. C. Childress, *Weierstrass division in quasianalytic local rings*, Canad. J. Math. 28 (1976), 938–953.

[5] J. Denef and L. van den Dries, *p-adic and real subanalytic sets*, Ann. of Math. (2) 128 (1988), 79–138.

[6] L. van den Dries, *On the elementary theory of restricted elementary functions*, J. Symbolic Logic 53 (1988), 796–808.

[7] L. van den Dries and P. Speissegger, *The field with multisummable series and the exponential function*, Proc. London Math. Soc. 81 (2000), 513–565.

[8] S. Łojasiewicz, *Ensembles semi-analytiques*, I.H.E.S., Bures-sur-Yvette, 1964.

[9] C. Miller, *Infinite differentiability in polynomially bounded o-minimal structures*, Proc. Amer. Math. Soc. 123 (1995), 2551–2555.

[10] ——, *Expansions of the real field with power functions*, Ann. Pure Appl. Logic 68 (1994), 79–94.

[11] J. C. Tougeron, *Idéaux de fonctions différentiables*, Ergeb. Math. Grenzgeb. 71, Springer, Berlin, 1972.

[12] A. Wilkie, *Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function*, J. Amer. Math. Soc. 9 (1996), 1051–1094.

Department of Mathematics
Faculty of Sciences
University Ibn Tofail
BP 133 Kénitra, Morocco
E-mail: elkhadiri@univ-ibntofail.ac.ma
hassansfouli@hotmail.com

Received 25.4.2005
and in final form 7.3.2006 (1577)