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Algebraic dependences of meromorphic mappings sharing few moving hyperplanes

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Abstract. We study algebraic dependences of three meromorphic mappings which share few moving hyperplanes without counting multiplicity.

1. Introduction. In 1926, R. Nevanlinna showed that two distinct nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$ cannot have the same inverse images for five distinct values, and that g is a special type of linear fractional transformation of f if they have the same inverse images counted with multiplicities for four distinct values.

Recently, motivated by the establishment of the second main theorem of value distribution theory for moving targets (e.g., Ru and Wang [RW], Thai and Quang [TQ2]) with truncated multiplicities, the finiteness problem of meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ intersecting a few moving hyperplanes (i.e, moving targets) regardless of multiplicity has been studied intensively. We recall the recent results of Thai and Quang [TQ1] which are the best results available at present.

Let $a_1, \dots, a_q$ $(q \geq N+1)$ be meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ with reduced representations $a_i = (a_{i0} : \dots : a_{iN})$ $(1 \leq i \leq q)$. We say that $a_1, \dots, a_q$ are *in general position* if $\det(a_{i_k j}) \not\equiv 0$ for any $1 \leq i_0 < i_1 < \dots < i_N \leq q$.

Throughout this paper, we denote by $\mathcal{M}$ the field of all meromorphic functions on $\mathbb{C}^n$ and denote by $\mathcal{R}(\{a_i\}_{i=1}^q) \subset \mathcal{M}$ the smallest subfield of $\mathcal{M}$ which contains $\mathbb{C}$ and all a_{jk}/a_{jl} with $a_{jl} \not\equiv 0$.

Let f be a meromorphic mapping of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ with reduced representation $f = (f_0 : \dots : f_N)$. We say that f is *linearly nondegenerate* over $\mathcal{R}(\{a_i\}_{j=1}^q)$ if $f_0, \dots, f_N$ are linearly independent over $\mathcal{R}(\{a_i\}_{i=1}^q)$.

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Let f, a be two meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ with reduced representations $f = (f_0 : \cdots : f_N)$, $a = (a_0 : \cdots : a_N)$ respectively. We say that a is *small with respect to* f if $\| \ T_a(r) = o(T_f(r))$ as $r \to \infty$. Put $(f, a) = \sum_{j=0}^{N} a_j f_j$.

Let $f : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be a meromorphic mapping. Let d be a positive integer. Let $\{a_j\}_{j=1}^q$ be small (with respect to f) meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ in general position such that

$$\dim\{z \in \mathbb{C}^n : (f, a_i)(z) = (f, a_j)(z) = 0\} \leq n - 2 \quad (1 \leq i < j \leq q).$$

Consider the set $\mathcal{F}(f, \{a_j\}_{j=1}^q, d)$ of all meromorphic maps $g : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ satisfying the conditions:

(i) $\min\,(\nu_{(f,a_j)}, d) = \min(\nu_{(g,a_j)}, d)$ $(1 \leq j \leq q)$,

(ii) $f(z) = g(z)$ on $\bigcup_{j=1}^q \{z \in \mathbb{C}^n : (f, a_j)(z) = 0\}$.

Denote by $\sharp S$ the cardinality of the set S. In [TQ1] Thai and Quang proved the following.

THEOREM A ([TQ1, Theorem 1.2]). *Assume that f is linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=1}^q)$.*

(a) *If $q = 2N^2 + 4N$ and $N \geq 2$, then $\sharp\mathcal{F}(f, \{a_j\}_{j=1}^q, 1) = 1$.*

(b) *If $q = (3N^2 + 7N + 2)/2$ and $N \geq 2$, then $\sharp\mathcal{F}(f, \{a_j\}_{j=1}^q, 2) \leq 2$.*

Note that in the original paper [TQ1], the authors assume that all maps g in the definition of the family $\mathcal{F}(f, \{a_j\}_{j=1}^q, 1)$ are linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=1}^q)$. Actually, in this paper we will show that if f is linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=1}^q)$ then so is each $g \in \mathcal{F}(f, \{a_j\}_{j=1}^q, 1)$, for $q > N(N + 2)$.

As far as we know, there has been no result on the family $\mathcal{F}(f, \{a_j\}_{j=1}^q, 1)$ in the case where $q < (3N^2 + 7N + 2)/2$.

Our purpose in the present work is to handle this case. We will prove a theorem on algebraic dependence of three maps in $\mathcal{F}(f, \{a_j\}_{j=1}^q, 1)$ as follows.

MAIN THEOREM 1.1. *Let $f : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ $(N \geq 2)$ be a meromorphic mapping. Let $\{a_j\}_{j=1}^q$ be small (with respect to f) meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ in general position such that*

$$\dim \mathrm{Zero}(f, a_i) \cap \mathrm{Zero}(f, a_j) \leq n - 2 \quad (1 \leq i < j \leq q).$$

Let $f_1, f_2, f_3 \in \mathcal{F}(f, \{a_j\}_{j=1}^q, 1)$.

(a) *If $q > 3N^2 + 3/2$ then $f_1 \wedge f_2 \wedge f_3 \equiv 0$.*

(b) *If f is linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=1}^q)$ and $q > (3N^2 + 3N + 3)/2$ then $f_1 \wedge f_2 \wedge f_3 \equiv 0$.*

Thoan–Duc [PP] and Min Ru [R] have given some results on algebraic dependence of meromorphic mappings. In the case of three maps, the main theorem of the present paper is an improvement of their results.

2. Basic notions and auxiliary results from Nevanlinna theory

2.1. We set $\|z\| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, and

$$B(r) := \{z \in \mathbb{C}^n : \|z\| < r\}, \quad S(r) := \{z \in \mathbb{C}^n : \|z\| = r\} \quad (0 < r < \infty).$$

Define

$$\begin{aligned} v_{n-1}(z) &:= (dd^c\|z\|^2)^{n-1}, \\ \sigma_n(z) &:= d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{n-1} \qquad \text{on } \mathbb{C}^n \setminus \{0\}. \end{aligned}$$

2.2. Let F be a nonzero holomorphic function on a domain Ω in $\mathbb{C}^n$. For a sequance $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\mathcal{D}^\alpha F = \partial^{|\alpha|} F / \partial^{\alpha_1} z_1 \cdots \partial^{\alpha_n} z_n$. We define a map $\nu_F : \Omega \to \mathbb{Z}$ by

$$\nu_F(z) := \max\{m : \mathcal{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < m\} \quad (z \in \Omega).$$

By a *divisor* on a domain Ω in $\mathbb{C}^n$ we mean a map $\nu : \Omega \to \mathbb{Z}$ such that, for each $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighborhood $U \subset \Omega$ of a such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\leq n-2$. Two divisors are regarded to be the same if they are identical outside an analytic set of dimension $\leq n-2$. For a divisor ν on Ω we set $|\nu| := \overline{\{z : \nu(z) \neq 0\}}$, which is a purely $(n-1)$-dimensional analytic subset of Ω or empty.

Take a nonzero meromorphic function φ on a domain Ω in $\mathbb{C}^n$. For each $a \in \Omega$, we choose nonzero holomorphic functions F and G on a neighborhood $U \subset \Omega$ such that $\varphi = F/G$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq n-2$, and we define divisors ν_φ, ν_φ^∞ by $\nu_\varphi := \nu_F$, $\nu_\varphi^\infty := \nu_G$, which are independent of the choices of F and G and so globally well-defined on Ω.

2.3. For a divisor ν on $\mathbb{C}^n$ and for a positive integer M or $M = \infty$, we define the counting function of ν by

$$\nu^{(M)}(z) = \min\{M, \nu(z)\}.$$

Moreover, we set

$$n(t) = \begin{cases} \displaystyle\int_{|\nu| \cap B(t)} \nu(z) v_{n-1} & \text{if } n \geq 2, \\ \displaystyle\sum_{|z| \leq t} \nu(z) & \text{if } n = 1. \end{cases}$$

Similarly, we define $n^{(M)}(t)$.

Set

$$N(r,\nu) = \int_1^r \frac{n(t)}{t^{2n-1}}\,dt \qquad (1 < r < \infty).$$

Similarly, we define $N(r,\nu^{(M)})$, which we also denote by $N^{(M)}(r,\nu)$.

Let $\varphi : \mathbb{C}^n \to \mathbb{C}$ be a meromorphic function. Define

$$N_\varphi(r) = N(r,\nu_\varphi), \qquad N_\varphi^{(M)}(r) = N^{(M)}(r,\nu_\varphi).$$

For brevity we will omit the superscript $^{(M)}$ if $M = \infty$.

2.4. Let $f : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be a meromorphic mapping. For fixed homogeneous coordinates $(w_0 : \cdots : w_N)$ on $\mathbb{P}^N(\mathbb{C})$, we take a reduced representation $f = (f_0 : \cdots : f_N)$, which means that each f_i is a holomorphic function on $\mathbb{C}^n$ and $f(z) = (f_0(z) : \cdots : f_N(z))$ outside the analytic set $\{f_0 = \cdots = f_N = 0\}$ of codimension ≥ 2. Set $\|f\| = (|f_0|^2 + \cdots + |f_N|^2)^{1/2}$.

The *characteristic function* of f is defined by

$$T_f(r) = \int_{S(r)} \log\|f\|\,\sigma_n - \int_{S(1)} \log\|f\|\,\sigma_n.$$

Let a be a meromorphic mapping of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ with reduced representation $a = (a_0 : \cdots : a_N)$. We define

$$m_{f,a}(r) = \int_{S(r)} \log\frac{\|f\|\cdot\|a\|}{|(f,a)|}\,\sigma_n - \int_{S(1)} \log\frac{\|f\|\cdot\|a\|}{|(f,a)|}\,\sigma_n,$$

where $\|a\| = (|a_0|^2 + \cdots + |a_N|^2)^{1/2}$.

If $f, a : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ are meromorphic mappings such that $(f,a) \not\equiv 0$, then the first main theorem for moving targets in value distribution theory states that

$$T_f(r) + T_a(r) = m_{f,a}(r) + N_{(f,a)}(r).$$

Let φ be a nonzero meromorphic function on $\mathbb{C}^n$, which is occasionally regarded as a meromorphic map into $\mathbb{P}^1(\mathbb{C})$. The *proximity function* of φ is defined by

$$m(r,\varphi) := \int_{S(r)} \log\max(|\varphi|,1)\,\sigma_n.$$

2.5. As usual, the notation "$\|\ P$" means the assertion P holds for all $r \in [0,\infty)$ excluding a Borel subset E of the interval $[0,\infty)$ with $\int_E dr < \infty$.

2.6. Let V be a complex vector space of dimension $N \geq 1$. For two vectors α and β in V, we write $\alpha \cong \beta$ if they are linearly dependent, and $\alpha \not\cong \beta$ otherwise.

2.7. We will need two theorems:

Theorem 2.1 (Second Main Theorem for moving targets [TQ2, Corollary 1]). *Let $f : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be a meromorphic mapping. Let $\{a_i\}_{i=1}^q$ $(q \geq 2N+1)$ be a set of q small (with respect to f) meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ in general position such that $(f, a_i) \not\equiv 0$ $(1 \leq i \leq q)$. Then*

$$\Big\| \quad \frac{q}{2N+1} T_f(r) \leq \sum_{i=1}^{q} N^{(N)}_{(f,a_i)}(r) + o(T_f(r)).$$

Theorem 2.2 (Second Main Theorem for moving targets [TQ1, Lemma 3.1]). *Let $f : \mathbb{C}^n \to \mathbb{P}^N(\mathbb{C})$ be a meromorphic mapping. Let $\{a_i\}_{i=1}^q$ $(q \geq N+2)$ be a set of q small (with respect to f) meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ in general position. Assume that f is linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=1}^q)$. Then*

$$\Big\| \quad \frac{q}{N+2} T_f(r) \leq \sum_{i=1}^{q} N^{(N)}_{(f,a_i)}(r) + o(T_f(r)).$$

3. Proof of Main Theorem. In order to prove the main theorem, we need the following lemmas.

Lemma 3.1. *Let f be a meromorphic mapping of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$. Let $\{a_i\}_{i=1}^q$ $(q > N(N+2))$ be a set of q small (with respect to f) meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ in general position. Assume that f is linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=1}^q)$. Then each $g \in \mathcal{F}(f, \{a_i\}_{i=1}^q, 1)$ is linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=1}^q)$.*

Proof. Assume f, g and a_i $(1 \leq i \leq q)$ have reduced representations

$$f = (f_0 : \cdots : f_N), \qquad g = (g_0 : \cdots : g_N),$$
$$a_i = (a_{i0} : \cdots : a_{iN}) \qquad (1 \leq i \leq q).$$

Suppose that g is linearly degenerate over $\mathcal{R}\{a_i\}_{i=1}^q$. Then there exist functions $c_i \in \mathcal{R}\{a_i\}_{i=1}^q$ $(0 \leq i \leq N)$, not all zeros, such that

$$c_0 g_0 + c_1 g_1 + \cdots + c_N g_N = 0.$$

We consider a meromorphic mapping c with a reduced representation $c = (hc_0 : \cdots : hc_N)$, where h is a meromorphic function on $\mathbb{C}^n$. It is clear that c is small with respect to f and

$$(g, c) := \sum_{j=0}^{N} hc_j g_j \equiv 0.$$

Since f is linearly nondegenerate over $\mathcal{R}\{a_i\}_{i=1}^q$, we have

$$(f, c) := \sum_{j=0}^{N} hc_j f_j \not\equiv 0.$$

On the other hand $f(z) = g(z) = 0$ for all $z \in \bigcup_{i=1}^q \mathrm{Zero}(f, a_i)$, hence $(f, c)(z) = (g, c)(z) = 0$ for all $z \in \bigcup_{i=1}^q \mathrm{Zero}(f, a_i)$. This implies that

$$N_{(f,c)}(r) \geq \sum_{i=1}^{q} N^{(1)}_{(f,a_i)}(r).$$

Consequently,

$$\begin{aligned}\| \ T_f(r) \geq N_{(f,c)}(r) &\geq \sum_{i=1}^{q} N^{(1)}_{(f,a_i)}(r)\\ &\geq \sum_{i=1}^{q} \frac{1}{N} N^{(N)}_{(f,a_i)}(r) \geq \frac{q}{N(N+2)} T_f(r) + o(T_f(r)).\end{aligned}$$

Letting $r \to \infty$, we get $q \leq N(N+2)$. This is a contradiction.

Hence g is linearly nondegenerate over $\mathcal{R}\{a_i\}_{i=1}^q$. ■

LEMMA 3.2. *Let $f : \mathbb{C}^n \longrightarrow \mathbb{P}^N(\mathbb{C})$ be a meromorphic mapping and let $\{a_i\}_{i=1}^q$ $(q \geq 3N+3)$ be a family of q small (with respect to f) meromorphic mappings of $\mathbb{C}^n$ into $\mathbb{P}^N(\mathbb{C})$ in general position with*

$$\dim \mathrm{Zero}(f, a_i) \cap \mathrm{Zero}(f, a_j) \leq n-2 \qquad (1 \leq i < j \leq q).$$

Let $f_1, f_2, f_3 \in \mathcal{F}(f, \{H_i\}_{i=1}^q, 1)$. Assume that $f_1 \wedge f_2 \wedge f_3 \not\equiv 0$. Then

$$q \sum_{u=1}^{3} T_{f_u}(r) \geq \frac{2q+3N-3}{3N} \sum_{i=1}^{q} N^{(N)}_{(f_u,a_i)}(r) + o(T_f(r)).$$

Proof. We consider $\mathcal{M}^3$ as a vector space over the field $\mathcal{M}$. For each $i = 1, \ldots, q$, we set

$$v_i = ((f_1, a_i), (f_2, a_i), (f_3, a_i)) \in \mathcal{M}^3.$$

By changing the indices if necessary, we may assume that

$$\underbrace{v_1 \cong \cdots \cong v_{k_1}}_{\text{group 1}} \not\cong \underbrace{v_{k_1+1} \cong \cdots \cong v_{k_2}}_{\text{group 2}} \not\cong \cdots \not\cong \underbrace{v_{k_{s-1}+1} \cong \cdots \cong v_{k_s}}_{\text{group } s},$$

where $k_s = q$.

For each $1 \leq i \leq q$, we set

$$I_i = \begin{cases} \{i+N, i+N+1, \ldots, i+3N-1\} & \text{if } i+3N-1 \leq q,\\ \{i+N, \ldots, q, 1, 2, \ldots, i+3N-q-1\} & \text{if } i+N \leq q < i+3N-1,\\ \{i+N-q, \ldots, i+3N-q-1\} & \text{if } i+N > q.\end{cases}$$

Since $f_1 \wedge f_2 \wedge f_3 \not\equiv 0$, the number of elements of each group is at most N. Hence v_i and v_j belong to distinct groups for all $j \in I_i$ and $i = 1, \ldots, q$. This means that $v_i \wedge v_j \neq 0$ $(j \in I_i, 1 \leq i \leq q)$.

Claim. *For every $1 \le i \le q$, we have*

$$\sum_{u=1}^{3} T_{f_u}(r) \ge \sum_{u=1}^{3}\Big(\Big[N^{(N)}_{(f_u,a_i)}(r) - \frac{2N+1}{3}N^{(1)}_{(f_u,a_i)}(r)\Big]$$
$$+ \sum_{j\in I_i}\Big[\frac{1}{N}N^{(N)}_{(f_u,a_j)}(r) - \frac{2N+1}{3N}N^{(1)}_{(f_u,a_j)}(r)\Big] + \frac{2}{3}\sum_{v=1}^{q}N^{(1)}_{(f_u,a_v)}(r)\Big)$$
$$+ o(T_f(r)).$$

We now prove the claim. It suffices to prove it for $i = 1$.

For $1 \le j \le 2N$, we set $V_j = v_1 \wedge v_{j+N} \ne 0$. By changing the indices if necessary, we may assume that

$$\underbrace{V_1 \cong \cdots \cong V_{l_1}}_{\text{group 1}} \not\cong \underbrace{V_{l_1+1} \cong \cdots \cong V_{l_2}}_{\text{group 2}} \not\cong \cdots \not\cong \underbrace{V_{l_{t-1}+1} \cong \cdots \cong V_{l_t}}_{\text{group } t},$$

where $l_t = 2N$.

For each $1 \le j \le N$, we set

$$P_j = \det\begin{pmatrix} (f_1,a_1) & (f_1,a_{j+N}) & (f_1,a_{j+2N}) \\ (f_2,a_1) & (f_2,a_{j+N}) & (f_2,a_{j+2N}) \\ (f_3,a_1) & (f_3,a_{j+N}) & (f_3,a_{j+2N}) \end{pmatrix}.$$

Since again $f_1 \wedge f_2 \wedge f_3 \ne 0$, the number of elements of each group is at most N. Hence V_j and V_{j+N} belong to distinct groups, so v_1, v_{j+N}, v_{j+2N} are linearly independent over $\mathcal{M}$ for every $j = 1,\dots,N$. This means that $P_j \not\equiv 0$ $(1 \le i \le N)$.

Fix $1 \le j \le N$. For $z \notin \bigcup_{u=1}^3 I(f_u) \cup \bigcup_{i'\ne j'}(\mathrm{Zero}(f,a_{i'}) \cap \mathrm{Zero}(f,a_{j'}))$, we consider the following four cases:

Case 1: z is a zero of (f,a_1). We set

$$m = \min\{\nu_{(f_1,a_1)}(z), \nu_{(f_2,a_1)}(z), \nu_{(f_2,a_1)}(z)\}.$$

Then there exist a neighborhood U of z and a holomorphic function h defined on U such that $\mathrm{Zero}(h) = U \cap \mathrm{Zero}(f,a_1)$ and dh has no zero. Moreover we may assume that

$$U \cap \Big(\bigcup_{u=1}^{3} I(f_u) \cup \bigcup_{i'\ne j'}\big(\mathrm{Zero}(f,a_{i'}) \cap \mathrm{Zero}(f,a_{j'})\big)\Big) = \emptyset.$$

We see that there exist holomorphic functions $\varphi_1,\varphi_2,\varphi_3$ defined on U such that

$$(f_u,a_1) = h^m\varphi_u \quad \text{on } U \text{ for } 1 \le u \le 3.$$

On the other hand, since $f_1 = f_2 = f_3$ on $\mathrm{Zero}(f, a_1)$, we have

$$\frac{(f_u, a_{j+N})}{(f_1, a_{j+N})} = \frac{(f_u, a_{j+2N})}{(f_1, a_{j+2N})} \quad \text{on } \mathrm{Zero}(f, a_1),\ u = 2, 3.$$

Therefore, there exist holomorphic functions ψ_2 and ψ_3 satisfying

$$\frac{(f_u, a_{j+N})}{(f_1, a_{j+N})} - \frac{(f_u, a_{j+2N})}{(f_1, a_{j+2N})} = h\psi_u \quad \text{on } \mathrm{Zero}(f, a_1),\ u = 2, 3.$$

We rewrite P_j on U as follows:

$$\begin{aligned}
P_j &= h^m \det \begin{pmatrix} \varphi_1 & (f_1, a_{j+N}) & (f_1, a_{j+2N}) \\ \varphi_2 & (f_2, a_{j+N}) & (f_2, a_{j+2N}) \\ \varphi_3 & (f_3, a_{j+N}) & (f_3, a_{j+2N}) \end{pmatrix} \\
&= h^m (f_1, a_{j+N})(f_1, a_{j+2N}) \det \begin{pmatrix} \varphi_1 & 1 & 1 \\ \varphi_2 & \frac{(f_2, a_{j+N})}{(f_1, a_{j+N})} & \frac{(f_2, a_{j+2N})}{(f_1, a_{j+2N})} \\ \varphi_3 & \frac{(f_3, a_{j+N})}{(f_1, a_{j+N})} & \frac{(f_3, a_{j+2N})}{(f_1, a_{j+2N})} \end{pmatrix} \\
&= -\, h^{m+1} (f_1, a_{j+N})(f_1, a_{j+2N}) \det \begin{pmatrix} \varphi_1 & 1 & 0 \\ \varphi_2 & \frac{(f_2, a_{j+N})}{(f_1, a_{j+N})} & \psi_2 \\ \varphi_3 & \frac{(f_3, a_{j+N})}{(f_1, a_{j+N})} & \psi_3 \end{pmatrix}.
\end{aligned}$$

This yields

$$\nu_{P_j}(z) \geq m + 1 = \min\{\nu_{(f_1, a_1)}(z), \nu_{(f_2, a_1)}(z), \nu_{(f_2, a_1)}(z)\} + 1.$$

Case 2: z is a zero of (f, a_{j+N}). Repeating the same argument as in Case 1, we have

$$\nu_{P_j}(z) \geq \min\{\nu_{(f_1, a_{j+N})}(z), \nu_{(f_2, a_{j+N})}(z), \nu_{(f_2, a_{j+N})}(z)\} + 1.$$

Case 3: z is a zero of (f, a_{j+2N}). Repeating the same argument as in Case 1, we have

$$\nu_{P_j}(z) \geq \min\{\nu_{(f_1, a_{j+2N})}(z), \nu_{(f_2, a_{j+2N})}(z), \nu_{(f_2, a_{j+2N})}(z)\} + 1.$$

Case 4: z is a zero point of (f, a_v) with $v \notin \{1, j+N, j+2N\}$. We have

$$P_j = \det \begin{pmatrix} (f_1, a_1) & (f_1, a_{j+N}) & (f_1, a_{j+2N}) \\ (f_2, a_1) & (f_2, a_{j+N}) & (f_2, a_{j+2N}) \\ (f_3, a_1) & (f_3, a_{j+N}) & (f_3, a_{j+2N}) \end{pmatrix} \tag{3.1}$$

$$= \prod_{t=1,j+N,j+2N} (f_1, a_t) \det \begin{pmatrix} 1 & 1 & 1 \\ \frac{(f_2,a_1)}{(f_1,a_1)} & \frac{(f_2,a_{j+N})}{(f_1,a_{j+N})} & \frac{(f_2,a_{j+2N})}{(f_1,a_{j+2N})} \\ \frac{(f_3,a_1)}{(f_1,a_1)} & \frac{(f_3,a_{j+N})}{(f_1,a_{j+N})} & \frac{(f_3,a_{j+2N})}{(f_1,a_{j+2N})} \end{pmatrix}$$

$$= \prod_{t=1,j+N,j+2N} (f_1, a_t) \det \begin{pmatrix} \frac{(f_2,a_{j+N})}{(f_1,a_{j+N})} - \frac{(f_2,a_1)}{(f_1,a_1)} & \frac{(f_2,a_{j+2N})}{(f_1,a_{j+2N})} - \frac{(f_2,a_1)}{(f_1,a_1)} \\ \frac{(f_3,a_{j+N})}{(f_1,a_{j+N})} - \frac{(f_3,a_1)}{(f_1,a_1)} & \frac{(f_3,a_{j+2N})}{(f_1,a_{j+2N})} - \frac{(f_3,a_1)}{(f_1,a_1)} \end{pmatrix}.$$

Since $f_1(z) = f_2(z) = f_3(z)$, we have

$$\frac{(f_2, a_{j+N})}{(f_1, a_{j+N})}(z) - \frac{(f_2, a_1)}{(f_1, a_1)}(z) = \frac{(f_2, a_{j+2N})}{(f_1, a_{j+2N})}(z) - \frac{(f_2, a_1)}{(f_1, a_1)}(z) = 0,$$
$$\frac{(f_3, a_{j+N})}{(f_1, a_{j+N})}(z) - \frac{(f_3, a_1)}{(f_1, a_1)}(z) = \frac{(f_3, a_{j+2N})}{(f_1, a_{j+2N})}(z) - \frac{(f_3, a_1)}{(f_1, a_1)}(z) = 0.$$

Therefore, (3.1) implies that z is a zero of P_j with multiplicity at least 2.

Thus, from the above four cases we have

$$\nu_{P_j}(z) \geq \sum_{v=1,j+N,j+2N} (\min\{\nu_{(f_1,a_v)}, \nu_{(f_2,a_v)}, \nu_{(f_3,a_v)}\} + 1) + 2 \sum_{\substack{v=1 \\ v\neq 1,\, j+N,\, j+2N}}^{q} \nu^{(1)}_{(f,a_v),\leq k}(z)$$

for all z outside the analytic set

$$I(f_1) \cup I(f_2) \cup I(f_3) \cup \bigcup_{i'\neq j'} f^{-1}(a_{i'} \cap a_{j'})$$

of codimension two.

Since $\min\{a, b, c\} \geq \min\{a, N\} + \min\{b, N\} + \min\{c, N\} - 2N$ for all positive integers a, b and c, the above inequality implies that

$$\nu_{P_j}(z) \geq \sum_{v=1,j+N,j+2N} \big(\min\{\nu_{(f_1,a_v)}(z), N\} + \min\{\nu_{(f_2,a_v)}(z), N\} + \min\{\nu_{(f_3,a_v)}(z), N\} - (2N-1) \min\{\nu_{(f,a_v)}(z), 1\}\big) + 2 \sum_{\substack{v=1 \\ v\neq 1,j+N,j+2N}}^{q} \nu^{(1)}_{(f,a_v)}(z)$$

for all z outside an analytic subset of codimension two in $\mathbb{C}^n$.

Integrating both sides of the above inequality, we get

$$N_{P_j}(r) \geq \sum_{v=1,j+N,j+2N} \Big(\sum_{u=1}^{3} N^{(N)}_{(f_u,a_v)}(r) - (2N-1) N^{(1)}_{(f,a_v)}(r) \Big) + 2 \sum_{\substack{v=1 \\ v\neq 1,j+N,j+2N}}^{q} N^{(1)}_{(f,a_v)}(r).$$

On the other hand, by Jensen's formula and the definition of the characteristic function we have

$$\begin{aligned} N_{P_i}(r) &= \int\limits_{S(r)} \log |P_i|\, \eta + O(1) \\ &\leq \sum_{u=1}^{3} \int\limits_{S(r)} \log (|(f_u, a_1)|^2 + |(f_u, a_{j+N})|^2 + |(f_u, a_{j+2N})|)^{1/2}\, \eta \\ &\leq \sum_{u=1}^{3} \int\limits_{S(r)} \log \|f\|\, \eta + O\Big(\max_{v=1,j+N,j+2N} T_{a_v}(r) \Big) = \sum_{u=1}^{3} T_{f_u}(r) + o(T_f(r)). \end{aligned}$$

This implies that

$$\sum_{u=1}^{3} T_{f_u}(r) \geq \sum_{v=1,j+N,j+2N} \Big(\sum_{u=1}^{3} N^{(N)}_{(f_u,a_v)}(r) - (2N-1) N^{(1)}_{(f,a_v)}(r) \Big) + 2 \sum_{\substack{v=1 \\ v\neq 1,j+N,j+2N}}^{q} N^{(1)}_{(f,a_v)}(r) + o(T_f(r)).$$

Summing both sides of the above inequality over $j = 1, \ldots, N$, we have

$$\begin{aligned} N \sum_{u=1}^{3} T_{f_u}(r) &\geq \sum_{j=1}^{N} \Big(\sum_{v=1,j+N,j+2N} \Big(\sum_{u=1}^{3} N^{(N)}_{(f_u,a_v)}(r) - (2N-1) N^{(1)}_{(f,a_v)}(r) \Big) \\ &\qquad + 2 \sum_{\substack{v=1 \\ v\neq 1,j+N,j+2N}}^{q} N^{(1)}_{(f,a_v)}(r) \Big) + o(T_f(r)) \\ &\geq \sum_{j=1}^{N} \Big(\sum_{v=1,j+N,j+2N} \Big(\sum_{u=1}^{3} N^{(N)}_{(f_u,a_v)}(r) - (2N+1) N^{(1)}_{(f,a_v)}(r) \Big) \\ &\qquad + 2 \sum_{v=1}^{q} N^{(1)}_{(f,a_v)}(r) \Big) + o(T_f(r)) \end{aligned}$$

$$= N\Big(\sum_{u=1}^{3} N^{(N)}_{(f_u,a_1)}(r) - (2N+1)N^{(1)}_{(f,a_1)}(r)\Big)$$

$$+ \sum_{j\in I_1}\Big(\sum_{u=1}^{3} N^{(N)}_{(f_u,a_j)}(r) - (2N+1)N^{(1)}_{(f,a_j)}(r)\Big) + 2N\sum_{v=1}^{q} N^{(1)}_{(f,a_v)}(r) + o(T_f(r))$$

$$= \sum_{u=1}^{3}\Bigg(N\Big[N^{(N)}_{(f_u,a_1)}(r) - \frac{2N+1}{3}N^{(1)}_{(f_u,a_1)}(r)\Big]$$

$$+ \sum_{j\in I_1}\Big[N^{(N)}_{(f_u,a_j)}(r) - \frac{2N+1}{3}N^{(1)}_{(f_u,a_j)}(r)\Big] + \frac{2N}{3}\sum_{v=1}^{q} N^{(1)}_{(f_u,a_v)}(r)\Bigg) + o(T_f(r)).$$

Dividing both sides by N, we get the inequality of the claim.

We continue the proof of the lemma.

By the claim, for every $1 \le i \le q$, we have

$$\sum_{u=1}^{3} T_{f_u}(r) \ge \sum_{u=1}^{3}\Bigg(\Big[N^{(N)}_{(f_u,a_i)}(r) - \frac{2N+1}{3}N^{(1)}_{(f_u,a_i)}(r)\Big]$$

$$+ \sum_{j\in I_i}\Big[\frac{1}{N}N^{(N)}_{(f_u,a_j)}(r) - \frac{2N+1}{3N}N^{(1)}_{(f_u,a_j)}(r)\Big] + \frac{2}{3}\sum_{v=1}^{q} N^{(1)}_{(f_u,a_v)}(r)\Bigg) + o(T_f(r)).$$

Thus, by summing them up, we have

$$q\sum_{u=1}^{3} T_{f_u}(r) \ge \sum_{u=1}^{3}\sum_{i=1}^{q}\big(3N^{(N)}_{(f_u,a_i)}(r) + (2q/3 - 2N - 1)N^{(1)}_{(f_u,a_i)}(r)\big) + o(T_f(r)). \tag{3.2}$$

It is easy to see that

$$N^{(1)}_{(f_u,a_i)}(r) \ge \frac{1}{N}N^{(N)}_{(f_u,a_i)}(r), \qquad \forall i, u.$$

Therefore, the inequality (3.2) implies that

$$q\sum_{u=1}^{3} T_{f_u}(r) \ge \frac{2q+3N-3}{3N}\sum_{u=1}^{3}\sum_{i=1}^{q} N^{(N)}_{(f_u,a_i)}(r) + o(T_f(r)).$$

The lemma is proved. ■

Proof of Main Theorem 1.1. With the assumption $q > 3N^2 + 3/2$ in (a) or $q > (3N^2 + 3N + 3)/2$ in (b), we have $q \ge 2N + 1$. Then, for each

$f_u \in \mathcal{F}(f, \{a_j\}_{j=1}^q, 1)$ $(1 \le u \le 3)$ we have

$$\Big\| \frac{q}{2N+1} T_{f_u}(r) \le \sum_{i=1}^{q} N^{(N)}_{(f_u,a_i)}(r) + O\Big(\max_{1\le i\le q} T_{a_i}(r)\Big) + o(T_{f_u}(r))$$
$$\le N \sum_{i=1}^{q} N^{(1)}_{(f_u,a_i)}(r) + o(T_{f_u}(r)) + o(T_f(r))$$
$$= N \sum_{i=1}^{q} N^{(1)}_{(f,a_i)}(r) + o(T_{f_u}(r)) + o(T_f(r))$$
$$\le NqT_f(r) + +o(T_{f_u}(r)) + o(T_f(r)).$$

This yields $\| \ T_{f_u}(r) = O(T_f(r))$ $(1 \le u \le 3)$. Similarly, we have $\| \ T_f(r) = O(T_{f_u}(r))$ $(1 \le u \le 3)$.

We now prove the two assertions of the theorem.

(a) Suppose that $f_1 \wedge f_2 \wedge f_3 \not\equiv 0$. Then by Lemma 3.2 we have

$$q \sum_{u=1}^{3} T_{f_u}(r) \ge \frac{2q+3N-3}{3N} \sum_{u=1}^{3}\sum_{i=1}^{q} N^{(N)}_{(f_u,a_i)}(r) + o(T_f(r)).$$

By using the Second Main Theorem (Theorem 2.1) for meromorphic mappings with moving targets, we have

$$\sum_{u=1}^{3} \frac{q}{2N+1} T_{f_u}(r) \le \sum_{u=1}^{3}\sum_{i=1}^{q} N^{(N)}_{(f_u,a_i)}(r) + o(T_f(r))$$
$$\le \frac{3Nq}{2q+3N-3} \sum_{u=1}^{3} T_{f_u}(r) + o(T_f(r)).$$

Since $\| \ T_f(r) = O(T_{f_u}(r))$ $(1 \le u \le 3)$, letting $r \to +\infty$, we get $q \le 3N^2 + 3/2$. This is a contradiction. Thus, $f_1 \wedge f_2 \wedge f_3 = 0$.

(b) Suppose that $f_1 \wedge f_2 \wedge f_3 \not\equiv 0$. By Lemma 3.1, f_1, f_2, f_3 are linearly nondegenerate over $\mathcal{R}(\{a_i\}_{i=1}^q)$.

By the Second Main Theorem (Theorem 2.2) and Lemma 3.1,

$$\sum_{u=1}^{3} \frac{q}{N+2} T_{f_u}(r) \le \sum_{u=1}^{3}\sum_{i=1}^{q} N^{(N)}_{(f_u,a_i)}(r) + o(T_f(r))$$
$$\le \frac{3Nq}{2q+3N-3} \sum_{u=1}^{3} T_{f_u}(r) + o(T_f(r)).$$

Letting $r \to +\infty$, we get $q \le (3N^2 + 3N + 3)/2$. This is a contradiction. Thus, $f_1 \wedge f_2 \wedge f_3 = 0$. ∎

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