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Zeros of solutions of certain higher order linear differential equations

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Abstract. We investigate the exponent of convergence of the zero-sequence of solutions of the differential equation

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f' + D(z)f = 0, \tag{1}$$

where $D(z) = Q_1(z)e^{P_1(z)} + Q_2(z)e^{P_2(z)} + Q_3(z)e^{P_3(z)}$, $P_1(z), P_2(z), P_3(z)$ are polynomials of degree $n \geq 1$, $Q_1(z), Q_2(z), Q_3(z), a_j(z)$ $(j = 1, \ldots, k-1)$ are entire functions of order less than n, and $k \geq 2$.

1. Introduction and results. We shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [5, 8]). We will use the notation $\sigma(f)$ to denote the order of growth of a meromorphic function $f(z)$ and $\lambda(f)$ to denote the exponent of convergence of the zero-sequence of $f(z)$.

K. Ishizaki and K. Tohge [6, 7] have studied the exponent of convergence of the zero-sequence of solutions of the equation

$$f'' + (e^{P_1(z)} + e^{P_2(z)} + Q_0(z))f = 0, \tag{2}$$

where $P_1(z), P_2(z)$ are non-constant polynomials

$$P_1(z) = \zeta_1 z^n + \cdots, \qquad P_2(z) = \zeta_2 z^m + \cdots, \qquad \zeta_1\zeta_2 \neq 0 \qquad (n, m \in \mathbb{N}).$$

and $Q_0(z)$ is an entire function of order less than $\max\{n, m\}$, and $e^{P_1(z)}$ and $e^{P_2(z)}$ are linearly independent. They have obtained the following results:

Theorem A ([7]). *Suppose that $n = m$, and that $\zeta_1 \neq \zeta_2$ in (2). If ζ_1/ζ_2 is non-real, then for any solution $f \not\equiv 0$ of (2), we have $\lambda(f) = \infty$.*

Theorem B ([6]). *Suppose that $n = m$, and that $\zeta_1/\zeta_2 = \rho > 0$ in (2). If $0 < \rho < 1/2$ or $Q_0(z) \equiv 0$, $3/4 < \rho < 1$, then for any solution $f \not\equiv 0$ of (2), we have $\lambda(f) \geq n$.*

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Recently, J. Tu and Z. X. Chen [9] investigated the exponent of convergence of the zero-sequence of solutions of the differential equation

$$f'' + (Q_1(z)e^{P_1(z)} + Q_2(z)e^{P_2(z)} + Q_3(z)e^{P_3(z)})f = 0, \tag{3}$$

and obtained the following theorem:

THEOREM C ([9]). *Let Q_1, Q_2, Q_3 be entire functions of order less than n, and $P_1(z), P_2(z), P_3(z)$ be polynomials of degree $n \geq 1$,*

$$P_1(z) = \zeta_1 z^n + \cdots, \qquad P_2(z) = \zeta_2 z^n + \cdots, \qquad P_3(z) = \zeta_3 z^n + \cdots,$$

where $\zeta_1, \zeta_2, \zeta_3$ are complex numbers.

(i) *If ζ_1/ζ_2 is non-real and $0 < \lambda = \zeta_3/\zeta_2 < 1/2$, then for any solution $f \not\equiv 0$ of (3), we have $\lambda(f) = \infty$.*
(ii) *If $0 < \zeta_1/\zeta_2 < 1/4$ and $0 < \lambda = \zeta_3/\zeta_2 < 1$, then for any solution $f \not\equiv 0$ of (3), we have $\lambda(f) \geq n$.*

It is natural to ask about the exponent of convergence of the zero-sequence of solutions of the higher order linear differential equation (1). In the present paper we shall investigate this problem and obtain the following result which improves all the theorems mentioned earlier.

THEOREM 1.1. *Let $P_1(z), P_2(z), P_3(z)$ be as in Theorem C and $Q_1(z)$, $Q_2(z), Q_3(z), a_j(z)$ $(j = 1, \ldots, k-1)$ be entire functions of order less than n and $k \geq 2$.*

(i) *If ζ_1/ζ_2 is non-real and $0 < \lambda = \zeta_3/\zeta_2 < 1/k$, then for any solution $f \not\equiv 0$ of (1), we have $\lambda(f) = \infty$.*
(ii) *If $0 < \zeta_1/\zeta_2 < 1/2k$ and $0 < \lambda = \zeta_3/\zeta_2 < 1$, then for any solution $f \not\equiv 0$ of (1), we have $\lambda(f) \geq n$.*

2. Notations and some lemmas. To prove the theorem, we need some notations and a series of lemmas. Let $P_j(z)$ $(j = 1, 2, 3)$ be polynomials of degree $n \geq 1$, $P_j(z) = (\alpha_j + i\beta_j)z^n + \cdots$, $\alpha_j, \beta_j \in \mathbb{R}$. Define

$$\delta(P_j, \theta) = \delta_j(\theta) = \alpha_j \cos n\theta - \beta_j \sin n\theta, \qquad \theta \in [0, 2\pi)\ (j = 1, 2, 3),$$
$$S_j^+ = \{\theta : \delta_j(\theta) > 0\}, \qquad S_j^- = \{\theta : \delta_j(\theta) < 0\} \qquad (j = 1, 2, 3).$$

Let $f(z)$ and $a(z)$ be meromorphic functions in the plane that satisfy

$$T(r, a) = o\{T(r, f)\},$$

except possibly for a set of r having finite linear measure. We then say that $a(z)$ is a *small function* with respect to $f(z)$.

LEMMA 2.1 ([4]). *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f) = \sigma < \infty$, and $\Gamma = \{(k_1, j_1), \ldots, (k_m, j_m)\}$ be a finite set of distinct pairs of integers which satisfy $k_i > j_i \geq 0$ for $i = 1, \ldots, m$. Let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset [0, 2\pi)$ of linear measure zero*

ν such that if $\varphi \in [0, 2\pi) \setminus E$ then there is a constant $R_1 = R_1(\varphi) > 1$ such that for all z satisfying $\arg z = \varphi$ *and $|z| = r > R_1$, and for all $(k, j) \in \Gamma$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

LEMMA 2.2 ([2, 9]). *Suppose that $P(z) = (\alpha + \beta i)z^n + \cdots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) is a polynomial of degree $n \ge 1$, and $A(z)$ ($\not\equiv 0$) is an entire function with $\sigma(A) < n$. Set $g(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then for any given $\varepsilon > 0$, there exists a set $H_1 \subset [0, 2\pi)$ of linear measure zero such that for any $\theta \in [0, 2\pi) \setminus (H_1 \cup H_2)$, where $H_2 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ is a finite set, there is $R > 0$ such that for $|z| = r > R$, we have:*

(i) *If $\delta(P, \theta) > 0$, then*

$$\exp\{(1-\varepsilon)\delta(P, \theta)r^n\} < |g(re^{i\theta})| < \exp\{(1+\varepsilon)\delta(P, \theta)r^n\}.$$

(ii) *If $\delta(P, \theta) < 0$, then*

$$\exp\{(1+\varepsilon)\delta(P, \theta)r^n\} < |g(re^{i\theta})| < \exp\{(1-\varepsilon)\delta(P, \theta)r^n\}.$$

LEMMA 2.3 ([1]). *Suppose $\pi(z)$ is the canonical product formed with the zeros $\{z_n : n = 1, 2, \dots,\}$ ($z_n \neq 0$) of an entire function $f(z)$. Set $O_n = \{z : |z - z_n| < |z_n|^{-\alpha}\}$ (α ($> \lambda(f)$) is a constant). Then for any given $\varepsilon > 0$,*

$$|\pi(z)| \ge \exp\{-|z|^{\lambda(f)+\varepsilon}\} \quad \text{for } z \notin \bigcup_{n=1}^{\infty} O_n.$$

LEMMA 2.4 ([3]). *Let $f(z)$ be an entire function of order $\sigma(f) = \alpha < +\infty$. Then for any given $\varepsilon > 0$, there is a set $E \subset [1, \infty)$ of finite linear measure and finite logarithmic measure such that for all z satisfying $|z| \notin [0, 1] \cup E$, we have*

$$\exp\{-r^{\alpha+\varepsilon}\} \le |f(z)| \le \exp\{r^{\alpha+\varepsilon}\}.$$

LEMMA 2.5 ([10]). *Let $P_j(z)$ ($j = 1, 2, 3$) be polynomials of degree $n \ge 1$,*

$$P_1(z) = \zeta z^n + B_1(z), \qquad P_2(z) = \rho_1 \zeta z^n + B_2(z), \qquad P_3(z) = \rho_2 \zeta z^n + B_3(z),$$

where $\zeta = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, $|\alpha| + |\beta| \neq 0$, $0 < \rho_1 < 1$, $0 < \rho_2 < 1$, and $B_1(z), B_2(z)$, $B_3(z)$ are polynomials of degree at most $n-1$. Let $Q_1(z) \not\equiv 0$, $Q_2(z), Q_3(z)$ be entire functions of order less than n. Then for any given $\varepsilon > 0$, there exist a set E of finite linear measure and a constant ξ ($n - 1 < \xi < n$) such that

$$m(r, Q_1 e^{P_1} + Q_2 e^{P_2} + Q_3 e^{P_3}) \ge (1-\varepsilon)m(r, e^{P_1}) + O(r^{\xi}), \qquad r \to \infty \ (r \notin E).$$

LEMMA 2.6 ([11]). *Let $f(z)$ be an entire function and write $f(z)=\pi e^h$. Then*

(i) $$\frac{f^{(k)}}{f}=(h')^k+k\,\frac{\pi'}{\pi}(h')^{k-1}+\frac{k(k-1)}{2}(h')^{k-2}h''+H_{k-2}(h') \quad (k\ge 2),$$

where $H_{k-2}(h')$ is a differential polynomial of degree no more than $k-2$ in h', and its coefficients are terms of the type $c(\pi'/\pi)^{s_1}\cdots(\pi^{(k)}/\pi)^{s_k}$, where c is a constant and $s_1,\dots,s_k$ are non-negative integers; and

(ii) $$\frac{f^{(k+1)}}{f}-\frac{f^{(k)}}{f}\frac{f'}{f}=k(h')^{k-1}h''+H_{k-1}(h') \quad (k\ge 1),$$

where $H_{k-1}(h')$ is a differential polynomial of degree no more than $k-1$ in h', and its coefficients are terms of the type $c(\pi'/\pi)^{s_1}\cdots(\pi^{(k+1)}/\pi)^{s_{k+1}}$, where c is a constant and $s_1,\dots,s_{k+1}$ are non-negative integers.

LEMMA 2.7 ([11]). *Let $U_1(z),h(z),Q_1(z),P_1(z)$ be entire functions satisfying $U_1=Q_1h''-\frac{1}{k}(Q_1'+Q_1P_1')h'$. Then*

$$Q_1^{n-1}h^{(n)}=A_{1,n-2}(U_1,Q_1)+B_{n-1}(Q_1)h' \quad (n\ge 2),$$

where $A_{1,n-2}(U_1,Q_1)$ is an algebraic expression in $U_1^{(j)}$, $Q_1^{(j)}$, $P_1^{(j)}$ $(j=0,1,\dots,l)$, involving addition, subtraction and multiplication, where the degree of $U_1^{(j)}$ is no more than 1 and the degree of $Q_1^{(j)}$ is no more than l; $B_d(Q_1)$ is a differential polynomial of degree no more than d in Q_1, its coefficients are algebraic expressions in $P_1^{(i)}$ $(i=1,\dots,d)$ and $1/k$, involving addition, subtraction and multiplication.

LEMMA 2.8. *Let $h(z)$ and $c_j(z)$ $(j=0,1,\dots,k-1)$ be meromorphic functions satisfying*

$$c_{k-1}(z)(h')^{k-1}+c_{k-2}(z)(h')^{k-2}+\cdots+c_1(z)h'+c_0(z)=0.$$

Then

$$m(r,h')\le\sum_{j=0}^{k-1}T(r,c_j(z))+O(1).$$

LEMMA 2.9. *Let h be a meromorphic function of finite order, and $E_{k-1}(h')$ a differential polynomial of degree no more than $k-1$, whose coefficients are meromorphic functions $a_j(z)$ $(j=0,1,\dots,k-1)$ with $\sigma(a_j)<n$. Then for sufficiently large r,*

$$m(r,(h')^k+E_{k-1}(h'))\le km(r,h')+O(r^{\xi}),$$

where $0<\max\{\sigma(a_j):j=0,1,\dots,k-1\}<\xi<n$.

REMARK 2.1. Lemmas 2.8 and 2.9 are immediate consequences of the Valiron–Mohon'ko theorem (see [8]) and/or Clunie technique.

3. Proof of Theorem 1.1. Since $\zeta_3 = \lambda\zeta_2$, $\lambda > 0$, we have $S_2^+ = S_3^+$ and $S_2^- = S_3^-$. We see that S_j^+ and S_j^- have n components S_{jk}^+ and S_{jk}^- respectively ($j = 1, 2, 3$; $k = 1, \dots, n$). Hence we write

$$S_j^+ = \bigcup_{k=1}^{n} S_{jk}^+, \qquad S_j^- = \bigcup_{k=1}^{n} S_{jk}^- \qquad (j = 1, 2, 3).$$

Furthermore, we define

$$D_{12} = \left\{\theta \in S_1^+ \cap S_2^+ : \delta_1(\theta) > \frac{k(\lambda+1)}{k-1}\,\delta_2(\theta)\right\},$$
$$D_{21} = \left\{\theta \in S_1^+ \cap S_2^+ : \delta_2(\theta) > \frac{\lambda+1}{\lambda}\,\delta_1(\theta)\right\}.$$

(i) Let $f \not\equiv 0$ be a solution of (1). Suppose that $\lambda(f) < \infty$. Write $f = \pi e^h$, where π is the canonical product of the zeros of f, and h is an entire function. From our hypothesis, we have $\sigma(\pi) = \lambda(\pi) < \infty$. From (1), we get

$$\frac{f^{(k)}}{f} + a_{k-1}\,\frac{f^{(k-1)}}{f} + \cdots + a_1\,\frac{f'}{f} + D(z) = 0, \tag{4}$$

By Lemma 2.6(i), we get

$$(h')^k = E_{k-1}(h') - Q_1(z)e^{P_1(z)} - Q_2(z)e^{P_2(z)} - Q_3(z)e^{P_3(z)}, \tag{5}$$

where $E_{k-1}(h')$ is a differential polynomial of degree no more than $k-1$ in h', and its coefficients are terms of the type $ca_j^p(z)(\pi'/\pi)^{s_1}\cdots(\pi^{(k)}/\pi)^{s_k}$ ($j = 1, \dots, k-1$), where c is a constant, $s_1, \dots, s_k$ are non-negative integers and p is 0 or 1.

Eliminating e^{P_1} from (4), we have

$$\begin{aligned}
Q_1\biggl(\frac{f^{(k+1)}}{f} - \frac{f^{(k)}}{f}\frac{f'}{f}\biggr) &+ a_{k-1}Q_1\biggl(\frac{f^{(k)}}{f} - \frac{f^{(k-1)}}{f}\frac{f'}{f}\biggr) + a_1Q_1\biggl(\frac{f''}{f} - \frac{f'}{f}\frac{f'}{f}\biggr)\\
&- (Q_1' + Q_1P_1')\biggl(\frac{f^{(k)}}{f} + a_{k-1}\,\frac{f^{(k-1)}}{f} + \cdots + a_1\,\frac{f'}{f} + Q_2e^{P_2} + Q_3e^{P_3}\biggr)\\
&+ Q_1\biggl[a_{k-1}'\,\frac{f^{(k-1)}}{f} + \cdots + a_1'\,\frac{f'}{f}\biggr] + Q_1(Q_2' + Q_2P_2')e^{P_2}\\
&+ Q_1(Q_3' + Q_3P_3')e^{P_3} = 0.
\end{aligned}$$

By Lemma 2.6(ii), we can write this as

$$\begin{aligned}
kU_1(h')^{k-1} = F_{k-1}^1(h') &+ e^{P_2}[Q_2(Q_1' + Q_1P_1') - Q_1(Q_2' + Q_2P_2')]\\
&+ e^{P_3}[Q_3(Q_1' + Q_1P_1') - Q_1(Q_3' + Q_3P_3')],
\end{aligned} \tag{6}$$

where

$$U_1 = Q_1h'' - \frac{1}{k}\,(Q_1' + Q_1P_1')h', \tag{7}$$

and $F^1_{k-1}(h')$ is a differential polynomial of degree no more than $k-1$ in h', with coefficients of the type $c(a_j(z))^p(a'_j(z))^q(Q_1)^l(Q'_1)^t(P'_1)^u(\pi'/\pi)^{s_1}\cdots(\pi^{(k)}/\pi)^{s_k}$, where c is a constant, $s_1,\ldots,s_k$ are non-negative integers and each of p,q,l,t,u is 0 or 1. Similarly, we obtain

$$\begin{aligned}(8)\qquad kU_2(h')^{k-1} = F^2_{k-1}(h') &+ e^{P_1}[Q_1(Q'_2+Q_2P'_2)-Q_2(Q'_1+Q_1P'_1)]\\ &+ e^{P_3}[Q_3(Q'_2+Q_2P'_2)-Q_2(Q'_3+Q_3P'_3)],\end{aligned}$$

where

$$(9)\qquad U_2 = Q_2h'' - \frac{1}{k}\,(Q'_2+Q_2P'_2)h',$$

and $F^2_{k-1}(h')$ is a differential polynomial of degree no more than $k-1$ in h', with coefficients of the type $c(a_j(z))^p(a'_j(z))^q(Q_2)^l(Q'_2)^t(P'_2)^u(\pi'/\pi)^{s_1}\cdots(\pi^{(k)}/\pi)^{s_k}$, where c is a constant, $s_1,\ldots,s_k$ are non-negative integers and each of p,q,l,t,u is 0 or 1.

Let $\max\{\sigma(Q_i),\sigma(a_j): i=1,2,3;\ j=1,\ldots,k-1\}<\xi_1<\xi_2<\xi_3<n$. From Lemma 2.4 we get

$$|Q_i(re^{i\theta}|\le\exp(r^{\xi_1})\quad(i=1,2,3),\qquad |a_j(z)|\le\exp(r^{\xi_1})\quad(j=1,\ldots,k-1),$$

for sufficiently large r and for any $\theta\in[0,2\pi)$. Applying the Clunie Lemma [5, Lemma 3.3] to (5), for any given $\varepsilon>0$ we get

$$\begin{aligned}T(r,h') &= m(r,h')\\ &\le m(r,Q_1e^{P_1}+Q_2e^{P_2}+Q_3e^{P_3})\\ &\quad+O\Big(\sum_{j=1}^{k}m(r,\pi^{(j)}/\pi)+\sum_{i=1}^{k-1}m(r,a_i)\Big)+S(r,h')\\ &\le O(r^{n+\varepsilon})+S(r,h'),\end{aligned}$$

which implies $\sigma(h')\le n$. It follows from (7) and (9) that $\sigma(U_1)\le n$ and $\sigma(U_2)\le n$ respectively.

In the following, we will show that there exists a set $E_0\subset[0,2\pi)$ with $m(E_0)=0$ such that if $\theta\in S_2^-\setminus E_0$, then

$$(10)\qquad |U_1(re^{i\theta})|\le O(\exp\{r^{\xi_2}\}),\qquad r\to\infty.$$

If $|h'(re^{i\theta})|\le 1$, from Lemmas 2.1, 2.2 and 2.4 and (7), we have

$$\begin{aligned}(11)\qquad |U_1(re^{i\theta})| &\le \frac{|h''(re^{i\theta})|}{|h'(re^{i\theta})|}\,|Q_1(re^{i\theta})|+\frac{1}{k}\,|P'_1(re^{i\theta})|\,|Q_1(re^{i\theta})|\\ &\quad+\frac{1}{k}\,\frac{|Q'_1(re^{i\theta})|}{|Q_1(re^{i\theta})|}\,|Q_1(re^{i\theta})|\\ &\le O(\exp\{r^{\xi_2}\}),\qquad r\to\infty.\end{aligned}$$

Assume $|h'(re^{i\theta})| \geq 1$. Since $F^1_{k-1}(h')$ is the sum of a finite number of terms of the type

$$H(z) = c(a_j(z))^p(a_j'(z))^q(Q_1)^l(Q_1')^t(P_1')^u\left(\frac{\pi'}{\pi}\right)^{s_1}\cdots\left(\frac{\pi^{(k)}}{\pi}\right)^{s_k} \times (h')^{l_0}(h'')^{l_1}\cdots(h^{(v)})^{l_{v-1}},$$

where $l_0, l_1, \ldots, l_{v-1}$ are non-negative integers and $l_0+l_1+\cdots+l_{v-1} \leq k-1$, from Lemma 2.1 we get

$$\begin{aligned}(12)\quad &\frac{|H(re^{i\theta})|}{|h'(re^{i\theta})|^{k-1}}\\ &\leq |c|\,|a_j(re^{i\theta})|^p|a_j'(re^{i\theta})|^q|Q_1(re^{i\theta})|^l|Q_1'(re^{i\theta})|^t|P_1'(re^{i\theta})|^u\\ &\quad\times\left|\frac{\pi'(re^{i\theta})}{\pi(re^{i\theta})}\right|^{s_1}\cdots\left|\frac{\pi^{(k)}(re^{i\theta})}{\pi(re^{i\theta})}\right|^{s_k}\frac{|h''(re^{i\theta})|^{l_1}}{|h'(re^{i\theta})|}\cdots\frac{|h^{(v)}(re^{i\theta})|^{l_{v-1}}}{|h'(re^{i\theta})|}\\ &\leq O(\exp\{r^{\xi_2}\}).\end{aligned}$$

Thus

$$\frac{|F^1_{k-1}(re^{i\theta})|}{|h'(re^{i\theta})|^{k-1}} \leq O(\exp\{r^{\xi_2}\}). \tag{13}$$

From (8), (13) and Lemma 2.2, we get

$$\begin{aligned}(14)\quad &k|U_1(re^{i\theta})|\\ &\leq \frac{|F^1_{k-1}(re^{i\theta})|}{|h'(re^{i\theta})|^{k-1}} + |e^{P_2(re^{i\theta})}|\,\big|Q_2(re^{i\theta})(Q_1'(re^{i\theta}) + Q_1(re^{i\theta})P_1'(re^{i\theta}))\\ &\quad - Q_1(re^{i\theta})((Q_2'(re^{i\theta}) + Q_2(re^{i\theta})P_2'(re^{i\theta}))\big|\\ &\quad + |e^{P_3(re^{i\theta})}|\,\big|Q_3(re^{i\theta})(Q_1'(re^{i\theta}) + Q_1(re^{i\theta})P_1'(re^{i\theta}))\\ &\quad - Q_1(re^{i\theta})(Q_3'(re^{i\theta}) + Q_3(re^{i\theta})P_3'(re^{i\theta})\big|\\ &\leq O(\exp\{r^{\xi_2}\}), \qquad r \to \infty.\end{aligned}$$

From (11) and (14), we obtain (10).

We note that there exist $\bar{\theta}_j$ $(j = 1, 2, 3)$ satisfying $\delta_j(\theta) = 0$ on the rays $\arg z = \bar{\theta}_j + q\pi/n$, where $q = 0, \ldots, 2n-1$, which form $2n$ sectors of opening π/n each, thus we may assume that $\bar{\theta}_j \in [0, \pi/n)$. Since $\zeta_2 = \lambda\zeta_3$, $\lambda > 0$, we have $\bar{\theta}_2 = \bar{\theta}_3$. Write $\bar{\theta}_{jq} = \bar{\theta}_j + q\pi/n$, $j = 1, 2$. If there are some integers q_1 and q_2 such that $\bar{\theta}_{1q_1} = \bar{\theta}_{2q_2}$, then $\bar{\theta}_1 - \bar{\theta}_2 + (q_1 - q_2)\pi/n = 0$, and we have $\tan n\bar{\theta}_j = \alpha_j/\beta_j$, $j = 1, 2$. This gives

$$0 = \tan(n\bar{\theta}_1 - n\bar{\theta}_2 + (q_1 - q_2)\pi) = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\alpha_1\alpha_2 + \beta_1\beta_2}.$$

This contradicts the assumption that ζ_1/ζ_2 is non-real. Hence each component of S_1^+ and S_2^+ contains a component of $S_1^+ \cap S_2^+$. The boundaries of the components of $S_1^+ \cap S_2^+$ are some of the rays $\arg z = \bar{\theta}_{jq}$. We fix a component of $S_1^+ \cap S_2^+$, say S^*. We may write

$$S^* = \{\theta \in S_1^+ \cap S_2^+ : \theta_1^* < \theta < \theta_2^*,\ \delta_1(\theta_1^*) = \delta_2(\theta_2^*) = 0\}$$

or

$$S^* = \{\theta \in S_1^+ \cap S_2^+ : \theta_2^* < \theta < \theta_1^*,\ \delta_1(\theta_1^*) = \delta_2(\theta_2^*) = 0\}.$$

Since every component of S_1^+ and S_2^+ has opening π/n, the rays $\arg z = \theta_1^*$ and $\arg z = \theta_2^*$ are contained in S_2^+ and S_1^+ respectively. We handle the first case, the proof of the second being similar. Then there exist $\eta_1, \eta_2 > 0$ such that

$$\{\theta : \theta_1^* < \theta < \theta_1^* + \eta_1\} \subset D_{21}, \qquad \{\theta : \theta_2^* - \eta_2 < \theta < \theta_2^*\} \subset D_{12}.$$

Hence there exists a $\theta \in (S_{2k}^+ \cap D_{12}) \setminus E_0$ for any $k = 1, \ldots, n$. Take $0 < \frac{k(\lambda+1)}{k-1}\delta_2 < \sigma_2 < \sigma_1 < \delta_1$, $0 < \varepsilon_1 < 1 - \frac{\sigma_1}{\delta_1}$, $0 < \varepsilon_2 < \frac{(k-1)\sigma_2}{k\delta_2} - 1$, $0 < \varepsilon_3 < \frac{(k-1)\sigma_2}{k\lambda\delta_2} - 1$. By Lemma 2.2, we have

$$\begin{aligned}
(15)\qquad & |Q_1 e^{P_1(re^{i\theta})} + Q_2 e^{P_2(re^{i\theta})} + Q_3 e^{P_3(re^{i\theta})}| \\
&\geq |Q_1 e^{P_1(re^{i\theta})}| \left| 1 - \left| \frac{Q_2}{Q_1}\, e^{P_2(re^{i\theta}) - P_1(re^{i\theta})} \right| - \left| \frac{Q_3}{Q_1}\, e^{P_3(re^{i\theta}) - P_1(re^{i\theta})} \right| \right| \\
&\geq \exp\{(1-\varepsilon_1)\delta_1 r^n\}(1 - o(1)) \\
&\geq \exp\{\sigma_1 r^n\}(1 - o(1)), \qquad r \to \infty.
\end{aligned}$$

We assume that there exists an unbounded sequence $\{r_q\}$ such that $0 < |h'(r_q e^{i\theta})| \leq 1$. From (5), (15) and Lemma 2.1, we get

$$\begin{aligned}
\exp\{\sigma_1 r_q^n\}(1 - o(1)) &\leq |h'(r_q e^{i\theta})|^k + |E_{k-1}(h'(r_q e^{i\theta}))| \\
&\leq 1 + \sum |c|\, |a_j(r_q e^{i\theta})|^p \left| \frac{\pi'(r_q e^{i\theta})}{\pi(r_q e^{i\theta})} \right|^{s_1} \cdots \left| \frac{\pi^{(k)}(r_q e^{i\theta})}{\pi(r_q e^{i\theta})} \right|^{s_k} \\
&\quad \times |h'(r_q e^{i\theta})|^{l_0} \cdots |h^{(v)}(r_q e^{i\theta})|^{l_{v-1}} \\
&\leq 1 + \sum |c|\, |a_j(r_q e^{i\theta})|^p \left| \frac{\pi'(r_q e^{i\theta})}{\pi(r_q e^{i\theta})} \right|^{s_1} \cdots \left| \frac{\pi^{(k)}(r_q e^{i\theta})}{\pi(r_q e^{i\theta})} \right|^{s_k} \\
&\quad \times \left| \frac{h''(r_q e^{i\theta})}{h'(r_q e^{i\theta})} \right|^{l_1} \cdots \left| \frac{h^{(v)}(r_q e^{i\theta})}{h'(r_q e^{i\theta})} \right|^{l_{v-1}} \\
&\leq O(\exp\{r_q^{\xi_2}\}) \qquad (q \to \infty),
\end{aligned}$$

which is not true. Hence we may assume that $|h'(re^{i\theta})| \geq 1$ for all r sufficiently large. From (5), (15) and Lemma 2.2, we get

$$\begin{aligned}\exp\{\sigma_1 r_q^n\}(1-o(1)) &\le |h'(r_q e^{i\theta})|^k + |E_{k-1}(h'(r_q e^{i\theta}))| \\ &\le |h'(r_q e^{i\theta})|^k \Bigg[1 + \sum |c|\,|a_j(r_q e^{i\theta})|^p \left|\frac{\pi'(r_q e^{i\theta})}{\pi(r_q e^{i\theta})}\right|^{s_1} \\ &\qquad \cdots \left|\frac{\pi^{(k)}(r_q e^{i\theta})}{\pi(r_q e^{i\theta})}\right|^{s_k} \left|\frac{h''(r_q e^{i\theta})}{h'(r_q e^{i\theta})}\right|^{l_1} \cdots \left|\frac{h^{(v)}(r_q e^{i\theta})}{h'(r_q e^{i\theta})}\right|^{l_{v-1}}\Bigg] \\ &\le |h'(r_q e^{i\theta})|^k (1 + O(\exp\{r_q^{\xi_2}\})) \quad (q\to\infty),\end{aligned}$$

i.e.

$$|h'(re^{i\theta})|^k \ge \frac{1-o(1)}{1+O(\exp\{r^{\xi_2}\})}\exp\{\sigma_1 r^n\} \quad (r\to\infty).$$

Then we obtain for all r large enough

$$|h'(re^{i\theta})| \ge \exp\left\{\frac{1}{k}\,\sigma_2 r^n\right\}. \tag{16}$$

From Lemma 2.1, (6) and (16), we get

$$\begin{aligned}(17)\quad & k|U_1(re^{i\theta})| \\ &\le \frac{|F^1_{k-1}(re^{i\theta})|}{|h'(re^{i\theta})|^{k-1}} \\ &\quad + \frac{|e^{P_2(re^{i\theta})}|}{|h'(re^{i\theta})|^{k-1}}\Bigg[|Q_2(re^{i\theta})|\left(\frac{|Q_1'(re^{i\theta}|}{|Q_1(re^{i\theta})|}\,|Q_1(re^{i\theta})| + |Q_1(re^{i\theta})|\,|P_1'(re^{i\theta})|\right) \\ &\quad + |Q_1(re^{i\theta})|\left(\frac{|Q_2'(re^{i\theta})|}{|Q_2(re^{i\theta})|}\,|Q_2(re^{i\theta})| + |Q_2(re^{i\theta})|\,|P_2'(re^{i\theta})|\right)\Bigg] \\ &\quad + \frac{|e^{P_3(re^{i\theta})}|}{|h'(re^{i\theta})|^{k-1}}\Bigg[|Q_3(re^{i\theta})|\left(\frac{|Q_1'(re^{i\theta})|}{|Q_1(re^{i\theta})|}\,|Q_1(re^{i\theta})| + |Q_1(re^{i\theta})|\,|P_1'(re^{i\theta})|\right) \\ &\quad + |Q_1(re^{i\theta})|\left(\frac{|Q_3'(re^{i\theta})|}{|Q_3(re^{i\theta})|} + |Q_3(re^{i\theta})|\,|P_3'(re^{i\theta})|\right)\Bigg] \\ &\le O(\exp\{r^{\xi_2}\}) + (1+o(1))\exp\left\{\left(\delta_2(1+\varepsilon_2) - \frac{(k-1)\sigma_2}{k}\right)r^n\right\} \\ &\quad + (1+o(1))\exp\left\{\left(\lambda\delta_2(1+\varepsilon_3) - \frac{(k-1)\sigma_2}{k}\right)r^n\right\} \quad (r\to\infty).\end{aligned}$$

Since $\delta_2(1+\varepsilon_2) - (k-1)\sigma_2/k < 0$ and $\lambda\delta_2(1+\varepsilon_3) - (k-1)\sigma_2/k < 0$, this gives that for all sufficiently large r,

$$|U_1(re^{i\theta})| \le O(\exp\{r^{\xi_2}\}). \tag{18}$$

Now we fix a γ $(= \gamma_{2k}) \in (S_{2k}^+ \cap D_{12}) \setminus E_0$, $k = 1, \dots, n$. Then we find $\gamma_1, \gamma_2 \in S_2^- \setminus E_0$ with $\gamma_1 < \gamma < \gamma_2$ such that $\gamma - \gamma_1 < \pi/n$ and $\gamma_2 - \gamma < \pi/n$.

We first prove that for any θ with $\gamma_1 \le \theta \le \gamma$, we have

$$|U_1(re^{i\theta})| \le O(\exp\{r^{\xi_3}\}) \quad (r \to \infty). \tag{19}$$

Write $\gamma - \gamma_1 = \pi/(n+\tau_1)$ with $\tau_1 > 0$. Since $\sigma(U_1) \le n$, we have $|U_1(re^{i\theta})| \le e^{r^{n+\tau_2}}$ with $0 < \tau_2 < \tau_1$ for sufficiently large r. Set

$$g(z) = U_1(z)/\exp((ze^{-(\gamma+\gamma_1)/2})^{\xi_3}).$$

Then $g(z)$ is regular in the region $\{z : \gamma_1 \le \arg z \le \gamma\}$. Since $\gamma_1 \le \arg z = \theta \le \gamma$ and $\gamma - \gamma_1 < \pi/n$, we infer that $\cos\arg((ze^{-(\gamma+\gamma_1)/2})^{\xi_3}) \ge K$ for some $K > 0$. In fact,

$$-\frac{\pi}{2} < -\frac{\pi\xi_3}{2n} \le -\xi_3\frac{\gamma-\gamma_1}{2} \le \arg((ze^{-(\gamma+\gamma_1)/2})^{\xi_3}) \le \xi_3\,\frac{\gamma-\gamma_1}{2} \le \frac{\pi\xi_3}{2n} < \frac{\pi}{2}.$$

Hence for $\gamma_1 < \theta < \gamma$,

$$|g(re^{i\theta})| \le \left|\frac{U_1(re^{i\theta})}{\exp\{Kr^{\xi_3}\}}\right| \le O(\exp\{r^{n+\tau_2}\}) \quad (r \to \infty).$$

It follows from (10) and (18) that for some $M > 0$, as $r \to \infty$,

$$|g(re^{i\gamma_1})| \le \frac{O(e^{r^{\xi_2}})}{\exp\{Kr^{\xi_3}\}} \le M$$

and

$$|g(re^{i\gamma})| \le \frac{O(e^{r^{\xi_2}})}{\exp\{Kr^{\xi_3}\}} \le M.$$

By the Phragmén–Lindelöf theorem, we obtain (19). Similarly we see that (19) holds for $\gamma < \theta < \gamma_2$. Hence we conclude that (19) holds for any $\theta \in [0, 2\pi)$.

By a similar proof as before we can prove that for any $\theta \in [0, 2\pi)$,

$$|U_2(re^{i\theta})| \le O(\exp\{r^{\xi_3}\}) \quad (r \to \infty). \tag{20}$$

By (7) and (9), we have

$$Q_2U_1 - Q_1U_2 = \frac{1}{k}\,h'[Q_1(Q_2' + Q_2P_2') - Q_2(Q_1' + Q_1P_1')]. \tag{21}$$

Since $\sigma(Q_j) < \xi_2 < \xi_3$ $(j = 1, 2, 3)$, by (5), (10), (20) and Lemma 2.9,

$$\begin{aligned} m(r, Q_1e^{P_1(z)} &+ Q_2e^{P_2(z)} + Q_3e^{P_3(z)}) \\ &\le km(r, h') + O(\log r) \\ &\le km(r, U_1 - U_2) + O(\log r) \le O(r^{\xi_3}) \quad (r \to \infty). \end{aligned} \tag{22}$$

Since ζ_1/ζ_2 is non-real, $S_1^+ \cap S_2^-$ contains an interval $I = [\varphi_1, \varphi_2]$ satisfying $\min_{\theta\in I}\delta_1(\theta) = s > 0$. By Lemma 2.2, there exists an $R(I)$ (> 0) such that for any $\theta \in I$ and $r \geq R(I)$,

$$|Q_1 e^{P_1(re^{i\theta})}| \geq \exp((1-\varepsilon)\delta_1 r^n),$$
$$|Q_2 e^{P_2(re^{i\theta})}| \leq \exp((1-\varepsilon)\delta_2 r^n),$$
$$|Q_3 e^{P_3(re^{i\theta})}| \leq \exp((1-\varepsilon)\lambda\delta_2 r^n).$$

Hence, we have

$$\begin{aligned}(23)\qquad & m(r, Q_1 e^{P_1(z)} + Q_2 e^{P_2(z)} + Q_3 e^{P_3(z)}) \\ &\geq \int_{\varphi_1}^{\varphi_2} \log^+ |Q_1 e^{P_1(z)} + Q_2 e^{P_2(z)} + Q_3 e^{P_3(z)}|\, d\theta \\ &\geq \int_{\varphi_1}^{\varphi_2} (1-o(1)) \log^+ |Q_1 e^{P_1(z)}|\, d\theta \\ &\geq \int_{\varphi_1}^{\varphi_2} (1-o(1))(1-\varepsilon) s r^n\, d\theta \\ &\geq (1-o(1))(1-\varepsilon) s r^n (\varphi_2 - \varphi_1) \qquad (r \to \infty).\end{aligned}$$

Combining (22) and (23) and recalling that $\xi_3 < n$, we get a contradiction. Hence, $\lambda(f) = \infty$.

(ii) Let $f \not\equiv 0$ be a solution of (1). Write $f = \pi e^h$, suppose that $\lambda(f) < n$. From our hypothesis, we have $\sigma(\pi) = \lambda(\pi) < n$. Eliminating e^{P_1} from (5), we have

$$\begin{aligned}(24)\qquad kU(h')^{k-1} = F_{k-1}(h') &+ e^{P_2}[Q_2(Q_1' + Q_1 P_1') - Q_1(Q_2' + Q_2 P_2')] \\ &+ e^{P_3}[Q_3(Q_1' + Q_1 P_1') - Q_1(Q_3' + Q_3 P_3')],\end{aligned}$$

where

$$(25)\qquad U = Q_1 h'' - \frac{1}{k}(Q_1' + Q_1 P_1')h',$$

From (24), (25) and Lemma 2.7, we have

$$\begin{aligned}(26)\qquad & c_{k-1}(z)(h')^{k-1} + c_{k-2}(h')^{k-2} + \cdots + c_1(z)h' \\ &= c_0(z) + e^{P_2}[Q_2(Q_1' + Q_1 P_1') - Q_1(Q_2' + Q_2 P_2')] \\ &\quad + e^{P_3}[Q_3(Q_1' + Q_1 P_1') - Q_1(Q_3' + Q_3 P_3')],\end{aligned}$$

where $c_j(z)$ $(j = 0, 1, \ldots, k-1)$ is an algebraic expression in $U^{(l)}$ $(l = 0, 1, \ldots, k-2)$, $Q_1^{(i)} (i = 0, 1 \ldots, k-1)$, $P_1^{(s)} (s = 0, 1, \ldots, l-1)$, $1/k$, $1/Q_1$ and a_j, a_j' $(j = 1, \ldots, k-1)$, involving addition, subtraction and multiplication.

Now we suppose that at least one of $c_j(z)$ $(j = 1, \ldots, k-1)$ is not identically vanishing and the right hand side of (26) does not vanish identically. Without loss of generality, suppose $c_{k-1}(z) \not\equiv 0$. Then from (26) and

Lemma 2.8, we have

$$
\begin{aligned}
(27)\qquad T(r,h') &= m(r,h')\\
&\le \sum_{i=0}^{k-1} T(r,c_i(z)) + m(r, e^{P_2}[Q_2(Q_1'+Q_1P_1')-Q_1(Q_2'+Q_2P_2')]\\
&\quad + e^{P_3}[Q_3(Q_1'+Q_1P_1')-Q_1(Q_3'+Q_3P_3')]) + O(1).
\end{aligned}
$$

Take $\max\{\sigma(Q_1),\sigma(Q_2),\sigma(Q_3),\lambda(f)\} < \xi_2 < \xi_3 < n$. From (5), we obtain

$$(28)\qquad T(r, Q_1e^{P_1(z)}+Q_2e^{P_2(z)}+Q_3e^{P_3(z)}) \le kT(r,h') + O(\log r).$$

By Lemma 2.5, we have

$$
\begin{aligned}
(29)\qquad & m(r, Q_1e^{P_1(z)}+Q_2e^{P_2(z)}+Q_3e^{P_3(z)})\\
&\qquad \ge (1-\varepsilon)m(r,e^{P_1}) + O(r^{\xi_3}) \quad (r\to\infty,\ r\notin E).
\end{aligned}
$$

where E has finite linear measure. From (28) and (29), we obtain

$$(30)\qquad T(r,h') \ge \frac{1-\varepsilon}{k} T(r,e^{P_1}) + O(r^{\xi_3}) \quad (r\to\infty,\ r\notin E).$$

Since $0<\rho=\zeta_2/\zeta_1<1/2k$, $\zeta_3=\lambda\zeta_2$, $0<\lambda<1$, we get

$$\delta(P_2,\theta)=\rho\delta(P_1,\theta),\quad S_{1k}^+=S_{2k}^+=S_{3k}^+,\quad S_{1k}^-=S_{2k}^-=S_{3k}^- \quad (k=1,\dots,n).$$

By the same reasoning as in (11) and (14), we have

$$(31)\qquad |U(re^{i\theta})| \le O(\exp\{r^{\xi_2}\}) \quad (r\to\infty)$$

for any $\theta\in S_1^-\setminus E_0, m(E_0)=0$. Also by the same reasoning as in (15)–(18), we have

$$(32)\qquad |U(re^{i\theta})| \le O(\exp\{r^{\xi_2}\}) \quad (r\to\infty)$$

for any $\theta\in S_1^+\setminus E_0, m(E_0)=0$. Since $\sigma(U)\le n$, by the Phragmén–Lindelöf theorem, we have

$$(33)\qquad |U(re^{i\theta})| \le O(\exp\{r^{\xi_3}\}) \quad (r\to\infty)$$

for any $\theta\in[0,2\pi)$.

In the following, we estimate $T(r,c_j)$.

From (33), Lemma 2.3 and the theorem on logarithmic derivatives, we have

$$(34)\qquad T(r,c_j) \le O\Big(\sum_{i=0}^{k-1} T(r,Q_1^{(i)}) + \sum_{j=0}^{k-1} m(r,a_j) + \sum_{j=0}^{k-1} m(r,a_j') + \sum_{s=0}^{k-1} m(r,P_1^{(s)}) + \sum_{t=1}^{k-2} m(r,U^{(t)}/U) + m(r,U) + \overline{N}(r,1/\pi) + O(\log r)\Big) \le O(r^{\xi_3}), \quad r\to\infty,\ j=0,1,\dots,k-1,$$

and

$$(35)\qquad T(r, e^{P_2}[Q_2(Q_1' + Q_1P_1') - Q_1(Q_2' + Q_2P_2')] + e^{P_3}[Q_3(Q_1' + Q_1P_1') - Q_1(Q_3' + Q_3P_3')]) \le O(r^{\xi_3}) + T(r,e^{P_2}) + T(r,e^{P_3}) = (1+\lambda)T(r,e^{P_2}) + O(r^{\xi_3}) \le (1+\lambda)\rho T(r,e^{P_1}) + O(r^{\xi_3}), \quad r\to\infty.$$

From (27), (30), (34) and (35), we get

$$(36)\qquad \frac{1-\varepsilon}{k} T(r,e^{P_1}) + O(r^{\xi_3}) \le T(r,h') \le (1+\lambda)\rho T(r,e^{P_1}) + O(r^{\xi_3}), \quad r\to\infty,\ r\notin E.$$

Thus (36) implies

$$(37)\qquad \Big(\frac{1-\varepsilon}{k} - (1+\lambda)\rho - o(1)\Big) T(e,e^{P_1}) \le 0, \quad r\to\infty,\ r\notin E.$$

Since $0 < \rho = \zeta_2/\zeta_1 < 1/2k$, $0 < \lambda < 1$, we get a contradiction. Hence $c_{k-1} = \dots = c_1 = c_0 + e^{P_2}[Q_2(Q_1' + Q_1P_1') - Q_1(Q_2' + Q_2P_2')] + e^{P_3}[Q_3(Q_1' + Q_1P_1') - Q_1(Q_3' + Q_3P_3')] \equiv 0$. From (26), we have

$$(38)\qquad -c_0(z) = e^{P_2}[Q_2(Q_1' + Q_1P_1') - Q_1(Q_2' + Q_2P_2')] + e^{P_3}[Q_3(Q_1' + Q_1P_1') - Q_1(Q_3' + Q_3P_3')].$$

We assume that the right hand side above is not identically zero; otherwise, we have

$$e^{P_2 - P_3} = -\frac{Q_3(Q_1' + Q_1P_1') - Q_1(Q_3' + Q_3P_3')}{Q_2(Q_1' + Q_1P_1') - Q_1(Q_2' + Q_2P_2')},$$

and since $\zeta_3 = \lambda\zeta_2$, $0 < \lambda < 1$, a simple order consideration leads to a contradiction. From (38), by (34) and Lemma 2.5, we obtain

$$(39)\qquad (1-\varepsilon)T(r,e^{P_2}) + O(r^{\xi_3}) \le O(r^{\xi_3}), \quad r\to\infty.$$

From (39), we have $\sigma(e^{P_2}) < \xi_3 < n$, a contradiction. Hence $\lambda(f) \ge n$.

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