ANNALES
POLONICI MATHEMATICI
88.3 (2006)

The symmetrized polydisc cannot be exhausted by domains biholomorphic to convex domains

by Nikolai Nikolov (Sofia)

Abstract. We prove that the symmetrized polydisc cannot be exhausted by domains biholomorphic to convex domains.

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$. Let $\sigma_n = (\sigma_{n,1}, \ldots, \sigma_{n,n}) : \mathbb{C}^n \to \mathbb{C}^n$ be defined as follows:

$$\sigma_{n,k}(z_1, \ldots, z_n) = \sum_{1 \le j_1 < \cdots < j_k \le n} z_{j_1} \cdots z_{j_k}, \quad 1 \le k \le n.$$

The set $\mathbb{G}_n = \sigma_n(\mathbb{D}^n)$ is called the *symmetrized n-disc*. The symmetrized bidisc $\mathbb{G}_2$ is the first example of a bounded pseudoconvex domain which is not biholomorphic to any convex domain and on which the Carathéodory and Kobayashi distances coincide (see [1]). Moreover, it cannot be exhausted by domains biholomorphic to convex domains (see [2]). It has been asked in [4] whether the last result remains true for $\mathbb{G}_n$, $n \ge 3$. The aim of this note is to give a positive answer to the above question.

Let us begin with the following definition. Let $k_1 \le \ldots \le k_n$ be positive integers and

$$\pi_\lambda(z_1, \ldots, z_n) = (\lambda^{k_1} z_1, \ldots, \lambda^{k_n} z_n).$$

A domain D in $\mathbb{C}^n$ is called $(k_1, \ldots, k_n)$-*balanced* if $\pi_\lambda(z) \in D$ for $z \in D$, $\lambda \in \overline{\mathbb{D}}$. For such a domain D one has

$$D = \{z \in \mathbb{C}^n : h(z) < 1\},$$

where

$$h(z) = \inf\{\lambda > 0 : \pi_{1/\lambda}(z) \in D\}, \quad z \in \mathbb{C}^n.$$

It is easy to see that h is an upper semicontinuous, non-negative function

2000 *Mathematics Subject Classification*: Primary 32H35.

Key words and phrases: symmetrized polydisc.

This paper was started during the author's stay at the Jagiellonian University, Kraków, in February 2005. He would like to thank Peter Pflug, Marek Jarnicki and Włodzimierz Zwonek for helpful discussions.

on $\mathbb{C}^n$ with

$$h(\pi_\lambda(z)) = |\lambda| h(z), \qquad \lambda \in \mathbb{C},\ z \in \mathbb{C}^n.$$

Note that the $(1,\dots,1)$-balanced domains are exactly the balanced domains in the usual sense (cf. [3]). As in the case of balanced domains one has the following

PROPOSITION 1. *A $(k_1,\dots,k_n)$-balanced domain D is pseudoconvex if and only if $\log h$ is a plurisubharmonic function.*

Proof. It is clear that if $\log h$ is a plurisubharmonic function, then D is a pseudoconvex domain.

To prove the converse, define $\Phi : \mathbb{C}^n \ni (z_1,\dots,z_n) \mapsto (z_1^{k_1},\dots,z_n^{k_n}) \in \mathbb{C}^n$ and set $\widetilde{D} := \Phi^{-1}(D)$, $\widetilde{h} = h \circ \Phi$. Note that $\widetilde{D} = \{z \in \mathbb{C}^n : \widetilde{h}(z) < 1\}$ and $\widetilde{h}(\lambda z) = |\lambda| h(z)$, $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$. Therefore $\widetilde{D}$ is a pseudoconvex balanced domain whose Minkowski functional is equal to $\widetilde{h}$. Consequently, $\log \widetilde{h}$ is a plurisubharmonic function (cf. [3]). On the other hand, one has $h(z) = \widetilde{h}(\sqrt[k_1]{z_1},\dots,\sqrt[k_n]{z_n})$, $z \in \mathbb{C}^n_*$, where the roots are arbitrarily chosen. Thus $\log h$ is a plurisubharmonic function on $\mathbb{C}^n_*$ and hence, by the removable singularities theorem (cf. [3]), it is plurisubharmonic on $\mathbb{C}^n$. ∎

The crucial step in the proof of our main result is the following

PROPOSITION 2. *Let D be a $(k_1,\dots,k_n)$-balanced domain which can be exhausted by domains biholomorphic to convex domains. If $2k_{m+1} > k_n$ for some m, $0 \le m \le n-1$, then the intersection $D_m = D \cap \{z_1 = \dots = z_m = 0\}$ is a convex set (we assume that $D_m = D$ if $m = 0$).*

Proof. The proof is similar to that of Theorem 1 in [2].

Take two points $a, b \in D_m$. We may find a domain $D' \subset D$ which is biholomorphic to a convex domain G and such that $\lambda a, \lambda b \in D'$ for $\lambda \in \overline{\mathbb{D}}$. Let $\Psi : D' \to G$ be the corresponding biholomorphic mapping. We may assume that $\Psi(0) = 0$ and $\Psi'(0) = \mathrm{id}$. If

$$g_{ab}(\lambda) = \frac{\Psi(\pi_\lambda(a)) + \Psi(\pi_\lambda(b))}{2},$$

then $\Psi^{-1} \circ g_{ab}$ is a holomorphic mapping from a neighborhood of $\overline{\mathbb{D}}$ into D. Set $f_{ab}(\lambda) = \pi_{1/\lambda} \circ \Psi^{-1} \circ g_{ab}(\lambda)$. We shall see later that $f_{ab}(\lambda)$ can be extended at 0 by proving that

$$\lim_{\lambda \to 0} f_{ab}(\lambda) = \frac{a+b}{2}. \tag{1}$$

If (1) holds, then $h \circ f_{ab}$ is a subharmonic function by Proposition 1, and the maximum principle implies that

$$h(f_{ab}(0)) \le \max_{|\lambda|=1} h(f_{ab}(\lambda)) < 1.$$

Hence $(a+b)/2 \in D_m$ if $a, b \in D_m$, i.e. D_m is a convex set.

To prove (1), note that $\Psi^{-1}(0) = 0$ and $(\Psi^{-1})'(0) = \mathrm{id}$ imply that, for any $j = 1, \ldots, n$, one has

$$\Psi_j^{-1} \circ g_{ab}(\lambda) = g_{abj}(\lambda) + O(|g_{ab}(\lambda)|^2).$$

Since $\Psi(0) = 0$, $\Psi'(0) = \mathrm{id}$ and $a, b \in D_m$, it follows that

$$g_{abj}(\lambda) = \frac{a_j + b_j}{2}\,\lambda^{k_j} + O(|\lambda|^{2k_{m+1}}).$$

Now the inequality $2k_{m+1} > k_n$ shows that

$$\frac{\Psi_j^{-1} \circ g_{ab}(\lambda)}{\lambda^{k_j}} = \frac{a_j + b_j}{2} + O(|\lambda|)$$

and letting $\lambda \to 0$ we obtain (1). ∎

A consequence of Proposition 2 is that any balanced domain which can be exhausted by domains biholomorphic to convex domains is convex itself.

Note also that the condition $2k_{m+1} > k_n$ is essential, as the following simple example shows. The $(1,2)$-balanced domain

$$D = \{z \in \mathbb{C}^2 : |z_1|^2 + |z_2 + z_1^2| < 1\}$$

is not convex, but it is biholomorphic to the $(1,2)$-balanced convex domain

$$G = \{z \in \mathbb{C}^2 : |z_1|^2 + |z_2| < 1\}.$$

Now we are ready to prove our main result. To do this, we shall apply Proposition 2 and the Cohn criterion which states that (see e.g. [5]) all the roots of a polynomial $f(\zeta) = \sum_{j=0}^{n} a_j \zeta^{n-j}$, $n \ge 2$, $a_0 \ne 0$, belong to $\mathbb{D}$ if and only if $|a_0| > |a_n|$ and all the roots of the polynomial

$$f^\star(\zeta) = \frac{\overline{a}_0 f(\zeta) - a_n \zeta^n \overline{f}(1/\overline{\zeta})}{\zeta}$$

belong to $\mathbb{D}$.

Proposition 3. *The symmetrized n-disc $\mathbb{G}_n$, $n \ge 3$, cannot be exhausted by domains biholomorphic to convex domains.*

Proof. Note that $\mathbb{G}_n$ is a $(1,\ldots,n)$-balanced domain. Hence, by Proposition 2, it is enough to show that if $m = [n/2]$, then the set G_n of points $(a_{m+1}, \ldots, a_n) \in \mathbb{C}^{n-m}$ such that all the zeros of the polynomial $f_n(\zeta) = \zeta^n + \sum_{j=m+1}^{n} a_j \zeta^{n-j}$ belong to $\mathbb{D}$ is not convex.

We shall first settle the cases $n = 3$ and $n = 4$, and then reduce the general case to them.

The case $n = 3$. For $f_3(\zeta) = \zeta^3 + p\zeta + q$ one has

$$f_3^\star(\zeta) = \frac{f_3(\zeta) - q\zeta^3 \overline{f}_3(1/\overline{\zeta})}{\zeta} = (1 - |q|^2)\zeta^2 - \overline{p}q\zeta + p,$$

$$f_3^{\star\star}(\zeta) = \frac{(1-|q|^2)f_3^\star(\zeta) - p\zeta^2\overline{f_3^\star}(1/\overline{\zeta})}{\zeta}$$
$$= ((1-|q|^2)^2 - |p|^2)\zeta - \overline{p}q(1-|q|^2) + p^2\overline{q}.$$

It follows from the Cohn criterion that

$$G_3 = \{(p,q) \in \mathbb{C}^2 : |q| < 1,\, r(p,q) < 0\},$$

where

$$r(p,q) = |\overline{p}q(1-|q|^2) - p^2\overline{q}| + |p|^2 - (1-|q|^2)^2.$$

It is easy to see that if $q' \in (-1,1)$ and $p' = 1 - q'^2$, then $(p_1,q_1) = (p'e^{2\pi i/3}, q')$ and $(p_2,q_2) = (p'e^{\pi i/3}, q'e^{\pi i/2})$ are boundary points of D, since $r(p',q') = 0$ and $r(p,q') < 0$ if $p \in (|q'|-1, p')$. Then for

$$(p_0,q_0) = \left(\frac{p_1+p_2}{2}, \frac{q_1+q_2}{2}\right) = \left(p'\cos\frac{\pi}{6}\,e^{\pi i/2}, q'\cos\frac{\pi}{4}\,e^{\pi i/4}\right)$$

one has

$$|\overline{p}_0 q_0(1-|q_0|^2) - p_0^2\overline{q}_0| = |p_0q_0|(1-|q_0|^2+|p_0|).$$

Therefore

$$r(p_0,q_0) = (1-|q_0|^2+|p_0|)(1+|q_0|)(|p_0|+|q_0|-1).$$

So $r(p_0,q_0) > 0$ if and only if $|p_0|+|q_0| > 1$. For $q' = 1/2$ it follows that

$$|p_0|+|q_0| = \frac{3\sqrt{3}+2\sqrt{2}}{8} > 1.$$

Thus $(p_0,q_0) \notin \overline{G}_3$ and hence G_3 is not a convex set.

The case $n = 4$. Calculations similar to the previous case lead to

$$G_4 = \{(p,q) \in \mathbb{C}^2 : |p| + |q|^2 < 1,\, s(p,q) < 0\},$$

where

$$s(p,q) = (1-|q|^2)|\overline{p}q((1-|q|^2)^2-|p|^2) - p^3\overline{q}^2| + |p|^4|q|^2 - ((1-|q|^2)^2-|p|^2)^2.$$

It is easy to see that if $q' \in [0,1)$ and $p' = (1-q')\sqrt{1+q'}$, then $(p_1,q_1) = (p'e^{\pi i/2}, q') \in \partial D$ and $(p_2,q_2) = (p'e^{\pi i/4}, q'e^{\pi i/3}) \in \partial D$, since $s(p',q') = 0$ and $s(p',q) < 0$ if $p \in (-p',p')$. Then for

$$(p_0,q_0) = \left(\frac{p_1+q_1}{2}, \frac{p_2+q_2}{2}\right) = \left(p'\cos\frac{\pi}{8}\,e^{3\pi i/8}, q'\cos\frac{\pi}{6}\,e^{\pi i/6}\right)$$

one has

$$|\overline{p}_0q_0((1-|q_0|^2)^2-|p_0|^2) - p_0^3\overline{q}_0^2| = |p_0q_0|((1-|q_0|^2)^2 - |p_0|^2 + |p_0|^2|q_0|).$$

Therefore

$$s(p_0,q_0) = (1-|q_0|^2)(1-|q_0|^2)(1+|q_0|) - |p_0|^2)(1+|p_0|-|q_0|^2)(|p_0|+|q_0|-1).$$

So $s(p_0, q_0) > 0$ if and only if $|p_0| + |q_0| > 1$. For $q' = 2/5$ it follows that

$$|p_0| + |q_0| = \frac{1}{10}\left(3\sqrt{\frac{7(2+\sqrt{2})}{5}} + 2\sqrt{3}\right) > 1.$$

Thus $(p_0, q_0) \notin \overline{G}_4$ and hence G_4 is not a convex set.

The case $n \geq 5$. Let $j \in \{0, 1, 2\}$. Observe that the non-convex set G_3 coincides with the set of points $(p, q) \in \mathbb{C}^2$ such that all the zeros of the polynomial $z^j f_3(z^k)$, $k \geq 1$, belong to the unit disc. It follows that if either $n = 3k + 2$ and $k \geq 3$, $n = 3k + 1$ and $k \geq 2$, or $n = 3k$ and $k \geq 1$, then G_3 can be considered as an intersection of G_n and a complex hyperplane. Therefore G_n is not a convex set in these cases.

In the remaining cases $n = 5$ and $n = 8$ it is enough to observe that the non-convex set G_4 coincides with the set of points $(p, q) \in \mathbb{C}^2$ such that all the zeros of either of the polynomials $\zeta f_4(\zeta)$ and $f_4(\zeta^2)$ belong to the unit disc and then to complete the proof as above. ∎

References

[1] C. Costara, *The symmetrized bidisc and Lempert's theorem*, Bull. London Math. Soc. 36 (2004), 656–662.

[2] A. Edigarian, *A note on Costara's paper*, Ann. Polon. Math. 83 (2004), 189–191.

[3] M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, de Gruyter Exp. Math. 9, de Gruyter, Berlin, 1993.

[4] —, —, *Invariant distances and metrics in complex analysis—revisited*, Dissertationes Math. 430 (2005).

[5] Q. I. Rahman and G. Schmeisser, *Analytic Theory of Polynomials*, London Math. Soc. Monogr. (N.S.) 26, Oxford Univ. Press, Oxford, 2002.

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
1113 Sofia, Bulgaria
E-mail: nik@math.bas.bg

Received 6.3.2006 (1662)