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A domain whose envelope of holomorphy is not a domain

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Abstract. We construct a domain of holomorphy in $\mathbb{C}^N$, $N \geq 2$, whose envelope of holomorphy is not diffeomorphic to a domain in $\mathbb{C}^N$.

The envelope of holomorphy plays a central role in multivariate complex analysis; the standard textbooks in several complex variables give various constructions for it. Given a domain D in $\mathbb{C}^N$, the *envelope of holomorphy* of D is a Riemann domain (D^*, π) so that $\pi : D^* \to \mathbb{C}^N$ is a locally biholomorphic map. Roughly speaking, (D^*, π) is the largest Riemann domain to which all functions holomorphic on D extend. The manifold D^* is presented as an abstractly given complex manifold, and, in general, the map π is not injective. Thus, the manifold D^* is not presented as a domain in $\mathbb{C}^N$. The question arose in a recent discussion with some colleagues whether D^* might, nonetheless, be biholomorphically equivalent to a domain in $\mathbb{C}^N$.

It is the purpose of this note to exhibit a domain Ω in $\mathbb{C}^N$, $N \geq 2$, whose envelope of holomorphy is not biholomorphically equivalent to a domain in $\mathbb{C}^N$. The construction of Ω requires the following lemma, in which $\mathbb{B}_n$ denotes the open unit ball in $\mathbb{C}^n$ centered at the origin.

Lemma. *In $\mathbb{C}^n$, $n \geq 2$, let*

$$D_\varepsilon = \{z = x + iy \in \mathbb{C}^n = \mathbb{R}^n_x + i\mathbb{R}^n_y : z \in \mathbb{B}_n \text{ and } |y| > \varepsilon\}.$$

If K is a compact subset of $\mathbb{B}_n$, then for sufficiently small $\varepsilon > 0$ there is a domain $\widetilde{D}_\varepsilon \subset \mathbb{B}_n$ that contains D_ε and K, and with the property that each function holomorphic on D_ε continues holomorphically into $\widetilde{D}_\varepsilon$.

Precisely: The fixed domain $\widetilde{D}_\varepsilon$ has the property that for each function f holomorphic on D_ε there is a corresponding (single-valued) function $\widetilde{f}$ holomorphic on $\widetilde{D}_\varepsilon$ that is an extension of f.

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Proof. The set $b\mathbb{B}_n \cap \{z : y = 0\}$ is the $(n-1)$-sphere $\mathbb{S}^{n-1} = b\mathbb{B}_n \cap \mathbb{R}^n_x$, which, as a compact subset of $\mathbb{R}^n$, is polynomially convex. Consequently, there is $\varepsilon > 0$ small enough that the polynomially convex hull of the set $S_\varepsilon = b\mathbb{B}_n \cap \{z : |y| \leq \varepsilon\}$ is contained in $\overline{\mathbb{B}}_n \setminus K$. Thus, if $\varepsilon > 0$ is small, then $\widehat{S}_\varepsilon$, the polynomially convex hull of the set S_ε, is disjoint from K. A function holomorphic on D_ε continues holomorphically into $\mathbb{B}_n \setminus \widehat{S}_\varepsilon$. For this continuation, one can consult the appendix to [5] or [1]. For the $\widetilde{D}_\varepsilon$ of the lemma, we can take the latter set.

We now proceed to the construction of Ω.

Fix an integer $N \geq 2$. The space $\mathbb{C}^N$ contains N-dimensional compact totally real submanifolds, e.g., the unit torus $\mathbb{T}^N$, which is the distinguished boundary of the unit polydisc in $\mathbb{C}^N$. The paper [6]—see in particular Lemmas 3, 4, and 5—therefore provides in $\mathbb{C}^N$ a pair M_1 and M_2 of compact, connected, totally real N-dimensional submanifolds of class $\mathscr{C}^\infty$ with the following properties:

(a) $M_1 \cap M_2$ consists of two points, say p_1 and p_2.
(b) In a neighborhood of p_1 and in a neighborhood of p_2 the manifolds M_1 and M_2 coincide with their tangent planes and these tangent planes are transversal.
(c) For small balls B_j centered at p_j, each of the intersections $\overline{B}_j \cap (M_1 \cup M_2)$ is polynomially convex.

It is further shown that

(d) The union $M_1 \cup M_2$ has a neighborhood basis consisting of Stein domains.

Let us denote the union $M_1 \cup M_2$ by Σ. That Σ has a Stein neighborhood basis implies that it is holomorphically convex in the sense of Harvey and Wells [2], i.e., that every nonzero complex homomorphism of the algebra $\mathscr{O}(\Sigma)$ of germs of functions holomorphic on Σ is of the form $f \mapsto f(p)$ for some necessarily unique point $p \in \Sigma$. Alternatively phrased, Σ is, in a natural way, the spectrum of the algebra $\mathscr{O}(\Sigma)$. As a consequence of the holomorphic convexity of Σ, we can invoke an approximation theorem of O'Farrell, Preskenis, and Walsh [4, Theorem 2] to conclude that every continuous function on Σ can be approximated uniformly on Σ by functions holomorphic on varying neighborhoods of Σ.

Without loss of generality, we can suppose that the point p_1 from property (a) above is the origin. We can then choose coordinates so that the tangent space T_0M_1 is $\mathbb{R}^N$ and so that in the unit ball $\mathbb{B}_N$ centered at 0, the manifolds M_1 and M_2 both coincide with their tangent planes, which means that $M_1 \cap \overline{\mathbb{B}}_N = \mathbb{R}^N \cap \overline{\mathbb{B}}_N$ and that $M_2 \cap \overline{\mathbb{B}}_N = T_0M_2 \cap \overline{\mathbb{B}}_N$.

Let $\gamma : \mathbb{T} \to \Sigma$ map the unit circle $\mathbb{T}$ in the plane into Σ homeomorphically and in such a way that $\gamma(1) = 0$, $\gamma(-1) = p_2$, and so that γ carries the upper half of $\mathbb{T}$ into M_1, and the lower half into M_2. The map γ is not homotopically trivial in Σ.

We construct a continuous function $\varphi : \Sigma \to \mathbb{T}$ with $\varphi \circ \gamma(e^{it}) = e^{it}$ for all points $e^{it} \in \mathbb{T}$. There is such a function: By replacing φ by $\varphi/|\varphi|$ if necessary, we see that it suffices to find a zero-free function φ on Σ such that $\varphi \circ \gamma(e^{it}) = e^{it}$ for all points $e^{it} \in \mathbb{T}$. To do this, let λ_1 and λ_2 be continuous real-valued functions on M_1 and M_2, respectively, that satisfy $\lambda_1 \circ \gamma(e^{it}) = t$ when $0 \leq t \leq \pi$ and $\lambda_2 \circ \gamma(e^{it}) = t$ when $\pi \leq t \leq 2\pi$. The function φ that agrees on M_1 with $e^{i\lambda_1}$ and on M_2 with $e^{i\lambda_2}$ is continuous and zero-free on Σ and satisfies $\varphi \circ \gamma(e^{it}) = e^{it}$ for all t. By construction, $\varphi|M_1$ is homotopic to a constant as is $\varphi|M_2$. (That $\varphi|M_1$ is homotopic to a constant, is immediate: The map $H : [0,1] \times M_1 \to \mathbb{T}$ given by $H(t,x) = e^{it\lambda_1(x)}$ is a homotopy connecting $H(1,\cdot) = \varphi = e^{i\lambda_1}$ to the constant map $H(0,\cdot)$.)

By the approximation theorem of O'Farrell, Preskenis, and Walsh, the function φ can be approximated uniformly on Σ by functions holomorphic on varying neighborhoods of Σ. There is, therefore, a neighborhood Ω_0 of Σ on which there is a zero-free holomorphic function f_0 such that the map $f_0 \circ \gamma : \mathbb{T} \to \mathbb{C} \setminus \{0\}$ is homotopic to the inclusion $\mathbb{T} \hookrightarrow \mathbb{C} \setminus \{0\}$, such that $\Omega_0 = \Omega_{0,1} \cup \Omega_{0,2}$ with $\Omega_{0,1}$ and $\Omega_{0,2}$ domains that contain M_1 and M_2, respectively, such that $f_0|\Omega_{0,1}$ and $f_0|\Omega_{0,2}$ have holomorphic logarithms, say ℓ_1 and ℓ_2. These logarithms can be chosen so that $\ell_1(p_2) = \ell_2(p_2)$. With this choice, $\ell_1(0) = \ell_2(0) \pm 2\pi i$.

Let $\Omega \subset \Omega_0$ be a domain of the form $W_1 \cup W_2$, where W_1 is a thin ribbon around M_1 that is contained in $\Omega_{0,1}$ and that satisfies $W_1 \cap \mathbb{B}_N \subset \{z = x + iy \in \mathbb{R}^N_x + i\mathbb{R}^N_y : |y| < \eta\}$ for a small $\eta > 0$ whose size will be specified further below. The domain W_2 is constructed in the following way. The domain $\Omega_{0,2}$ contains a ball $\mathbb{B}(0,r)$ of some radius $r > 0$ centered at the origin. We take r to be less than 1 so that at points of $\mathbb{B}_N(0,r)$ the manifold M_2 agrees with its tangent space. Having fixed r, we introduce the set

$$\Delta_{r,\varepsilon} = \{z = x + iy \in (\mathbb{R}^N_x + i\mathbb{R}^N_y) \cap \mathbb{B}_N(0,r) : y > |\varepsilon|\}.$$

We insist that our ε be so small that the tangent space T_0M_2 meets the sphere $b\mathbb{B}_N(0,r)$ in a set that is contained in $\overline{\Delta}_{r,\varepsilon}$. The Lemma proved above shows that there is a domain D_0 containing $\Delta_{r,\varepsilon}$ and $\mathbb{B}_N(0,r) \cap T_0M_2$ such that every function f that is holomorphic on $\Delta_{r,\varepsilon}$ continues holomorphically into D_0.

Our domain W_2 is defined by

$$W_2 = (V \setminus \mathbb{B}_N(0,r)) \cup \Delta_{r,\varepsilon},$$

in which $V \subset \Omega_0$ is a thin tube around the manifold M_2 that is contained in $\Omega_{0,2}$.

We now choose the η used in the definition of W_1 to be smaller than the ε used in the definition of W_2. This has the effect that W_1 and $\Delta_{r,\varepsilon}$ are disjoint and, indeed, have disjoint closures.

As noted already, our domain Ω is the union $W_1 \cup W_2$.

The envelope of holomorphy of Ω is a Riemann domain (Ω^*, π). It has the property that $\pi(\Omega^*) \supset \Omega$, but more than that, $\pi(\Omega^*)$ also contains the domain D_0.

Let $\iota : \Omega \to \Omega^*$ be the canonical injection, so that $\pi \circ \iota$ is the identity map on Ω, and for every f holomorphic on Ω, there is a unique f^* holomorphic on Ω^* that satisfies $f = f^* \circ \iota$.

The map ι carries the manifold M_1 onto a submanifold M_1^* of Ω^*. It also carries the open subset $M_{2,+} = M_2 \setminus (\mathbb{B}_N(0,r) \cap \{z : |y| \leq \varepsilon\})$ onto a locally closed (1) submanifold $M_{2,+}^*$ of Ω^* that meets M_1^* at a single point, viz., at the point $\iota(p_2)$, and the intersection there is transversal.

Because the manifold Ω^* is a Stein manifold, there is a map $j : D_0 \to \Omega^*$ that agrees on $\Delta_{r,\varepsilon}$ with ι.

The existence of j is seen as follows. Since Ω^* is a Stein manifold, we can assume it to be a complex submanifold of $\mathbb{C}^m$ for a sufficiently large m. The map ι is defined on $\Delta_{r,\varepsilon}$, and, with $\Omega^* \subset \mathbb{C}^m$, it is given by an m-tuple $(\iota_1, \ldots, \iota_m)$ of holomorphic functions. Each ι_s extends to a holomorphic function j_s on D_0. The map j is then the m-tuple $(j_1, \ldots, j_m)$.

It carries the totally real disc $\Gamma = T_0 M_2 \cap \mathbb{B}_N(0,r)$ onto a locally closed submanifold Γ^* of Ω^*. The set $\Gamma^* \cup M_{2,+}^*$ is a smooth submanifold—call it M_2^*—of Ω^* that is diffeomorphic to the manifold M_2.

The manifolds M_1^* and M_2^* meet only at the point $\iota(p_2)$, which we denote by p_2^*. Because the map ι is injective, the intersection $M_1^* \cap M_2^*$ is necessarily contained in the set $\pi^{-1}(M_1 \cap M_2) = \pi^{-1}(0) \cup \{p_2^*\}$. The point now is that $\iota(0)$ is different from $j(0)$ as follows from the existence of the function f_0 that we constructed above. To prove this, we define a function ψ by the condition that $\psi = \ell_1$ on W_1 and $\psi = \ell_2$ on W_2. This function is well defined and holomorphic on Ω, and it is a branch of $\log f_0$ on Ω. Denote by ψ_0 the continuation of $\psi | \Delta_{r,\varepsilon}$ into the domain D_0. The value of ψ_0 at 0 differs by $\pm 2\pi i$ from the value of ψ at 0. This means that the points $\iota(0)$ and $j(0)$ differ. Therefore the manifolds M_1^* and M_2^* meet at a single point in Ω^*.

Intersection theory [3, middle of p. 132] shows that in $\mathbb{C}^N$ it is impossible for two N-dimensional compact smooth manifolds to intersect at a single point if the intersection is transversal, so the domain Ω^* is not biholomorphically equivalent to a domain in $\mathbb{C}^N$.

This completes our discussion of the announced domain Ω.

(1) A set is *locally closed* if it is a closed subset of an open subset of the ambient space.

This discussion gives rise to an obvious and probably difficult question: Can one give conditions under which the envelope of holomorphy of a domain in $\mathbb{C}^N$ *is* biholomorphically equivalent to a domain in $\mathbb{C}^N$? Note that this is not the question of when the envelope of holomorphy is schlicht. Precisely, the latter question, which is classical and not easy, is this: If D is a domain in $\mathbb{C}^N$ with envelope of holomorphy $(\widetilde{D}, \pi)$, what conditions on D guarantee that the projection π is injective?

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