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Minimal submanifolds in general (α, β)-spaces

by SONGTING YIN (Shanghai and Tongling), QUN HE (Shanghai) and DINGHE XIE (Shanghai)

Abstract. The volume forms of general (α, β)-metrics are studied. Some equations for minimal submanifolds in general (α, β)-spaces are established by using the normal frame field, and some minimal surfaces in general (α, β)-spaces with special curvature properties are constructed.

1. Introduction. In recent decades, Finsler geometry has been rapidly developed. The study of the geometry of submanifolds has also made some progress. As is well known, there are two commonly used volume forms, Buseman–Hausdorff volume form and Holmes–Thompson volume form, in Finsler geometry. By using the given volume form, some differential equations on minimal submanifolds were established and some examples of minimal (hyper)surfaces were also obtained ([CS2], [HY], [ST]). However, these results only focused on the minimal submanifolds in (α, β)-spaces and all previous examples of minimal surfaces were constructed in the Minkowski spaces. So it is meaningful to study minimal submanifolds in a more general context.

In the present paper, we will study the minimal submanifolds in general (α, β)-spaces. As defined in [YZ], *general* (α, β)*-metrics* can be expressed in the form $F = \alpha\phi(x, \beta/\alpha)$ for some C^∞ function $\phi(x, s)$, some Riemannian metric α and some 1-form β. Recall that the *navigation expression* of a Randers metric is

$$F = \frac{\sqrt{\lambda\alpha^2 + \beta^2}}{\lambda} - \frac{\beta}{\lambda},$$

where α is a Riemannian metric and β is a 1-form, $\lambda = 1 - \|\beta\|_\alpha^2$. Obviously it can be viewed as a general (α, β)-metric. In addition, the classification on this type of Finsler metrics with constant flag curvature has been solved completely. Apart from Finsler metrics with constant flag curvature, projec-

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tively flat Finsler metrics are also important in Finsler geometry. Therefore, it is significant to study the minimal submanifolds in general (α,β)-spaces with constant flag curvature or locally projectively flat general (α,β)-spaces. In this paper, we not only give some differential equations for minimal submanifolds but also construct some minimal surfaces in a 3-dimensional non-Minkowski space which is flat or locally projectively flat.

The content of the present paper is organized as follows. After introducing some definitions and basic concepts in Section 2, we give the relationship between the volume forms of general (α,β)-metrics and those of Riemannian metrics in Section 3. Based on this, we conclude that under certain conditions the minimal submanifolds in a general (α,β)-space (M,F) are just minimal submanifolds in the Riemannian manifold (M,α) (Theorem 3.2). In Section 4, with the help of the normal frame field with respect to F and α, we obtain a necessary and sufficient condition characterizing the minimal submanifolds in general (α,β)-spaces (Theorem 4.2). Finally, we give some differential equations and corresponding examples of minimal surfaces (Theorems 4.3–4.6, 4.8–4.9).

2. Preliminaries. Let M be an n-dimensional smooth manifold. A *Finsler metric* on M is a function $F : TM \to [0,\infty)$ with the following properties:

(i) F is smooth on $TM\backslash 0$;
(ii) $F(x,\lambda y) = \lambda F(x,y)$ for all $\lambda > 0$;
(iii) the induced quadratic form g is positive-definite, where

$$g := g_{ij}dx^i \otimes dx^j, \qquad g_{ij} = \frac{1}{2}[F^2]_{y^iy^j}. \tag{2.1}$$

Here and from now on, $[F]_{y^i}$, $[F^2]_{y^iy^j}$ mean $\frac{\partial F}{\partial y^i}$, $\frac{\partial^2 F}{\partial y^i \partial y^j}$, etc., and we will use the following convention for index ranges unless otherwise stated:

$$1 \le i,j \le \cdots \le n, \qquad 1 \le \alpha,\beta \le \cdots \le n+p.$$

The projection $\pi : TM \to M$ gives rise to the pull-back bundle π^*TM and its dual π^*T^*M over $TM\backslash 0$. In π^*T^*M there is a global section $\omega = [F]_{y^i}dx^i$, called the *Hilbert form*, whose dual is $l = l^i\frac{\partial}{\partial x^i}$, $l^i = \frac{y^i}{F}$, called the *distinguished field*.

The volume element dV_{SM} of the projective sphere bundle SM with respect to the Riemannian metric $\hat{g}$, the pull-back of the Sasaki metric from $TM\backslash 0$, can be expressed as

$$dV_{SM} = \varOmega d\tau \wedge dx, \tag{2.2}$$

where

$$\Omega := \det\left(\frac{g_{ij}}{F}\right), \qquad dx = dx^1 \wedge \cdots \wedge dx^n, \tag{2.3}$$

$$d\tau := \sum_{i=1}^{n} (-1)^{i-1} y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n. \tag{2.4}$$

The *volume form* of a Finsler n-manifold (M, F) is defined by

$$dV_M := \sigma(x) dx, \qquad \sigma(x) := \frac{1}{c_{n-1}} \int_{S_x M} \Omega \, d\tau, \tag{2.5}$$

where c_{n-1} denotes the volume of the unit Euclidean $(n-1)$-sphere S^{n-1}, and $S_x M = \{[y] \mid y \in T_x M\}$.

Let (M, F) and $(\widetilde{M}, \widetilde{F})$ be Finsler manifolds and $f : (M, F) \to (\widetilde{M}, \widetilde{F})$ be an immersion. Then f is called *isometric* if $F(x, y) = \widetilde{F}(f(x), df(y))$ for any $(x, y) \in TM \setminus 0$. It is clear that

$$g_{ij}(x, y) = \tilde{g}_{\alpha\beta}(\tilde{x}, \tilde{y}) f_i^\alpha f_j^\beta \tag{2.6}$$

for an isometric immersion f, where

$$\tilde{x}^\alpha = f^\alpha, \qquad \tilde{y}^\alpha = f_i^\alpha y^i, \qquad f_i^\alpha = \frac{\partial f^\alpha}{\partial x^i} \tag{2.7}$$

Let $(\pi^* TM)^\perp$ be the orthogonal complement of $\pi^* TM$ in $\pi^*(f^{-1} T\widetilde{M})$ with respect to $\tilde{g}$ and denote

$$\nu^* = \{\xi \in \Gamma(f^{-1} T^* \widetilde{M}) \mid \xi(df(X)) = 0, \forall X \in \Gamma(TM)\},$$

which is called the *normal bundle* of f ([S1]). Set

$$h^\alpha = f_{ij}^\alpha y^i y^j - f_k^\alpha G^k + \widetilde{G}^\alpha, \qquad h_\alpha = \tilde{g}_{\alpha\beta} h^\beta, \qquad h = \frac{h^\alpha}{F^2} \frac{\partial}{\partial \tilde{x}^\alpha}, \tag{2.8}$$

where $f_{ij}^\alpha = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}$, and G^k and $\widetilde{G}^\alpha$ are the geodesic coefficients of F and $\widetilde{F}$ respectively. We know from [HS1], [S1] that $h \in (\pi^* TM)^\perp$, which is called the *normal curvature*. The *mean curvature form* of f is defined by

$$\mu = \frac{1}{c_{n-1} \sigma} \left(\int_{S_x M} \frac{h_\alpha}{F^2} \Omega \, d\tau \right) d\tilde{x}^\alpha, \tag{2.9}$$

and $\mu \in \nu^*$. An isometric immersion $f : (M, F) \to (\widetilde{M}, \widetilde{F})$ is called *minimal* if any compact domain of M is the critical point of its volume functional with respect to any variation.

Lemma 2.1 ([HS1]). *Let $f : (M, F) \to (\widetilde{M}, \widetilde{F})$ be an isometric immersion. Then f is minimal if and only if $\mu = 0$.*

3. Volume element of a general (α,β)-metric. A general (α,β)-metric is defined by a Riemannian metric $\alpha=\sqrt{a_{ij}y^iy^j}$ and a 1-form $\beta=b_iy^i$. It can be expressed as

$$F=\alpha\phi(x,s),\qquad s=\beta/\alpha,$$

where $\phi(x,s)$ is a positive C^∞ function, $x\in M$, and $|s|\le b<b_0$ for some $0<b_0<\infty$. It is shown that F is positive-definite for any α and β with $b:=\|\beta\|_\alpha<b_0$ if and only if ϕ satisfies ([YZ])

$$\phi-s\phi_2+(b^2-s^2)\phi_{22}>0.$$

Furthermore,

$$g_{ij}=\rho a_{ij}+\rho_0b_ib_j+\rho_1(b_i\alpha_{y^j}+b_j\alpha_{y^i})-s\rho_1\alpha_{y^i}\alpha_{y^j},\tag{3.1}$$

$$\det(g_{ij})=\phi^nH(x,s)\det(a_{ij}),\tag{3.2}$$

$$g^{ij}=\rho^{-1}\{a^{ij}+\eta b^ib^j+\eta_0\alpha^{-1}(b^iy^j+b^jy^i)+\eta_1\alpha^{-2}y^iy^j\},\tag{3.3}$$

where

$$\begin{aligned}H(x,s)&=\phi(\phi-s\phi_2)^{n-2}(\phi-s\phi_2+(b^2-s^2)\phi_{22}),\\ \rho&=\phi(\phi-s\phi_2),\qquad \rho_0=\phi\phi_{22}+\phi_2\phi_2,\\ \rho_1&=(\phi-s\phi_2)\phi_2-s\phi\phi_{22},\\ (g^{ij})&=(g_{ij})^{-1},\qquad (a^{ij})=(a_{ij})^{-1},\qquad b^i=a^{ij}b_j,\\ \eta&=-\frac{\phi_{22}}{\phi-s\phi_2+(b^2-s^2)\phi_{22}},\\ \eta_0&=-\frac{(\phi-s\phi_2)\phi_2-s\phi\phi_{22}}{\phi(\phi-s\phi_2+(b^2-s^2)\phi_{22})},\\ \eta_1&=-\frac{(s\phi+(b^2-s^2)\phi_2)((\phi-s\phi_2)\phi_2-s\phi\phi_{22})}{\phi^2(\phi-s\phi_2+(b^2-s^2)\phi_{22})}.\end{aligned}\tag{3.4}$$

Let $b_{i|j}$ denote the coefficients of the covariant derivative of β with respect to α and write

$$r_{ij}=\frac{1}{2}(b_{i|j}+b_{j|i}),\qquad s_{ij}=\frac{1}{2}(b_{i|j}-b_{j|i}),\qquad r_{00}=r_{ij}y^iy^j,\qquad s^i_0=a^{ij}s_{jk}y^k,$$
$$r_i=b^jr_{ji},\qquad s_i=b^js_{ji},\qquad r_0=r_iy^i,\qquad s_0=s_iy^i,$$
$$r^i=a^{ij}r_j,\qquad s^i=a^{ij}s_j,\qquad r=b^ir_i.$$

It is easy to see that $\beta^\sharp=b^i\frac{\partial}{\partial x^i}$ is a Killing vector if and only if $r_{ij}=0$.

For an (α,β)-metric $F=\alpha\phi(\beta/\alpha)$, the Holmes–Thompson volume form was calculated in [CS1] and [CS2] respectively. Since the whole calculation does not involve the first variable in $H(x,s)$ defined by $(3.4)_1$ (see [CS1], [CS2] for details), we can obtain the following result analogously.

PROPOSITION 3.1. *Let (M,F) be a general (α,β)-space, where $F=\alpha\phi(x,\beta/\alpha)$. Then the Holmes–Thompson volume form dV_F and the Rie-*

mannian volume form dV_α satisfy

$$dV_F = \frac{\Gamma(n/2)}{\sqrt{\pi}\Gamma((n-1)/2)}\Big\{\int_0^\pi H(x, b\cos t)\sin^{n-2}(t)\,dt\Big\}\,dV_\alpha,$$

where $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}\,dx$ is the Gamma function.

If $\varphi(x,s) = H(x,s) - 1$ is odd in s, then

$$\int_0^\pi \varphi(x, b\cos t)\sin^{n-2}(t)\,dt = 0. \tag{3.5}$$

From (3.5) and Proposition 3.1 one can deduce that $dV_F = dV_\alpha$.

THEOREM 3.2. *Let (M,F) be a general (α,β)-space, where $F = \alpha\phi(x,\beta/\alpha)$. If $H(x,s)-1$ is odd in s, then the minimal submanifolds in (M,F) are just the minimal submanifolds in (M,α), and vice versa.*

REMARK. Noting that $H(x;s)-1 = s$ for a Randers metric $F = \alpha+\beta$, we reobtain the corresponding result in [HS2] from Theorem 3.2.

When $F = \sqrt{\lambda\alpha^2+\beta^2}/\lambda - \beta/\lambda$, we have $\phi(x,s) = \sqrt{\lambda+s^2}/\lambda - s/\lambda$, where $\lambda = 1-b^2$. By direct computation, we get

$$H(x,s) = \frac{\sqrt{\lambda+s^2}-s}{\lambda(\lambda+s^2)^{(n+1)/2}}.$$

Therefore,

$$\begin{aligned}\int_0^\pi H(x,b\cos t)\sin^{n-2}(t)\,dt &= \int_0^\pi \frac{\sqrt{\lambda+b^2\cos^2 t}-b\cos t}{\lambda(\lambda+b^2\cos^2 t)^{(n+1)/2}}\sin^{n-2}(t)\,dt\\ &= \frac{2}{\lambda}\int_0^{\pi/2}\frac{\sin^{n-2}(t)}{(1-b^2\sin^2 t)^{n/2}}\,dt.\end{aligned}$$

So we obtain the following

COROLLARY 3.3. *Let (M,F) be a general (α,β)-space, where $F = \sqrt{\lambda\alpha^2+\beta^2}/\lambda - \beta/\lambda$. Then the Holmes–Thompson volume form dV_F and the Riemannian volume form dV_α satisfy*

$$dV_F = \frac{2\Gamma(n/2)}{\lambda\sqrt{\pi}\,\Gamma((n-1)/2)}\Big\{\int_0^{\pi/2}\frac{\sin^{n-2}(t)}{(1-b^2\sin^2 t)^{n/2}}\,dt\Big\}\,dV_\alpha,$$

where $\lambda = 1-b^2$. In particular, when $n=2$,

$$dV_F = \frac{\sqrt{\pi}\,\Gamma(1)}{\lambda\sqrt{\lambda}\,\Gamma(1/2)}dV_\alpha.$$

REMARK. When $b \neq$ const, the minimal submanifolds of (M, F) are not necessarily the minimal submanifolds of (M, α). So it is reasonable to look for examples of minimal submanifolds in (M, F).

4. Minimal submanifolds of general (α, β)-spaces. Let $f : (M, F) \to (\widetilde{M}, \widetilde{F})$ be an isometric immersion. $\widetilde{F} = \tilde{\alpha}\phi(\tilde{x}, \tilde{\beta}/\tilde{\alpha})$, where

$$\tilde{\alpha} = \sqrt{\tilde{a}_{\alpha\beta}\tilde{y}^{\alpha}\tilde{y}^{\beta}}, \qquad \tilde{\beta} = \tilde{b}_{\alpha}\tilde{y}^{\alpha}.$$

Since f is isometric, we get

$$F = f^*\widetilde{F} = \alpha\phi(f(x), \beta/\alpha),$$

where

$$\alpha = \sqrt{a_{ij}y^i y^j}, \qquad a_{ij} = \tilde{a}_{\alpha\beta}f_i^{\alpha}f_j^{\beta}, \qquad \beta = b_i y^i, \qquad b_i = \tilde{b}_{\alpha}f_i^{\alpha}.$$

PROPOSITION 4.1. *Let $f : (M^n, F) \to (\widetilde{M}^{n+p}, \widetilde{F})$ be an isometric immersion where $\widetilde{F} = \tilde{\alpha}\phi(\tilde{x}, \tilde{\beta}/\tilde{\alpha})$. Denote by $\{\mathbf{n}_a\}_{a=n+1}^{n+p}$ a local orthonormal frame of the normal bundle $TM^{\perp}$ with respect to the Riemannian metric $\tilde{\alpha}$ such that $\mathbf{n}_{n+p}$ is parallel to $\tilde{\beta}^{\perp}$, and set*

$$\tilde{\mathbf{n}}_a = \sqrt{\frac{1}{\tilde{\rho}(1 + \tilde{\eta}\tilde{\beta}(\mathbf{n}_a)^2)}}\,[\mathbf{n}_a + \tilde{\eta}\tilde{\beta}(\mathbf{n}_a)\tilde{\beta}^{\sharp} + F\tilde{\eta}_0\tilde{\alpha}^{-1}\tilde{\beta}(\mathbf{n}_a)\tilde{l}]. \tag{4.1}$$

Then $\{\tilde{\mathbf{n}}_a\}_{a=n+1}^{n+p}$ is a local orthonormal frame of the normal bundle $(\pi^*TM)^{\perp}$ with respect to $\widetilde{F}$. Here $\tilde{\beta}^{\perp}$ is the projection of $\tilde{\beta}^{\sharp}$ into the normal bundle $TM^{\perp}$, $\tilde{\beta}^{\sharp} = \tilde{b}^{\alpha}\frac{\partial}{\partial \tilde{x}^{\alpha}}$, and $\tilde{\rho}, \tilde{\eta}, \tilde{\eta}_0$ are defined as in (3.4).

Proof. Let $\mathbf{n}_a = n_a^{\alpha}\frac{\partial}{\partial \tilde{x}^{\alpha}}$. Then

$$\tilde{a}(\mathbf{n}_a, \mathbf{n}_b) = \tilde{a}_{\alpha\beta}n_a^{\alpha}n_b^{\beta} = \delta_{ab}, \qquad \tilde{a}\left(\mathbf{n}_a, \frac{\partial}{\partial x^i}\right) = \tilde{a}_{\alpha\beta}n_a^{\alpha}f_i^{\beta} = 0. \tag{4.2}$$

Take $\tilde{\mathbf{n}}_a = \tilde{n}_a^{\alpha}\frac{\partial}{\partial \tilde{x}^{\alpha}}$ $(a = n+1, \ldots, n+p)$ satisfying

$$\tilde{g}_{\alpha\beta}\tilde{n}_a^{\alpha} = \xi_a\tilde{a}_{\alpha\beta}n_a^{\alpha}, \tag{4.3}$$

where

$$\xi_a = \sqrt{\frac{\tilde{\rho}}{1 + \tilde{\eta}\tilde{\beta}(\mathbf{n}_a)^2}}. \tag{4.4}$$

Then by (4.2), (4.3), (3.3), we have

$$\tilde{g}\left(\tilde{\mathbf{n}}_a, \frac{\partial}{\partial x^i}\right) = \tilde{g}_{\alpha\beta}\tilde{n}_a^{\alpha}f_i^{\beta} = \xi_a\tilde{a}_{\alpha\beta}n_a^{\alpha}f_i^{\beta} = 0,$$

and

$$\begin{aligned}\tilde g(\tilde{\mathbf{n}}_a,\tilde{\mathbf{n}}_b) &= \tilde g_{\alpha\beta}\tilde n_a^\alpha \tilde n_b^\beta = \xi_a \tilde a_{\alpha\beta} n_a^\alpha \xi_b \tilde g^{\beta\gamma}\tilde a_{\gamma\delta} n_a^\delta \\ &= \xi_a \tilde a_{\alpha\beta} n_a^\alpha \frac{\xi_b}{\tilde\rho}\{n_b^\beta + \tilde\eta\tilde\beta(\mathbf{n}_b)\tilde b^\beta + \tilde\eta_0\tilde\alpha^{-1}\tilde\beta(\mathbf{n}_b)\tilde y^\beta\} \\ &= \sqrt{\frac{1}{1+\tilde\eta\tilde\beta(\mathbf{n}_a)^2}}\sqrt{\frac{1}{1+\tilde\eta\tilde\beta(\mathbf{n}_b)^2}}\,\{\delta_{ab}+\tilde\eta\tilde\beta(\mathbf{n}_a)\tilde\beta(\mathbf{n}_b)\} = \delta_{ab}.\end{aligned}$$

The last equality holds since $\tilde\beta^\perp$ is parallel to $\mathbf{n}_{n+p}$. Therefore $\{\tilde{\mathbf{n}}_a\}_{a=n+1}^{n+p}$ is a local orthonormal frame of the normal bundle $(\pi^*TM)^\perp$ with respect to $\widetilde F$. Using (4.2), (4.3), (3.3) again, we obtain

$$\tilde{\mathbf{n}}_a = \sqrt{\frac{1}{\tilde\rho(1+\tilde\eta\tilde\beta(\mathbf{n}_a)^2)}}\,[\mathbf{n}_a + \tilde\eta\tilde\beta(\mathbf{n}_a)\tilde\beta^\sharp + F\tilde\eta_0\tilde\alpha^{-1}\tilde\beta(\mathbf{n}_a)\tilde l]. \ \blacksquare$$

REMARK. When $\widetilde F=\sqrt{\tilde\lambda\tilde\alpha^2+\tilde\beta^2}/\tilde\lambda-\tilde\beta/\tilde\lambda$, the relation between $\{\mathbf{n}_a\}_{a=n+1}^{n+p}$ and $\{\tilde{\mathbf{n}}_a\}_{a=n+1}^{n+p}$ can be expressed as

$$\tilde{\mathbf{n}}_a = \sqrt{\frac{\tilde\gamma}{\widetilde F(1-\tilde\beta(\mathbf{n}_a)^2)}}\,[\mathbf{n}_a + \tilde\beta(\mathbf{n}_a)(\tilde l - \tilde\beta^\sharp)],$$

where $\tilde\lambda = 1-\tilde b^2$, $\tilde\gamma = \sqrt{\tilde\lambda\tilde\alpha^2+\tilde\beta^2}$.

From Lemma 2.1, we know that $f:(M^n,F)\to(\widetilde M^{n+p},\widetilde F)$ is minimal if and only if

$$n_b^\alpha \int\limits_{S_xM} \frac{h_\alpha}{F^2}\varOmega\, d\tau = 0, \qquad \forall b. \tag{4.6}$$

Using (2.8), (4.2), (4.3) and (4.5), we get

$$\begin{aligned} h_\alpha &= \tilde g_{\alpha\gamma}h^\gamma = \sum_a \tilde g_{\alpha\gamma}\tilde g\Big(h^\beta\frac{\partial}{\partial\tilde x^\beta},\tilde{\mathbf{n}}_a\Big)\tilde n_a^\gamma \\ &= \sum_a \tilde g_{\alpha\gamma}[(f_{ij}^\beta y^i y^j - f_k^\beta G^k + \widetilde G^\beta)\tilde g_{\beta\delta}\tilde n_a^\delta]\tilde n_a^\gamma \\ &= \sum_a \xi_a^2[(f_{ij}^\beta y^i y^j + \widetilde G^\beta)\tilde a_{\beta\delta}n_a^\delta]\tilde a_{\alpha\gamma}n_a^\gamma \\ &= \sum_a \frac{\tilde\rho[(f_{ij}^\beta y^i y^j + \widetilde G^\beta)\tilde a_{\beta\delta}n_a^\delta]\tilde a_{\alpha\gamma}n_a^\gamma}{1+\tilde\eta\tilde\beta(\mathbf{n}_a)^2}. \end{aligned}\tag{4.7}$$

Plugging (2.3), (3.2) and (4.7) into (4.6) implies

$$n_b^\alpha \int\limits_{S_xM} \frac{h_\alpha}{F^2}\Omega\, d\tau = \int\limits_{S_xM} \frac{\tilde\rho[(f_{ij}^\beta y^i y^j + \widetilde{G}^\beta)\tilde a_{\beta\delta} n_b^\delta]\det(g_{ij})}{(1+\tilde\eta\tilde\beta(\mathbf{n}_b)^2)F^{n+2}}\, d\tau$$
$$= \det(a_{ij})\tilde a_{\beta\delta} n_b^\delta \int\limits_{S_xM} \frac{(f_{ij}^\beta y^i y^j + \widetilde{G}^\beta)(\phi - s\phi_2)H(x,s)}{\alpha^{n+2}(1+\tilde\eta\tilde\beta(\mathbf{n}_b)^2)\phi}\, d\tau,$$

where

$$H(x,s) = \phi(\phi - s\phi_2)^{n-2}[(\phi - s\phi_2) + (b^2 - s^2)\phi_{22}],$$
$$\tilde\eta = -\frac{\phi_{22}}{\phi - \tilde s\phi_2 + (\tilde b^2 - \tilde s^2)\phi_{22}}.$$

Theorem 4.2. *Let (M^n, F) be a submanifold in $(\widetilde{M}^{n+p}, \widetilde{F})$ where $\widetilde{F} = \tilde\alpha\phi(\tilde x, \tilde\beta/\tilde\alpha)$. Then $f : (M^n, F) \to (\widetilde{M}^{n+p}, \widetilde{F})$ is minimal if and only if*

$$\tilde a_{\beta\delta} n_a^\delta \int\limits_{\alpha=1} \frac{(f_{ij}^\beta y^i y^j + \widetilde{G}^\beta)(\phi - s\phi_2)H}{(1+\tilde\eta\tilde\beta(\mathbf{n}_a)^2)\phi}\, d\tau = 0, \qquad \forall a. \tag{4.8}$$

In what follows, we consider hypersurfaces in a general (α,β)-space $(\widetilde{M}^{n+1}, \widetilde{F})$ with $\widetilde{F} = \sqrt{\tilde\lambda\tilde\alpha^2 + \tilde\beta^2}/\tilde\lambda - \tilde\beta/\tilde\lambda$. By direct computation, one gets

$$H(x,s) = (\tilde\lambda + b^2)\frac{F}{\alpha}\left(\frac{\alpha}{\gamma}\right)^{n+1}, \qquad \tilde\eta = -1, \qquad \frac{\phi - s\phi_2}{\phi} = \frac{\alpha^2}{F\gamma}, \tag{4.9}$$

where $F = b\sqrt{\tilde\lambda\alpha^2 + \beta^2}/\tilde\lambda - \beta/\tilde\lambda$, $\gamma = \sqrt{\tilde\lambda\alpha^2 + \beta^2}$. Thus from (4.8), (4.9), we have

$$\tilde a_{\beta\delta} n^\delta \int\limits_{S_xM} \frac{(f_{ij}^\beta y^i y^j + \widetilde{G}^\beta)(\phi - s\phi_2)H}{\alpha^{n+2}(1+\tilde\eta\tilde\beta(\mathbf{n})^2)\phi}\, d\tau \tag{4.10}$$
$$= \frac{\tilde a_{\beta\delta} n^\delta(\tilde\lambda + b^2)}{1 - \tilde\beta(\mathbf{n})^2} \int\limits_{S_xM} \frac{f_{ij}^\beta y^i y^j + \widetilde{G}^\beta}{\left(\sqrt{\tilde\lambda\alpha^2 + \beta^2}\right)^{n+2}}\, d\tau = 0.$$

Note that the geodesic coefficient $\widetilde{G}^\beta$ is twice that in [R]. In particular, when $\tilde\alpha$ is a Riemannian metric with constant sectional curvature, i.e. $\tilde\alpha$ is projectively flat, and $\tilde\beta^\sharp = b^\alpha \frac{\partial}{\partial\tilde x^\alpha}$ is a Killing vector field, we know from [R] that

$$\widetilde{G}^\beta = \overline{\widetilde{G}^\beta} - \widetilde{F}^2\tilde s^\beta - 2\widetilde{F}\tilde s_0^\beta = P\tilde y^\beta - \widetilde{F}^2\tilde s^\beta - 2\widetilde{F}\tilde s_0^\beta,$$

where $\overline{\widetilde{G}^\beta}$ denote the geodesic coefficients of $\tilde\alpha$, and $P = \tilde\alpha_{\tilde x^\delta}\tilde y^\delta/\tilde\alpha$. Noting that $\tilde a_{\alpha\beta} n^\alpha \tilde y^\beta = 0$ and $F = (1-\beta)/\tilde\lambda$ when $\sqrt{\tilde\lambda\alpha^2 + \beta^2} = 1$, we obtain from (4.10) the following result.

THEOREM 4.3. *Let (M^n, F) be a hypersurface in $(\widetilde{M}^{n+1}, \widetilde{F})$ with $\widetilde{F} = \sqrt{\tilde{\lambda}\tilde{\alpha}^2+\tilde{\beta}^2}/\tilde{\lambda} - \tilde{\beta}/\tilde{\lambda}$. If $\tilde{\alpha}$ is a Riemannian metric with constant sectional curvature and $\tilde{\beta}^\sharp$ is a Killing vector, then $f : (M^n, F) \to (\widetilde{M}^{n+1}, \widetilde{F})$ is minimal if and only if*

$$\tilde{a}_{\beta\delta} n^\delta \int\limits_{\sqrt{\tilde{\lambda}\alpha^2+\beta^2}=1} \left[f^\beta_{ij} y^i y^j - \frac{\tilde{s}^\beta (1-\beta)^2}{\tilde{\lambda}^2} - \frac{2(1-\beta)}{\tilde{\lambda}} \tilde{s}^\beta_0 \right] d\tau = 0. \tag{4.11}$$

Now we will look for some minimal surfaces in a general (α,β)-space $(\widetilde{M}^3, \widetilde{F})$ with $\widetilde{F} = \sqrt{\tilde{\lambda}\tilde{\alpha}^2+\tilde{\beta}^2}/\tilde{\lambda} - \tilde{\beta}/\tilde{\lambda}$, where $\tilde{\alpha}$ is a Euclidean metric. If $\tilde{\beta}$ is parallel with respect to $\tilde{\alpha}$, then $\widetilde{F}$ is a Minkowski metric. Next, we will consider the case that $\tilde{\beta}$ is not parallel with respect to $\tilde{\alpha}$ any more. Let

$$\tilde{\alpha} = \sqrt{(\tilde{y}^1)^2 + (\tilde{y}^2)^2 + (\tilde{y}^3)^2}, \qquad \tilde{\beta} = k(\tilde{x}_2\tilde{y}^1 - \tilde{x}_1\tilde{y}^2), \qquad k = \text{const.} \tag{4.12}$$

Then $\widetilde{F}$ is a Finsler metric defined on $\widetilde{M}^3 := \{(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3) \in \mathbb{R}^3 \mid \tilde{x}_1^2 + \tilde{x}_2^2 < 1/k^2\}$ and $\tilde{\beta}^\sharp$ is a Killing vector. In this case, $\widetilde{F}$ is not Minkowskian, but its flag curvature still vanishes (see [BRS] for details). Let f be a rotation surface defined by $f(u,v) = (u\cos v, u\sin v, h(u))$, where $h(u)$ is a function to be determined. Then

$$\begin{aligned}
&(f^\alpha_i)_{2\times 3} = \begin{pmatrix} \cos v & \sin v & h' \\ -u\sin v & u\cos v & 0 \end{pmatrix}, \\
&(\tilde{y}^1 \quad \tilde{y}^2 \quad \tilde{y}^3) = (y^1 \quad y^2)(f^\alpha_i)_{2\times 3} \\
&\qquad\qquad = (y^1\cos v - uy^2\sin v \quad y^1\sin v + uy^2\cos v \quad y^1 h'), \\
&\tilde{\lambda}\circ f = 1 - \tilde{b}^2 = 1 - (k\tilde{x}_1)^2 - (k\tilde{x}_2)^2 = 1 - k^2u^2, \\
&\alpha = f^*\tilde{\alpha} = \sqrt{(1+h'^2)(y^1)^2 + u^2(y^2)^2}, \quad \beta = f^*\tilde{\beta} = -ku^2y^2.
\end{aligned} \tag{4.13}$$

Set

$$y^1 = \frac{\cos\theta}{\sqrt{(1-k^2u^2)(1+h'^2)}}, \quad y^2 = \frac{\sin\theta}{u}, \qquad \theta \in [0, 2\pi].$$

Then

$$\sqrt{\tilde{\lambda}\alpha^2+\beta^2} = \sqrt{(1-k^2u^2)(1+h'^2)(y^1)^2 + u^2(y^2)^2} = 1.$$

In this case, (4.11) is equivalent to

$$n^\beta \int\limits_{\sqrt{\tilde{\lambda}\alpha^2+\beta^2}=1} \left[f^\beta_{ii} (y^i)^2 - \frac{\tilde{s}^\beta (1+\beta^2)}{\tilde{\lambda}^2} + \frac{2\beta}{\tilde{\lambda}} \tilde{s}^\beta_0 \right] d\tau = 0. \tag{4.14}$$

Furthermore, a direct computation yields

$$(4.15)\qquad \begin{aligned} \tilde{s}^1 &= \tilde{a}^{1\alpha}\tilde{s}_{\beta\alpha}\tilde{b}^\beta = k^2u\cos v, \quad \tilde{s}^2 = k^2u\sin v, \quad \tilde{s}^3 = 0,\\ \tilde{s}_0^1 &= \tilde{a}^{1\alpha}\tilde{s}_{\alpha\beta}\tilde{y}^\beta = ky^1\sin v + kuy^2\cos v,\\ \tilde{s}_0^2 &= -ky^1\cos v + kuy^2\sin v, \qquad \tilde{s}_0^3 = 0, \end{aligned}$$

and

$$d\tau = y^1dy^2 - y^2dy^1 = \frac{d\theta}{u\sqrt{(1-k^2u^2)(1+h'^2)}},$$

$$(4.16)\qquad \begin{aligned} \int_{\sqrt{\tilde{\lambda}\alpha^2+\beta^2}=1} d\tau &= \frac{2\pi}{u\sqrt{(1-k^2u^2)(1+h'^2)}},\\ \int_{\sqrt{\tilde{\lambda}\alpha^2+\beta^2}=1} (y^1)^2\, d\tau &= \frac{\pi}{u[(1-k^2u^2)(1+h'^2]^{3/2}},\\ \int_{\sqrt{\tilde{\lambda}\alpha^2+\beta^2}=1} (y^2)^2\, d\tau &= \frac{\pi}{u^3\sqrt{(1-k^2u^2)(1+h'^2)}}. \end{aligned}$$

In (4.14), we set

$$W^\beta = \int_{\sqrt{\tilde{\lambda}\alpha^2+\beta^2}=1} \left[f_{ii}^\beta (y^i)^2 - \frac{\tilde{s}^\beta(1+\beta^2)}{\tilde{\lambda}^2} + \frac{2\beta}{\tilde{\lambda}}\tilde{s}_0^\beta\right] d\tau, \qquad \beta = 1,2,3.$$

Since

$$(4.17)\qquad (f_{ii}^\alpha)_{2\times 3} = \begin{pmatrix} 0 & 0 & h'' \\ -u\cos v & -u\sin v & 0 \end{pmatrix},$$

we deduce from (4.13)–(4.17) that

$$\begin{aligned} W^1 &= -\frac{\pi\cos v}{\sqrt{(1-k^2u^2)(1+h'^2)}}\left(\frac{1}{u^2} + \frac{2k^2+k^4u^2}{(1-k^2u^2)^2} + \frac{2k^2}{1-k^2u^2}\right),\\ W^2 &= -\frac{\pi\sin v}{\sqrt{(1-k^2u^2)(1+h'^2)}}\left(\frac{1}{u^2} + \frac{2k^2+k^4u^2}{(1-k^2u^2)^2} + \frac{2k^2}{1-k^2u^2}\right),\\ W^3 &= \frac{\pi h''}{u[(1-k^2u^2)(1+h'^2)]^{3/2}}. \end{aligned}$$

On the other hand, (4.14) is equivalent to

$$(4.18)\qquad \sum_{\beta=1}^{3} W^\beta n^\beta = 0.$$

The normal vector to the surface is

$$\mathbf{n} = \left(\frac{-h'\cos v}{\sqrt{1+h'^2}}\ \frac{-h'\sin v}{\sqrt{1+h'^2}}\ \frac{1}{\sqrt{1+h'^2}}\right).$$

Substituting the above formulas into (4.18), one gets

$$(2k^2u^2 + 1)h'(1 + h'^2) = u(k^2u^2 - 1)h''. \tag{4.19}$$

THEOREM 4.4. *Let $(\widetilde{M}^3, \widetilde{F})$ be a general (α, β)-space, where $\widetilde{F} = \sqrt{\tilde{\lambda}\tilde{\alpha}^2 + \tilde{\beta}^2}/\tilde{\lambda} - \tilde{\beta}/\tilde{\lambda}$, and $\tilde{\alpha}$ and $\tilde{\beta}$ are defined by* (4.12). *Then a rotation surface $f = (u\cos v, u\sin v, h(u))$ in $(\widetilde{M}^3, \widetilde{F})$ is minimal if and only if h satisfies* (4.19).

Let $w = h'^2$. Then (4.19) becomes

$$\frac{2k^2u^2 + 1}{u(k^2u^2 - 1)} = \frac{w'}{2w(1 + w)}.$$

By a direct computation, one obtains

$$w = \frac{C[1 - k^2u^2]^3}{u^2 - C[1 - k^2u^2]^3},$$

where C is a non-negative constant. Therefore,

$$h = \pm\int \sqrt{w}\, du = \pm\int \frac{\sqrt{C}\,[1 - k^2u^2]^{3/2}}{\sqrt{u^2 - C[1 - k^2u^2]^3}}\, du. \tag{4.20}$$

THEOREM 4.5. *Let $(\widetilde{M}^3, \widetilde{F})$ be a general (α, β)-space, where $\widetilde{F}$, $\tilde{\alpha}$ and $\tilde{\beta}$ are as in Theorem 4.4. Then there exists a minimal rotation surface in $(\widetilde{M}^3, \widetilde{F})$ which can be expressed as*

$$f = \left(u\cos v, u\sin v, \pm\int \frac{\sqrt{C}\,[1 - k^2u^2]^{3/2}}{\sqrt{u^2 - C[1 - k^2u^2]^3}}\, du\right).$$

REMARK. Noting that $\widetilde{F}$ is Euclidean when $k = 0$, one gets $h(u) = \cosh^{-1} u$ from (4.20), which is just the classical result in Euclidean space $\mathbb{R}^3$.

Now we study the second case, that is, $f = (u\cos v, u\sin v, h(v))$, with $\tilde{\alpha}$ and $\tilde{\beta}$ defined as in (4.12). We will show that although the minimal submanifolds of $(\widetilde{M}, \widetilde{F})$ are not necessarily minimal submanifolds of $(M, \tilde{\alpha})$ in the general case, there are still some exceptions. Analogously, we have

$$(f_i^\alpha)_{2\times 3} = \begin{pmatrix} \cos v & \sin v & 0 \\ -u\sin v & u\cos v & h' \end{pmatrix},$$

$$(\tilde{y}^1 \quad \tilde{y}^2 \quad \tilde{y}^3) = (y^1\cos v - uy^2\sin v \quad y^1\sin v + uy^2\cos v \quad y^2h'),$$

$$\alpha = \sqrt{(y^1)^2 + (u^2 + h'^2)(y^2)^2}, \qquad \beta = -ku^2y^2,$$

$$\tilde{\lambda}\circ f = 1 - k^2u^2, \qquad \tilde{s}^1 = k^2u\cos v, \qquad \tilde{s}^2 = k^2u\sin v,$$

$$\tilde{s}^3 = 0, \qquad \tilde{s}^1_0 = ky^1\sin v + kuy^2\cos v,$$

$$\tilde{s}^2_0 = -ky^1\cos v + kuy^2\sin v, \qquad \tilde{s}^3_0 = 0.$$

The normal vector to the surface is

$$\mathbf{n} = \left(\frac{h' \sin v}{\sqrt{u^2 + h'^2}} \quad \frac{-h' \cos v}{\sqrt{u^2 + h'^2}} \quad \frac{u}{\sqrt{u^2 + h'^2}} \right),$$

and

$$(f^\alpha_{ii})_{2\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ -u\cos v & -u \sin v & h'' \end{pmatrix}.$$

Set

$$y^1 = \frac{\cos\theta}{\sqrt{1-k^2u^2}}, \quad y^2 = \frac{\sin\theta}{\sqrt{u^2 + (1-k^2u^2)h'^2}}, \qquad \theta \in [0, 2\pi].$$

Then

$$\sqrt{\tilde{\lambda}\alpha^2 + \beta^2} = \sqrt{(1-k^2u^2)(y^1)^2 + (u^2 + (1-k^2u^2)h'^2)(y^2)^2} = 1,$$

$$d\tau = \frac{d\theta}{\sqrt{(1-k^2u^2)(u^2 + (1-k^2u^2)h'^2)}}.$$

Furthermore, we have

$$\int_{\sqrt{\tilde{\lambda}\alpha^2+\beta^2}=1} d\tau = \frac{2\pi}{\sqrt{(1-k^2u^2)(u^2 + (1-k^2u^2)h'^2)}},$$

$$\int_{\sqrt{\tilde{\lambda}\alpha^2+\beta^2}=1} (y^1)^2\, d\tau = \frac{\pi}{(1-k^2u^2)^{3/2}\sqrt{(u^2 + (1-k^2u^2)h'^2)}},$$

$$\int_{\sqrt{\tilde{\lambda}\alpha^2+\beta^2}=1} (y^2)^2\, d\tau = \frac{\pi}{\sqrt{(1-k^2u^2)}\,(u^2 + (1-k^2u^2)h'^2)^{3/2}}.$$

Plugging the formulas above into (4.14) yields

$$h'' = 0. \tag{4.21}$$

THEOREM 4.6. *Let $(\widetilde{M}^3, \widetilde{F})$ be a general (α,β)-space, where $\widetilde{F}, \tilde{\alpha}$ and $\tilde{\beta}$ are as in Theorem 4.4. Then the minimal conoid in $(\widetilde{M}^3, \widetilde{F})$ must be a helicoid or a plane.*

REMARK. In Theorem 4.6, we have obtained minimal surfaces for both $(\widetilde{M}^3, \widetilde{F})$ and $(\widetilde{M}^3, \tilde{\alpha})$. This shows that the minimal conoids in such a non-Minkowski space are also minimal in Euclidean space.

Finally, we will consider another general (α,β)-metric $F = \alpha\phi(b^2, \beta/\alpha)$, where $b^2 := \|\beta\|^2_\alpha$. We also assume that F is projectively flat. It is well known that a Riemannian metric is projectively flat if and only if it has constant sectional curvature. In [S2], the author proved that a Randers metric $F = \alpha + \beta$ is projectively flat if and only if α is projectively flat and β is closed.

Lemma 4.7 ([YZ]). *Let $F=\alpha\phi(b^2,\beta/\alpha)$ be a general (α,β)-metric on a manifold M of dimension n (≥ 2). Then F is locally projectively flat if the following conditions hold:*

(1) *The function $\phi(b^2,s)$ satisfies the partial differential equation*

$$\phi_{22}=2(\phi_1-s\phi_{12}); \tag{4.22}$$

(2) *α is locally projectively flat, and β is closed and conformal with respect to α.*

Let $\widetilde{F}=\tilde{\alpha}\phi(\tilde{b}^2,\tilde{\beta}/\tilde{\alpha})$ be a locally projectively flat metric and $f:(M^n,F)\to(\tilde{M}^{n+1},\tilde{F})$ be an isometric immersion. Then by (4.2), (4.7) we have

$$\begin{aligned} h_\alpha &= \xi^2[(f^\beta_{ij}y^iy^j+\tilde{G}^\beta)\tilde{a}_{\beta\delta}n^\delta]\tilde{a}_{\alpha\gamma}n^\gamma \\ &= \xi^2[(f^\beta_{ij}y^iy^j+\tilde{P}\tilde{y}^\beta)\tilde{a}_{\beta\delta}n^\delta]\tilde{a}_{\alpha\gamma}n^\gamma=\xi^2f^\beta_{ij}y^iy^j\tilde{a}_{\beta\delta}n^\delta\tilde{a}_{\alpha\gamma}n^\gamma, \end{aligned} \tag{4.23}$$

where ξ is defined by (4.4). On the other hand, a direct computation similar to that in [HS2] yields

$$\det(g_{ij})=\frac{\det(a_{ij})}{\xi^2\det(\tilde{a}_{\alpha\beta})}\det(\tilde{g}_{\alpha\beta})=\frac{\phi(\tilde{b}^2,\tilde{s})\widetilde{H}\det(a_{ij})}{\xi^2}. \tag{4.24}$$

Plugging (4.23) and (4.24) into (4.6), one gets

$$\begin{aligned} n^\alpha\int_{S_xM}\frac{h_\alpha}{F^2}\varOmega\,d\tau &= n^\alpha\int_{S_xM}\frac{\xi^2f^\beta_{ij}y^iy^j\tilde{a}_{\beta\delta}n^\delta\tilde{a}_{\alpha\gamma}n^\gamma\det(g_{ij})}{F^{n+2}}\,d\tau \\ &= \det(a_{ij})f^\beta_{ij}\tilde{a}_{\beta\delta}n^\delta\int_{S_xM}\frac{y^iy^j\phi(\tilde{b}^2,\tilde{s})\tilde{H}}{\widetilde{F}^{n+2}}\,d\tau \\ &= \det(a_{ij})f^\beta_{ij}\tilde{a}_{\beta\delta}n^\delta\int_{S_xM}y^iy^j(\phi-\tilde{s}\phi_2)^{n-1}\frac{\phi-\tilde{s}\phi_2+(\tilde{b}^2-\tilde{s}^2)\phi_{22}}{\tilde{\alpha}^{n+2}}\,d\tau \\ &= \det(a_{ij})f^\beta_{ij}\tilde{a}_{\beta\delta}n^\delta\int_{\alpha=1}y^iy^j(\phi-\tilde{\beta}\phi_2)^{n-1}[\phi-\tilde{\beta}\phi_2+(\tilde{b}^2-\tilde{\beta}^2)\phi_{22}]\,d\tau. \end{aligned}$$

Hence, by Lemma 2.1 we obtain

Theorem 4.8. *Let (M^n,F) be a hypersurface in $(\widetilde{M}^{n+1},\widetilde{F})$, where $\widetilde{F}=\tilde{\alpha}\phi(\tilde{b}^2,\tilde{\beta}/\tilde{\alpha})$ is locally projectively flat. Then $f:(M^n,F)\to(\widetilde{M}^{n+1},\widetilde{F})$ is minimal if and only if*

$$f^\beta_{ij}\tilde{a}_{\beta\delta}n^\delta\int_{\alpha=1}y^iy^j(\phi-\tilde{\beta}\phi_2)^{n-1}[\phi-\tilde{\beta}\phi_2+(\tilde{b}^2-\tilde{\beta}^2)\phi_{22}]\,d\tau=0. \tag{4.25}$$

In what follows, we consider the minimal surface $f = (u\cos v, u\sin v, h(v))$ in a general (α,β)-space $(\widetilde{M}^3, \widetilde{F})$, with $\widetilde{F} = \tilde{\alpha}\phi(\tilde{b}^2, \tilde{\beta}/\tilde{\alpha})$ and

$$
\tag{4.26}
\begin{aligned}
\tilde{\alpha} &= \sqrt{(\tilde{y}^1)^2 + (\tilde{y}^2)^2 + (\tilde{y}^3)^2}, \qquad \tilde{\beta} = \tilde{x}^\alpha d\tilde{x}^\alpha,\\
\phi(\tilde{b}^2, \tilde{s}) &= 1 + \tilde{b}^2 + \tilde{s}^2 + g(\tilde{b}^2)\tilde{s},
\end{aligned}
$$

where g is a smooth function. Obviously, $\tilde{\alpha}$ is projectively flat. It is easily checked that $\tilde{\beta}$ is closed and conformal with respect to $\tilde{\alpha}$ since $d\tilde{\beta} = 0$ and $\tilde{b}_{\alpha|\beta} = \tilde{a}_{\alpha\beta}$. In addition, $\phi(\tilde{b}^2, \tilde{s})$ satisfies (4.22) ([YZ]). Therefore $\widetilde{F}$ is locally projectively flat by Lemma 4.7.

Denote

$$
\tag{4.27}
W^{ij} = \int_{\alpha=1} y^i y^j (\phi - \tilde{\beta}\phi_2)[\phi - \tilde{\beta}\phi_2 + (\tilde{b}^2 - \tilde{\beta}^2)\phi_{22}]\, d\tau.
$$

By a simple computation, we have

$$
\phi_2 = 2\tilde{s} + g(\tilde{b}^2), \qquad \phi_{22} = 2, \qquad \tilde{\beta} = uy^1 + hh'y^2.
$$

Plugging the above formulas into (4.27), one gets

$$
\begin{aligned}
W^{ij} = \int_{\alpha=1} & y^i y^j [\Pi_0 + \Pi_1 y^1 y^2 + \Pi_2 (y^1)^2 + \Pi_3 (y^2)^2 + \Pi_4 (y^1)^3 y^2 \\
& + \Pi_5 (y^2)^3 y^1 + \Pi_6 (y^1)^2 (y^2)^2 + \Pi_7 (y^1)^4 + \Pi_8 (y^2)^4]\, d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
&\Pi_0 = 3\tilde{b}^4 + 4\tilde{b}^2 + 1, \quad \Pi_1 = -uhh'(6\tilde{b}^2 + 4), \quad \Pi_2 = -u^2(6\tilde{b}^2 + 4),\\
&\Pi_3 = -(hh')^2(6\tilde{b}^2 + 4), \quad \Pi_4 = 12u^3hh', \quad \Pi_5 = 12u(hh')^3,\\
&\Pi_6 = 81u^2(hh')^2, \quad \Pi_7 = 3u^4, \quad \Pi_8 = 3(hh')^4.
\end{aligned}
$$

A direct calculation yields

$$
\begin{aligned}
W^{12} &= \int_{\alpha=1} [\Pi_1 (y^1y^2)^2 + \Pi_4 (y^1)^4 (y^2)^2 + \Pi_5 (y^2)^4 (y^1)^2]\, d\tau \\
&= \int_0^{2\pi} \bigg[\Pi_1 \frac{(\sin\theta\cos\theta)^2}{u^2 + h'^2} + \Pi_4 \frac{(\sin\theta)^2(\cos\theta)^4}{u^2 + h'^2} \\
&\qquad + \Pi_5 \frac{(\sin\theta)^4(\cos\theta)^2}{u^2 + h'^2}\bigg] \frac{1}{\sqrt{u^2 + h'^2}}\, d\theta \\
&= \frac{\pi}{\sqrt{u^2 + h'^2}} \left[\frac{\Pi_1}{4(u^2 + h'^2)} + \frac{\Pi_4}{8(u^2 + h'^2)} + \frac{\Pi_5}{8(u^2 + h'^2)^2}\right] = W^{21},
\end{aligned}
$$

$$W^{22} = \int_{\alpha=1} [\varPi_0(y^2)^2 + \varPi_2(y^1)^2(y^2)^2$$
$$+ \varPi_3(y^2)^4 + \varPi_6(y^1)^2(y^2)^4 + \varPi_7(y^1)^4(y^2)^2 + \varPi_8(y^2)^6]\, d\tau$$
$$= \int_0^{2\pi} \left[\varPi_0 \frac{(\cos\theta)^2}{u^2+h'^2} + \varPi_2 \frac{(\sin\theta\cos\theta)^2}{u^2+h'^2} + \varPi_3 \frac{(\cos\theta)^4}{(u^2+h'^2)^2}\right.$$
$$\left. + \varPi_6 \frac{(\sin\theta)^2(\cos\theta)^4}{(u^2+h'^2)^2} + \varPi_7 \frac{(\sin\theta)^4(\cos\theta)^2}{u^2+h'^2} + \varPi_8 \frac{(\cos\theta)^6}{(u^2+h'^2)^2}\right] \frac{d\theta}{\sqrt{u^2+h'^2}}$$
$$= \frac{\pi}{\sqrt{u^2+h'^2}} \left[\frac{\varPi_0}{u^2+h'^2} + \frac{\varPi_2}{4(u^2+h'^2)} + \frac{3\varPi_3}{8(u^2+h'^2)^2}\right.$$
$$\left. + \frac{\varPi_6}{(u^2+h'^2)^2} + \frac{\varPi_7}{8(u^2+h'^2)} + \frac{5\varPi_8}{16(u^2+h'^2)^3}\right].$$

On the other hand, it is easy to see that the normal vector is

$$\mathbf{n} = \left(\frac{h'\sin v}{\sqrt{u^2+h'^2}} \quad \frac{-h'\cos v}{\sqrt{u^2+h'^2}} \quad \frac{u}{\sqrt{u^2+h'^2}} \right),$$

and

$$(f^1_{ij}) = \begin{pmatrix} 0 & -\sin v \\ -\sin v & -u\cos v \end{pmatrix}, \qquad (f^2_{ij}) = \begin{pmatrix} 0 & \cos v \\ \cos v & -u\sin v \end{pmatrix},$$

$$(f^3_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & h'' \end{pmatrix}.$$

Substituting the above formulas into (4.25), one gets

(4.28) $D_1u+D_2u^2+D_3u^3+D_4u^4+D_5u^5+D_6u^6+D_7u^7+D_8u^8 = 0$ for all u,

where

$$D_1 = 48h^3h'^8 - 16hh'(3\tilde{b}^2+2) - 15h^4h'^4h''$$
$$- (3\tilde{b}^4+4\tilde{b}^2+1)h'^4h'' + 12(3\tilde{b}^2+2)h^2h'^4h'',$$
$$D_2 = 16hh'^5(3\tilde{b}^2+2) - 48h^3h'^5,$$
$$D_3 = 48hh'^6 - 36h^2h'^4h'' + 48h^3h'^4 + 8(3\tilde{b}^2+2)h'^4h'' - 32(3\tilde{b}^2+2)hh'^4$$
$$+ 12h^2h'^2h''(3\tilde{b}^2+2) - 2(3\tilde{b}^4+4\tilde{b}^2+1)h'^2h'',$$
$$D_4 = -48h^3h'^5 - 48hh'^5 - 36h^2h'^4h'' + 32(3\tilde{b}^2+2)hh'^3,$$

$$
\begin{aligned}
D_5 &= -96hh'^3 - 36h^2h'^2h'' + 16(3\tilde{b}^2+2)hh'^3,\\
D_6 &= -96hh'^3 + 16(3\tilde{b}^2+2)hh',\\
D_7 &= 8(3\tilde{b}^2+2)h'' - 6h'' + 48hh'^2,\\
D_8 &= -48h^3h'^3.
\end{aligned}
$$

Clearly, D_i is a function of v for each i. Thus (4.28) holds if and only if $D_i = 0$ for all i, which means that h is constant. As a consequence, we have

THEOREM 4.9. *Let $(\widetilde{M}^3, \widetilde{F})$ be a general (α, β)-space where $\widetilde{F}$ is defined by* (4.26)*. Then the minimal conoid in $(\widetilde{M}^3, \widetilde{F})$ must be a plane.*

REMARK. Theorem 4.9 shows that there exists no non-trivial minimal conoid in $(\widetilde{M}^3, \widetilde{F})$, which is quite different from the Riemannian case.

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Songting Yin
Department of Mathematics
Tongji University
200092 Shanghai, China
and
Department of Mathematics
and Computer Science
Tongling University
244000 Tongling, Anhui, China
E-mail: yst419@163.com

Qun He (corresponding author), Dinghe Xie
Department of Mathematics
Tongji University
200092 Shanghai, China
E-mail: hequn@tongji.edu.cn
xdinghe8707@126.com

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