ANNALES
POLONICI MATHEMATICI
88.3 (2006)

On nonsingular polynomial maps of $\mathbb{R}^2$

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Abstract. We consider nonsingular polynomial maps $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ under the following regularity condition at infinity (J_∞): There does not exist a sequence $\{(p_k, q_k)\} \subset \mathbb{C}^2$ of complex singular points of F such that the imaginary parts $(\Im(p_k), \Im(q_k))$ tend to $(0, 0)$, the real parts $(\Re(p_k), \Re(q_k))$ tend to ∞ and $F(\Re(p_k), \Re(q_k))) \to a \in \mathbb{R}^2$. It is shown that F is a global diffeomorphism of $\mathbb{R}^2$ if it satisfies Condition (J_∞) and if, in addition, the restriction of F to every real level set $P^{-1}(c)$ is proper for values of $|c|$ large enough.

1. Introduction. This paper addresses the question of whether a polynomial map $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ which is a local diffeomorphism is also a global diffeomorphism. Pinchuk's example [P] of a polynomial local diffeomorphism which is not a global diffeomorphism proves the necessity of some extra condition. In this paper, we propose a condition, called Condition (J_∞) below, which we believe to be necessary and sufficient for a polynomial map which is a local diffeomorphism to be a global diffeomorphism. In this way, we provide a positive answer in a simple (but not trivial) topological situation. Everything indicates that the very important problem of characterizing polynomial maps of the real plane that are global diffeomorphisms is very difficult. Condition (J_∞) ensures the nice feature that the complex singularities of the complexification of the polynomial map defined on the real plane do not have a direct influence on the behavior of the map at infinity. Certainly, polynomial maps satisfying the well known constant Jacobian Keller condition satisfy the (J_∞) condition which we now define.

2000 *Mathematics Subject Classification*: 14R15, 14E40.

Key words and phrases: Jacobian Conjecture, polynomial diffeomorphism.

Research of Nguyen Van Chau supported in part by the National Basic Program on Natural Science Vietnam, ICTP, Trieste, Italy, and ICMC-USP, SP, Brazil.

Research of C. Gutierrez supported in part by FAPESP and CNPq, grant numbers TEMÁTICO 03/03107-9 and 306992/2003-5, respectively.

For a complex point $p = (a+ib, c+id) \in \mathbb{C}^2$, $a, b, c, d \in \mathbb{R}$, we denote by $\Im(p) = (b, d)$ and $\Re(p) := (a, c)$ the imaginary part and the real part of p. Condition (J_∞) can be formulated as:

(J_∞) There does not exist a sequence $\{p_k\} \subset \mathbb{C}^2$ of singular points of the complexification $\widetilde{F} : \mathbb{C}^2 \to \mathbb{C}^2$ of F such that $\Im(p_k) \to (0,0)$, $\Re(p_k) \to \infty$ and $F(\Re(p_k)) \to a \in \mathbb{R}^2$.

THEOREM 1.1. *Suppose $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ is a nonsingular polynomial map satisfying Condition (J_∞). Then F is a global diffeomorphism of $\mathbb{R}^2$ provided that, for all $|c| > 0$ large enough, either*

(i) *$P^{-1}(c)$ is connected or*
(ii) *$F|_{P^{-1}(c)}$ is proper.*

This theorem improves the main result of [CG], where the nonzero constant Jacobian case was considered. Its proof, presented in §3, is based on the examination, in §2, of the behavior at infinity of nonsingular polynomial maps of $\mathbb{R}^2$ satisfying (J_∞). Theorem 1.1 is not valid for analytic maps of $\mathbb{R}^2$. Indeed, in [CG], we have constructed a nonzero constant Jacobian analytic map $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ which is noninjective, nonsurjective, at most 2-to-1 and such that for all $|c|$ large enough, $P^{-1}(c)$ is connected and $F|_{P^{-1}(c)}$ is proper. This example will be presented in §4. In that section we will also see that Pinchuk's example [P] does not satisfy Condition (J_∞) and can be modified so that the resulting map $F = (P, Q)$ is a noninjective nonsingular polynomial map which, when restricted to every level set $P^{-1}(c), Q^{-1}(c)$, is proper for $c \ll 0$.

In the case of complex nonsingular polynomial mappings a result analogous to Theorem 1.1(ii) was obtained earlier by Drużkowski [Dru] and Chądzyński [Cha]. Namely, if a nonsingular polynomial mapping $F = (f, g)$ of $\mathbb{C}^2$ is proper on the set $g^{-1}(c)$ for some $c \in \mathbb{C}$, then F is injective. Theorem 1.1(ii) is, in some sense, a real counterpart (but not a consequence) of that result.

Before continuing, we wish to thank the referee whose comments have been appreciated and incorporated into this work. Also, we wish to mention some results related to ours. Fernandes, Gutierrez and Rabanal [FGR] (see also [CGL]) showed that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable map (not necessarily C^1) and if, for some $\varepsilon > 0$, $\mathrm{Spec}(f) \cap [0, \varepsilon) = \emptyset$, then f is injective; here $\mathrm{Spec}(f)$ denotes the set of (complex) eigenvalues of the derivative $Df(x)$ when x varies in $\mathbb{R}^2$. Campbell [Ca1] classified the C^1 maps $\mathbb{R}^2 \to \mathbb{R}^2$ whose eigenvalues are both 1; all such maps are diffeomorphisms having explicit inverse.

2. Condition (J_∞). In the following, the Euclidean space $\mathbb{R}^2$ will be viewed as a subset of $\mathbb{C}^2$. Let $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map which is *dominant* (i.e. $F(\mathbb{R}^2)$ is an open set). Recall that the *nonproper value set* A_F of F is the set of all values $a \in \mathbb{R}^2$ which have no neighborhood with compact inverse image under F. It is well known that A_F, if nonempty, is the union of the images of a finite number of nonconstant polynomial maps from $\mathbb{R}$ into $\mathbb{R}^2$ (see for example [J]). When F is a local homeomorphism, the map $F : \mathbb{R}^2 \setminus F^{-1}(A_F) \to \mathbb{R}^2 \setminus A_F$ is an unbranched covering and A_F is just the discontinuity set of the integer-valued function $\#F^{-1}(a)$ defined on $\mathbb{R}^2$. When $A_F = \emptyset$, F is a homeomorphism of $\mathbb{R}^2$.

By means of the Newton–Puiseux expansion, we can describe both the behavior of F at infinity and the set A_F. Let $(u_0, v_0) \in A_F$ be a smooth point of A_F. Take a line segment L that intersects A_F transversally at the point (u_0, v_0). Then $F^{-1}(L)$ has some branches at infinity along which F tends to the value (u_0, v_0). Let $\gamma \subset F^{-1}(L)$ be one of such branches. In suitable linear coordinates of $\mathbb{R}^2$ the branch γ can be given by $(R, \infty) \ni x \mapsto (x, \gamma(x)) \in \gamma$ with a Newton–Puiseux expansion at infinity of the form

$$\gamma(x) = \sum_{k=0}^{\infty} c_k x^{1-k/m},$$

where $c_k \in \mathbb{R}$, $\gcd(\{k : c_k \neq 0\} \cup \{m\}) = 1$ and the series $\sum_k c_k t^k$ is absolutely convergent in the complex disk $|t| < \varepsilon$. Then, following [C], we can find a unique finite fractional series $\varphi(x, \xi)$ with parameter ξ such that $\gamma(x) = \varphi(x, \xi_0) +$ lower order terms in x and

$$\begin{aligned} F(x, \varphi(x, \xi)) &= (P_\varphi(x, \xi), Q_\varphi(x, \xi)) \\ &= (p_\varphi(\xi), q_\varphi(\xi)) + \text{lower order terms in } x, \end{aligned}$$

with $p_\varphi, q_\varphi \in \mathbb{R}[\xi]$ and $\max(\deg p_\varphi, \deg q_\varphi) > 0$. Such a fractional power series $\varphi(x, \xi)$ is called a *real dicritical series* of F. Let us represent $\varphi(x, \xi)$ in the normal form

$$\varphi(x, \xi) = \sum_{j=0}^{n_\varphi - 1} a_j x^{1-j/m_\varphi} + \xi x^{1-n_\varphi/m_\varphi}, \quad \gcd(\{j : a_j \neq 0\} \cup \{m_\varphi, n_\varphi\}) = 1,$$

and define the maps $\Phi, F_\varphi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^2$ by $\Phi(t, \xi) := (t^{-m_\varphi}, \varphi(t^{-m_\varphi}, \xi))$ and $F_\varphi := F \circ \Phi$. One can easily check the following properties:

(i) Φ is an analytic homeomorphism from $\mathbb{R}^+ \times \mathbb{R}$ onto its image $U_\varphi := \Phi(\mathbb{R}^+ \times \mathbb{R})$ which is a neighborhood of the branch γ considered.

(ii) F_φ is a polynomial map in (t, ξ), and the image of the polynomial map $f_\varphi(\xi) = (p_\varphi(\xi), q_\varphi(\xi))$ contains the value (u_0, v_0) and is an irreducible component of A_F.

In other words, the map Φ gives local analytic coordinates (t, ξ) in the neighborhood U_φ of the branch γ, and the polynomial map F_φ gives a representation of F in these coordinates.

As noted above, the set A_F, if nonempty, is the union of the images of a finite number of nonconstant polynomial maps from $\mathbb{R}$ into $\mathbb{R}^2$. In this way one can construct a finite family Λ of dicritical series φ such that $A_F = \bigcup_{\varphi\in\Lambda} f_\varphi(\mathbb{R})$, and the corresponding family $\{\Phi(\mathbb{R}^+ \times \mathbb{R})\}$ of open sets covers all branch curves at infinity along which F tends to a value in A_F. This structure can be used to examine the behavior of F at infinity. In the following we will use this structure to study the geometry of a nonsingular polynomial map F satisfying Condition (J_∞).

Recall that the *exceptional value set* E_h of a polynomial $h : \mathbb{R}^2 \to \mathbb{R}$ is the smallest subset of $\mathbb{R}$ such that the restriction $h : \mathbb{R}^2 \backslash h^{-1}(E_h) \to \mathbb{R}\backslash E_h$ is a locally trivial fibration. When $E_h = \emptyset$, the map h gives a trivial fibration with fiber homeomorphic to $\mathbb{R}$. By a *local irreducible complex branch* of a real curve $V \subset \mathbb{R}^2$ we mean the complexification of a local irreducible real branch of V. A local irreducible complex branch of A_F can be seen as the image $f_\varphi(D(\xi_0,\varepsilon))$ of a small disk $D(\xi_0,\varepsilon) := \{\xi \in \mathbb{C} : |\xi-\xi_0| < \varepsilon\}$, where $\xi_0 \in \mathbb{R}$ and φ is a dicritical series of F.

THEOREM 2.1. *Suppose $F = (P,Q) : \mathbb{R}^2 \to \mathbb{R}^2$ is a nonsingular polynomial map satisfying Condition (J_∞). If the complex line $L := \{u = c\}$ intersects transversally all local irreducible complex branches of A_F located at $A_F \cap L$, then $c \notin E_P$.*

To prove this theorem we need the following elementary fact:

LEMMA 2.2. *Let $f(t,\xi) = (u,v)$ be a holomorphic map from a neighborhood $U \subset \mathbb{C}^2$ of $(0,0)$ into $\mathbb{C}^2$, $f(t,\xi) = (p(\xi), q(\xi)) +$ higher order terms in t, $f(0,0) = (0,0)$. Assume that for some $\delta > 0$ small, the line $u = 0$ intersects the local branch $\Gamma := f(\{0\}\times D)$ transversally, where $D := \{\xi \in \mathbb{C} : |\xi| < \delta\}$, and that $\{(t,\xi) \in U : \det(Df(t,\xi)) = 0\} \subset \{(t,\xi) \in U : t = 0\}$. Then $\dot p(0) \neq 0$.*

Proof. Since the line $u = 0$ intersects Γ transversally at $(0,0)$, we can assume that, locally around 0, Γ is a smooth branch of a curve parameterized by $v = h(u)$, where h is an analytic diffeomorphism, with $h(0) = 0$, defined in a small neighborhood of 0. Define the new coordinates $(\overline{u},\overline{v}) = (u, v - h(u))$ in a neighborhood of $(0,0)$. Let $\overline{f}$ be the representation of f in these coordinates. Then

$$\overline{f}(t,\xi) = (p(\xi), 0) + \text{higher order terms in } t,$$

$\overline{f}(0,0) = (0,0)$, $\overline{f}(\{t = 0\}) \subset \Gamma = \{\overline{v} = 0\}$ and $\det(D\overline{f}(t,\xi)) \neq 0$ for $t \neq 0$. By examining the Newton diagrams of $\overline{f}_1$, $\overline{f}_2$ and $\det D\overline{f}$ we can verify that

$$\overline{f}(t,\xi) = (\xi u_1(t,\xi) + tu_2(t,\xi), t^k v_1(t,\xi)),$$

where u_1, u_2 and v_1 are holomorphic functions defined in U, $u_1(0,0) \neq 0$ and $v_1(0,0) \neq 0$ (see e.g. [O, Lemma 4.1]). It follows that $\dot p(0) = u_1(0,0) \neq 0$. ∎

Proof of Theorem 2.1. Let $c \in \mathbb{R}$. Assume that the complex line $L := \{u = c\}$ intersects transversally all local irreducible complex branches of A_F located at $A_F \cap L$.

Let $\Delta \ni c$ be an open interval. We will see that it is enough to construct a smooth vector field V on $P^{-1}(\Delta)$ such that $\langle \operatorname{grad} P(z), V(z)\rangle = 1$, and the solutions of the differential equation $\dot z = V(z)$ do not tend to infinity. In fact, if Δ is small enough and $\Psi(z,t)$ is the local flow induced by such a vector field V, then $\Psi : P^{-1}(c) \times \Delta \to P^{-1}(\Delta)$ is well defined and it is a diffeomorphism satisfying $P(\Psi(z,t)) = c + t$. Hence, $P : P^{-1}(\Delta) \to \Delta$ is a trivial fibration and, by definition, $c \notin E_P$.

To construct the vector field V as above, first we consider a given branch γ at infinity of the real curve $P = c$. Let φ be the dicritical series with coordinates (t,ξ) associated to γ. Assume that in these coordinates γ locates at the point $(0,\xi_0)$. Taking the derivative of F_φ we have

$$\det(DF_\varphi(t,\xi)) = -m_\varphi \det(DF(\Phi(t,\xi)))t^{n_\varphi - 2m_\varphi - 1}$$

Condition (J_∞) ensures that $\det(DF_\varphi(t,\xi)) \neq 0$ for $|t| \neq 0$ small enough; in fact, otherwise, there would exist a holomorphic map $\overline{\xi} : \{t \in \mathbb{C} : |t| < \varepsilon\} \to \mathbb{C}$ with $\overline{\xi}(0) = \xi_0$ such that $\det(DF(\Phi(t,\overline{\xi}(t)))) \equiv 0$. Therefore, $\Re(\Phi(t,\overline{\xi}(t))) = \Phi(t,\xi_0) +$ higher order terms in t for $t \in \mathbb{R}$ and $F(\Re(\Phi(t,\overline{\xi}(t)))$ tends to $(p_\varphi(\xi_0), q_\varphi(\xi_0)) \in \mathbb{R}^2$ as t tends to zero. Hence, we can apply Lemma 2.1 to find positive numbers α and β and a neighborhood $W_\gamma = \{(t,\xi) \in \mathbb{R}^2 : 0 < t < \alpha,\ |\xi - \xi_0| < \beta\}$ of γ such that

$$\dot p_\varphi(\xi) \neq 0 \quad \text{for } |\xi - \xi_0| < \beta, \qquad \frac{\partial}{\partial \xi} P_\varphi(t,\xi) \neq 0 \quad \text{for } (t,\xi) \in W_\gamma.$$

The property we need here is that in W_γ the motions in the direction $V_\gamma := \big(\frac{\partial}{\partial\xi} P_\varphi, 0\big)$ cannot tend to the line $t = 0$ and $\langle \operatorname{grad} P(z), V_\gamma(z)\rangle \neq 0$ for $z \in W_\gamma$.

Summarizing, for every branch γ at infinity of the real curve $P = c$, we have obtained a neighborhood W_γ of γ and a vector field V_γ defined on W_γ. Since the curve $P = c$ has only finitely many branches at infinity, we can choose a real number $R > 0$, as large as necessary, and a small interval $\Delta \ni c$ so that the family $\{W_\gamma\}_\gamma$ together with the open ball B_R of radius R centered at $(0,0)$ is an open covering of $P^{-1}(\Delta)$. Let $V_R(z) := \operatorname{grad} P(z)$ defined on B_R. Note that $V_R(z) \neq 0$, since F has no singularities. Then by using a smooth partition of unity we can construct, from the fields V_γ and $V_R(z)$, a smooth vector field $V(z)$ defined on $P^{-1}(\Delta)$ with the desired properties. ∎

The following corollary, which is an immediate consequence of Theorem 2.1, will be used to prove Theorem 1.1.

COROLLARY 2.3. *Suppose $F = (P,Q) : \mathbb{R}^2 \to \mathbb{R}^2$ is a nonsingular polynomial map satisfying Condition (J_∞). Then the nonproper value set A_F cannot be a finite union of lines and semi-lines.*

Proof. Assume by contradiction that A_F is the union of finitely many lines and semi-lines. We can choose a direction $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ so that the lines $au + bv = c$, $c \in \mathbb{R}$, intersect transversally all complexifications of the lines and semi-lines in A_F. Then, applying Theorem 2.1 to the pair $(aP+bQ, Q)$ we see that the exceptional value set E_{aP+bQ} is empty and that the map $aP+bQ : \mathbb{R}^2 \to \mathbb{R}$ gives a trivial fibration with fiber diffeomorphic to $\mathbb{R}$. As F has no singularities, it is monotone along the fibers of $aP+bQ$. It follows that it is injective. Hence, as every injective polynomial map of $\mathbb{R}^2$ must be bijective [N, Ku], F is a diffeomorphism of $\mathbb{R}^2$ and $A_F = \emptyset$. This contradiction proves the corollary. ∎

REMARK 2.4. In the above-mentioned representation, $A_F = \bigcup_{\varphi \in \Lambda} f_\varphi(\mathbb{R})$ and no component $f_\varphi(\mathbb{R})$ is a semi-line. In fact, this can be proved by applying Lemma 2.2 to the map F_φ.

3. Proof of Theorem 1.1. The conclusion of the theorem will follow from Corollary 2.3 and the following lemmas.

LEMMA 3.1 (Lemma 2.2 in [CG]). *Let $F = (P,Q) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map. Assume that, for every $|c|$ large enough, the restriction of F to $P^{-1}(c)$ is proper. Then the nonproper value set A_F of F, if not empty, must be formed by some lines and semi-lines parallel to the vertical axis.*

Proof. Assume that, for $|c| > R > 0$, the restriction of F to $P^{-1}(c)$ is proper. From the definitions above, we can easily see that if $L \subset \mathbb{R}^2$ is a line and the restriction of F to $F^{-1}(L)$ is proper, then $L \cap A_F = \emptyset$. This implies that A_F must be contained in $\{(c,d) \in \mathbb{R}^2 : |c| \leq R\}$. On the other hand, by Proposition 2.1, A_F is the union of the images of some nonconstant polynomial maps $(p,q) : \mathbb{R} \to \mathbb{R}^2$. Therefore, the first component of every such polynomial map must be constant and so A_F must consist of some lines and semi-lines parallel to the vertical axis. ∎

LEMMA 3.2. *Let $F = (P,Q) : \mathbb{R}^2 \to \mathbb{R}^2$ be a nonsingular polynomial map and let R be a positive number. If, for every $|c| > R$, the level set $P^{-1}(c)$ is connected, then the restriction of F to $P^{-1}(c)$ is proper.*

Proof. Let $F = (P,Q) : \mathbb{R}^2 \to \mathbb{R}^2$ be a nonsingular polynomial map. Assume that $P^{-1}(c)$ is connected for every $|c| > R > 0$. We define $W := \{(c,d) \in \mathbb{R}^2 : |c| > R\}$. Since Q is monotone along each connected component of a level set of P, F takes injectively $F^{-1}(W)$ onto W. It follows from the definition of A_F that $W \cap A_F = \emptyset$. Therefore, for $|c| > R$, the restriction of F to $P^{-1}(c)$ is proper. ∎

Proof of Theorem 1.1. Combine Lemmas 3.1 and 3.2 and Corollary 2.3. ∎

4. Discussion and examples. The following discussion and examples will clarify the statements and assumptions of the preceding results.

4.1. *Condition* (J_∞). In the real case, this condition generalizes the well known Jacobian condition, $\det DF \equiv c \neq 0$, and ensures that the complex singularities of F do not directly influence the behavior of F at infinity. As a real analogue of the Jacobian Conjecture, it is natural to ask whether *a nonsingular polynomial map of* $\mathbb{R}^2$ *is a global diffeomorphism if it satisfies condition* (J_∞).

4.2. *Condition* (i) *of Theorem 1.1.* It is clear that the restriction of F to the inverse image $F^{-1}(\{(a,b) : \|a\| \gg 0\})$ is injective. For a local diffeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$, denote by $n(h)$ the minimal integer k such that there exists an open disk $D \subset \mathbb{R}^2$ with $\# h^{-1}(p) \equiv k$ for all $p \in D$. Condition (i) implies that $n(F) = 1$. In Pinchuk's example, $n(F) = 2$ (see [Ca2, Ca3]). We conjecture that *a nonsingular polynomial* F *must be a global diffeomorphism if* $n(F) = 1$.

4.3. *Pinchuk map and Condition* (J_∞). In 1994, Pinchuk [P] found a noninjective nonsingular polynomial map of $\mathbb{R}^2$. Here, we will show that it does not satisfy Condition (J_∞).

Pinchuk's map $F = (P, Q)$ can be constructed in the following way (see [E]). Let $g := xy - 1$, $h := g(xg + 1)$ and $f := ((h+1)/x)(xg+1)^2$. Then let

$$P := f + h$$

and

$$Q := -g^2 - 6gh(h+1) - 170fh - 91h^2 - 195fh^2 - 69h^3 - 75h^3 f - \tfrac{75}{4}h^4.$$

We have

$$\det(P, Q) = g^2 + (g + f(13 + 15h))^2 + f^2,$$

which is always positive on $\mathbb{R}^2$.

To check Condition (J_∞) for (P, Q) we choose the function

$$\varphi(x, \xi) := x^{-1} + \xi x^{-3/2}.$$

Then computing with Maple V we get

$$\begin{aligned}
P(t^{-2}, \varphi(t^{-2}, \xi)) &= (\xi^4 + 2\xi^2) + t(3\xi^3 + 3\xi) + t^2(3\xi^2 + 1) + t^3\xi, \\
Q(t^{-2}, \varphi(t^{-2}, \xi)) &= \left(-75\xi^{10} - \tfrac{1155}{4}\xi^8 - 434\xi^6 - 261\xi^4\right) \\
&\quad + t(-698\xi^3 - 1673\xi^5 - 1425\xi^7 - 450\xi^9) \\
&\quad + t^2(-608\xi^2 - 2404\xi^4 - 5625/2\xi^6 - 1125\xi^8) \\
&\quad + t^3(-170\xi - 1535\xi^3 - 2775\xi^5 - 1500\xi^7) \\
&\quad + t^4\left(-365\xi^2 - \tfrac{5475}{4}\xi^4 - 1125\xi^6\right) \\
&\quad + t^5(-270\xi^3 - 450\xi^5) - 75t^6\xi^4.
\end{aligned}$$

One can see that the exceptional complex curve of $F_{\mathbb{C}}$ has a component parameterized by $\mathbb{C} \ni \xi \mapsto \left(\xi^4 + 2\xi^2, -75\xi^{10} - \frac{1155}{4}\xi^8 - 434\xi^6 - 261\xi^4\right)$.

Now, considering the function $\det DF$ along the parameters $\Phi(t,\xi) := (t^{-2}, \varphi(t^{-2},\xi))$, we have

$$\det(D\Phi(t,\xi)) = -2t^{-6}$$

and

$$\begin{aligned}
&\det(DF(\Phi(t,\xi)))\det(D\Phi(t,\xi)) \\
&\quad = 706t^3\xi + 170\xi^4 + 706t\xi^3 + 1074t^2\xi^2 + 3706t\xi^5 + 7468t^2\xi^4 + 7468t^3\xi^3 \\
&\qquad + 730\xi^6 + 1175\xi^8 + 840\xi^{10} + 225\xi^{12} + 170t^4 + 3706t^4\xi^2 + 7080\xi^7 t \\
&\qquad + 17745\xi^6 t^2 + 23680t^3\xi^5 + 17745t^4\xi^4 + 7080t^5\xi^3 + 5880\xi^9 t \\
&\qquad + 17640\xi^8 t^2 + 29400\xi^7 t^3 + 29400t^4\xi^6 + 17640t^5\xi^5 + 5880t^6\xi^4 \\
&\qquad + 1800\xi^{11} t + 6300\xi^{10} t^2 + 12600\xi^9 t^3 + 15750\xi^8 t^4 + 12600t^5\xi^7 \\
&\qquad + 6300t^6\xi^6 + 1800t^7\xi^5 + 1175t^6\xi^2 + 840t^7\xi^3 + 730t^5\xi + 225t^8\xi^4.
\end{aligned}$$

This is a polynomial in $\mathbb{R}[\xi][t]$ with t-free term

$$\xi^4(170 + 730\xi^2 + 1175\xi^4 + 840\xi^6 + 225\xi^8)$$

which vanishes at $\xi = 0$ and may also vanish at some other real values a_i. It follows that $\det(DF(\Phi(t,\xi))) = 0$ for some fractional power series $t \mapsto \xi_i(t) = -a_i + \text{terms with } \deg t > 0$. Thus, for such series $\xi_i(t)$ we have $\det(D(P,Q)(t^{-2}, t^2 + \xi_i(t)t^3)) \equiv 0$. As above, we have

$$\begin{aligned}
P(t^{-2}, t^2 + \xi_i(t)t^3) &= a_i^4 + 2a_i^2 + \text{terms with } \deg t > 0, \\
Q(t^{-2}, t^2 + \xi_i(t)t^3) &= -75a_i^{10} - \tfrac{1155}{4}a_i^8 - 434a_i^6 - 261a_i^4 \\
&\qquad\qquad + \text{terms with } \deg t > 0,
\end{aligned}$$

each of which tends to a real value as t tends to zero. Hence, we can select arbitrarily small real values t so as to obtain a sequence for which Condition (J_∞) does not hold.

4.4. *Condition* (ii) *of Theorem 1.1.* Related to this condition, we may apply the following proposition to the noninjective nonsingular polynomial Pinchuk map [P] presented in Section 4.3 above.

PROPOSITION 4.1. *Suppose $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a nonsingular polynomial map with the closure of $F(\mathbb{R}^2)$ equal to $\mathbb{R}^2$. Then there is a polynomial diffeomorphism φ of $\mathbb{R}^2$ such that the restriction of $\varphi \circ F = (P,Q)$ to every level set $P^{-1}(c), Q^{-1}(c)$ is proper for $c \ll 0$.*

Proof. This proposition is obviously true when F is injective. Suppose that F is not injective. Then

$$A_F = \bigcup_{i=1}^{n} \{f_i(t) : t \in \mathbb{R}\}$$

for some polynomial maps $f_i(t) = (p_i(t), q_i(t))$. By using a linear change of coordinates if necessary, we can assume that $\deg p_i \geq 1$ and $\deg q_i \geq 1$. Choose a positive integer k such that $2k > \deg p_i$, $4k > \deg q_i$, and define

$$\varphi(u, v) := (u + v^{2k}, v + (u + v^{2k})^2),$$

which is an automorphism of $\mathbb{R}^2$. Then one can see that the branches at infinity of the set $A_{\varphi \circ F}$ are contained in the positive cone of $\mathbb{R}^2$. From this it is easy to obtain the conclusion.

4.5. *Nonzero constant Jacobian analytic maps.* There exists a nonzero constant Jacobian analytic map $F = (P, Q) : \mathbb{R}^2 \to \mathbb{R}^2$ which is noninjective, nonsurjective, at most 2-to-1 and such that for all $|c|$ large enough, $P^{-1}(c)$ has exactly one connected component and also the restriction of F to each level $P^{-1}(c)$ is proper. It will be seen that this map is a sort of "algebraic map".

To construct F, first consider the map $F_1 = (P_1, Q_1) : (0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ given by $P_1(x, y) = x(y^2 - 1)$ and $Q_1(x, y) = xy(y^2 - 4)$. Then:

(1) $\det(DF_1(x, y)) = x(y^4 + y^2 + 4) > 0$ everywhere;
(2) if $c < 0$ then $P_1^{-1}(c)$ is the connected set which is the graph of the map $y \mapsto x = c/(y^2 - 1)$ defined in $(-1, 1)$;
(3) if $c > 0$ then $P_1^{-1}(c)$ has two connected components which are the graphs of the maps $x \mapsto y = \sqrt{(c + x)/x}$ and $x \mapsto y = -\sqrt{(c + x)/x}$ defined in $(0, \infty)$;
(4) $P_1^{-1}(0)$ has two connected components: $\{y = 1\}$ and $\{y = -1\}$;
(5) F_1 is not injective because $F_1(1, 2) = F_1(1, -2) = (3, 0)$;
(6) F_1 is not surjective because $(0, 0) \notin F_1((0, \infty) \times \mathbb{R})$;
(7) for all $c \in \mathbb{R} \setminus \{0\}$, F_1 restricted to $P_1^{-1}(c)$ is a proper map.

Now consider the analytic diffeomorphism $H_1 : (0, \infty) \times \mathbb{R} \to (0, \infty) \times \mathbb{R}$ given by $H_1(x, y) = (\sqrt{2x}, h(y))$, where the diffeomorphism $h : \mathbb{R} \to \mathbb{R}$ is the solution of the differential equation

$$h' = \frac{1}{h^4 + h^2 + 4}, \qquad h(0) = 0.$$

We can see that $h(y)$ satisfies the algebraic equation $(h(y))^5/5 + (h(y))^3/3 + 4(h(y)) = y$. Let $F_2 = F_1 \circ H_1$. We can check that $\det DF_2 \equiv 1$.

As H_1 takes vertical lines onto vertical lines, there is a diffeomorphism $f : (0, \infty) \to \mathbb{R}$ such that $H_1(\{(x, f(x)) : x \in (0, \infty)\}$ is the connected component $\{(x, -\sqrt{(5 + x)/x}) : x \in (0, \infty)\}$ of $P^{-1}(5)$. Define the area preserving analytic diffeomorphism $H_2 : (0, \infty) \times \mathbb{R} \to (0, \infty) \times \mathbb{R}$ by $H_2(x, y) = (x, y + f(x))$. Observe that H_2 takes the positive first quadrant $\{(x, y) \in \mathbb{R}^2 :$

$x > 0,\ y > 0\}$ onto the set $\{(x, y) \in \mathbb{R}^2 : x > 0,\ y > f(x)\}$, which in turn is taken by H_1 onto $\{(x, y) \in \mathbb{R}^2 : x > 0,\ y > -\sqrt{(5+x)/x}\}$.

We conclude that $F_3 = (P_3, Q_3) = F_1 \circ H_1 \circ H_2$, restricted to the first quadrant, has the following properties:

(1) $\det(DF_3(x, y)) = 1$ everywhere;
(2) if $c < 0$ then $P_3^{-1}(c)$ is connected;
(3) if $c \in (0, 5)$ then $P_3^{-1}(c)$ has two connected components, and if $c \geq 5$ then $P_3^{-1}(c)$ is connected;
(4) $P_3^{-1}(0)$ has two connected components;
(5) F_3 is noninjective and nonsurjective;
(6) for all $c \in \mathbb{R} \setminus \{1\}$, F_3 restricted to $P_3^{-1}(c)$ is a proper map.

Let $H_3 : \mathbb{R}^2 \to (0, \infty) \times \mathbb{R}$ and $H_4 : \mathbb{R}^2 \to \mathbb{R} \times (0, \infty)$ be the following area preserving diffeomorphisms:

$$H_3(x, y) = \left(x + \sqrt{x^2 + 4},\, y\, \frac{\sqrt{x^2+4}}{x + \sqrt{x^2+4}}\right),$$
$$H_4(x, y) = \left(x\, \frac{\sqrt{y^2+4}}{y + \sqrt{y^2+4}},\, y + \sqrt{y^2+4}\right).$$

Observe that the function $k(x) = x + \sqrt{x^2 + 4}$ satisfies the algebraic equation

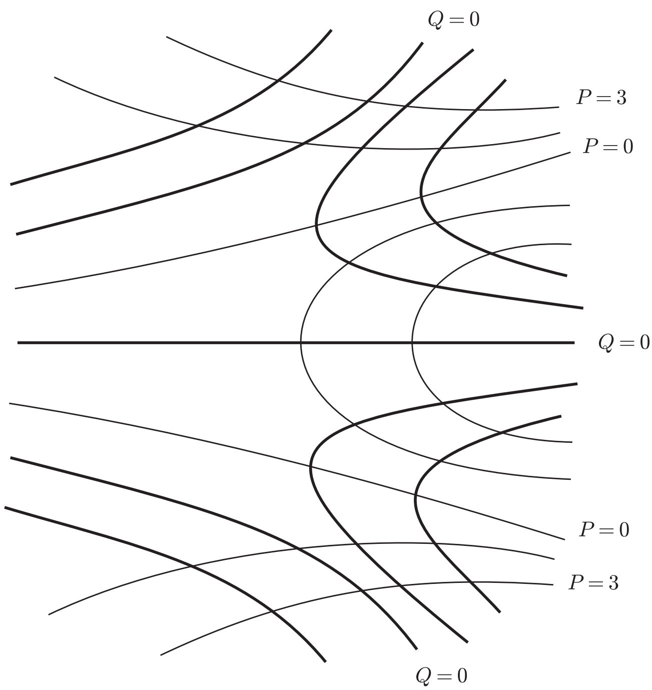

Fig. 1

$(k(x))^2 - xk(x) = 4$. It can be seen that $H_3 \circ H_4$ takes $\mathbb{R}^2$ onto the first quadrant and that $F = (P, Q) = F_3 \circ H_3 \circ H_4$ is as required at the beginning of this section.

Now we summarize the properties of the map $F = (P, Q)$ just constructed:

(1) $\det(DF(x, y)) = 1$ everywhere;
(2) if $c \in (-\infty, 0) \cup [5, \infty)$ then $P^{-1}(c)$ is connected, and if $c \in [0, 5)$ then $P^{-1}(c)$ has two connected components; in this way the foliation induced by the Hamiltonian vector field X_P has exactly one Reeb component, $\{P \leq 0\}$;
(3) if $c \in \mathbb{R} \setminus \{0\}$, then $Q^{-1}(c)$ has two connected components, and $Q^{-1}(0)$ has three connected components; in this way the foliation induced by X_Q has exactly two adjacent Reeb components with union $\{Q \leq 0\}$;
(4) F is noninjective and nonsurjective; more precisely, $F^{-1}(3, 0)$ consists of two points and $(0, 0) \notin F(\mathbb{R}^2)$;
(5) for all $c \in \mathbb{R} \setminus \{0\}$, F restricted to $P^{-1}(c)$ is a proper map.

Figure 1 shows the foliations induced by the Hamiltonian vector fields X_P and X_Q.

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Received 14.4.2005
and in final form 14.3.2006 (1569)