ANNALES
POLONICI MATHEMATICI
89.2 (2006)

A remark on semi-∇-flat functions

by Wojciech Kozłowski (Łódź)

Abstract. We give a pointwise characterization of semi-∇-flat functions on an affine manifold (M, ∇).

1. Introduction. Let (M, ∇) be an affine manifold with M connected. In [1] we introduced the concept of ∇-flat and pointwise-∇-flat functions on (M, ∇) as follows: Let f be a smooth function on M. Then f is called ∇-*flat* if $\nabla^k f = 0$ for some $k \geq 0$, and *pointwise*-∇-*flat* if for each $x \in M$ there exists $k = k(x) \geq 0$ such that $(\nabla^k f)(x) = 0$. Clearly, $\nabla^k = \nabla \circ \cdots \circ \nabla$ (k times).

Obviously, each ∇-flat function is pointwise-∇-flat. Conversely (Thm. 2.1 in [1]) we have

Theorem 1. *If* (M, ∇) *is real-analytic then any smooth pointwise*-∇-*flat function on* (M, ∇) *is real-analytic and* ∇-*flat.*

In this short note, we introduce a concept of semi-∇-flat functions and pointwise-semi-∇-flat functions, which slightly generalizes the concept of ∇-flat and pointwise-∇-flat functions. The main result (Thm. 2 in the next section), whose idea was suggested by Professor Michael Eastwood, asserts that a smooth function f is semi-∇-flat iff f is pointwise-semi-∇-flat. Now we define the objects we are interested in.

Let f be a smooth function on M. For any integer $k \geq 0$ and $p \in M$ let $b_p^k = b_p^{k,f}$ be defined by $b_p^k(v) = \nabla^k f(v, \ldots, v)$, $v \in T_pM$. Define the map $b^k \in C^\infty(TM)$ by putting $b^k(v) = b_p^k(v)$ if $v \in T_p$. The function f is called *semi*-∇-*flat* if there exists $k > 0$ such that $b^k = 0$, and *pointwise-semi*-∇-*flat* if for each $p \in M$ there exists $k = k(p) > 0$ such that $b_p^k = 0$. Clearly, each semi-∇-flat function is pointwise-semi-∇-flat.

2000 *Mathematics Subject Classification*: 53A15, 53C05.
Key words and phrases: affine manifold, ∇-flat function.

2. The main result. Before we formulate and prove the main result, we introduce some tools. Let (M,∇) be a smooth affine manifold with M connected. Take $p\in M$, and let G be an open and star-shaped neighbourhood of zero in T_pM such that the exponential mapping exp at p is defined on G. For any $v\in G$ let $\varphi_v : I\to M$ denote the geodesic curve such that $\varphi_v(I)\subset \exp(G)$, $\varphi_v(0)=p$ and $\dot\varphi_v(0)=v$. In the proof of Theorem 2 we will need the following

PROPOSITION 1. *Let $k\ge 0$. Suppose that f is a smooth function on M. If $\varphi_v^\star\nabla^k f=0$ for each $v\in G$, then $f\circ\exp$ is a polynomial function on G of degree $<k$.*

LEMMA 1. *Suppose that f is a pointwise-semi-∇-flat function on M. If $\varphi : I\to M$ is a geodesic then there exists $k\ge 0$ such that $\varphi^\star\nabla^k f=0$. Moreover, if $t\in I$ and $b^l=0$ in some neighbourhood of $\varphi(t)$ then $k\le l$.*

Proposition 1 is proved in [1]. To prove Lemma 1 observe the following: Let D be the canonical connection on I. Since $\varphi : (I,D)\to(M,\nabla)$ is an affine map,

$$(\star)\qquad \frac{d^m(f\circ\varphi)}{dt^m}=\varphi^\star\nabla^m f$$

for any $m\ge 0$. Lemma 1 is now a direct consequence of $(\star)$ and the following very well known fact: *Suppose $h : I\to\mathbb{R}$ is smooth. If for each $s\in I$ there exists $k=k(s)\ge 0$ such that $(d^kh/dt^k)(s)=0$ then h is a polynomial function.*

THEOREM 2. *If (M,∇) is an affine manifold with M connected then any smooth pointwise-semi-∇-flat function on (M,∇) is semi-∇-flat.*

REMARK. In contrast to Theorem 1 we do not require analyticity of (M,∇).

Proof of Theorem 2. Let f be pointwise-semi-∇-flat. For any $k\ge 0$ define the open subset V_k as follows; $p\in V_k$ if $b^k=0$ on some open neighbourhood of p. Put $V=\bigcup_{k=0}^\infty V_k$. Clearly, $V_l\subset V_{l+1}$, V is open, and using the Baire property one easily checks that V is dense in M.

Let $r=\min\{k : V_k\ne\emptyset\}$. We will show that $b^r=0$ on M. Since M is connected it suffices to show that V_r is closed.

Take any $q\in\overline{V}_r$. Let W be a small neighbourhood of q such that any $p\in W$ has a normal neighbourhood which contains W. Take $l\ge 0$ such that $V_l\cap W\ne\emptyset$. Let $p\in V_l\cap W$. Let G be as in Proposition 1, $\Omega=\exp(G)$ and $W\subset\Omega$. By Lemma 1 and Proposition 1, $P=f\circ\exp : G\to\mathbb{R}$ is a polynomial function of degree $<l$. Clearly, $\Omega\cap V_r\ne\emptyset$ and also by Lemma 1, for each geodesic $\varphi : I\to\Omega$ such that $\varphi(I)\cap V_r\ne\emptyset$, we have $\varphi^\star\nabla^r f=0$.

Let $U = \exp^{-1}(\Omega \cap V_r)$. By $(\star)$, for any $v \in U$ the map $t \mapsto P(tv)$ is a polynomial function (in one variable) of degree $< r$. Since U is non-empty and open, $\deg P < r$.

For any $v \in G$ let φ_v be the geodesic defined by $\varphi_v(t) = \exp(tv)$. Using $(\star)$ again we see that $\varphi_v^\star \nabla^r f = 0$, so $b_p^r = 0$. Since $p \in W \cap V_l$ was taken arbitrary, $W \cap V_l \subset W \cap V_r$. Since $V \cap \Omega$ is dense in Ω, $b^r = 0$ on Ω. Thus $\Omega \subset V_r$, so V_r is closed. ∎

Acknowledgements. The author would like to thank Professor Paweł Walczak for a helpful discussion.

References

[1] W. Kozłowski, *∇-flat functions on manifolds*, Ann. Polon. Math. 84 (2004), 177–180.

Institute of Mathematics
Polish Academy of Sciences
Łódź Branch
Banacha 22
90-238 Łódź, Poland
E-mail: wojciech@math.uni.lodz.pl

Received 12.12.2005 (1649)