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On isotropic Berwald metrics

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Abstract. We prove that every isotropic Berwald metric of scalar flag curvature is a Randers metric. We study the relation between an isotropic Berwald metric and a Randers metric which are pointwise projectively related. We show that on constant isotropic Berwald manifolds the notions of R-quadratic and stretch metrics are equivalent. Then we prove that every complete generalized Landsberg manifold with isotropic Berwald curvature reduces to a Berwald manifold. Finally, we study C-conformal changes of isotropic Berwald metrics.

1. Introduction. For a Finsler metric $F = F(x, y)$ on a smooth manifold M, geodesic curves are characterized by the system of second order differential equations

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where the local functions $G^i = G^i(x, y)$ are called the spray coefficients, and given by $G^i = \frac{1}{4}g^{il}\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\}$. In standard local coordinates (x^i, y^i) in TM, the vector field $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$ is called the spray of F [Sh1].

A Finsler metric F is called a Berwald metric if G^i are quadratic in $y \in T_xM$ for any $x \in M$ or equivalently the Berwald curvature ${B^i}_{jkl}$ vanishes. It is proved that on a Berwald manifold (M, F), the parallel translation along any geodesic preserves the Minkowski functionals. Thus, Berwald spaces can be viewed as Finsler spaces modeled on a single Minkowski space.

A Finsler metric F satisfying $F_{x^k} = FF_{y^k}$ is called a Funk metric. The standard Funk metric on the Euclidean unit ball $B^n(1)$ is denoted by Θ and defined by

$$\Theta(x, y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - |x|^2}, \qquad y \in T_xB^n(1) \simeq \mathbb{R}^n,$$

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where $\langle\, ,\rangle$ and $|\cdot|$ denote the Euclidean inner product and norm on $\mathbb{R}^n$, respectively. Chen–Shen introduce the notion of isotropic Berwald metrics [CS]. A Finsler metric F is said to be an isotropic Berwald metric if its Berwald curvature is of the form

$$B^i{}_{jkl} = c\{F_{y^jy^k}\delta^i{}_l + F_{y^ky^l}\delta^i{}_j + F_{y^ly^j}\delta^i{}_k + F_{y^jy^ky^l}y^i\} \tag{1.1}$$

for some scalar function $c = c(x)$ on M. Berwald metrics are trivially isotropic Berwald metrics with $c = 0$. Funk metrics are also non-trivial isotropic Berwald metrics.

The Riemann curvature $\mathbf{R}_y = R^i{}_k dx^k \otimes \frac{\partial}{\partial x^i}\big|_x : T_xM \to T_xM$ is a family of linear maps on tangent spaces, defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{1.2}$$

A Finsler metric F is said to be of scalar flag curvature if for some scalar function K on TM_0 the Riemann curvature has the form

$$R^i{}_k = KF^2\{\delta^i_k - F^{-1}F_{y^k}y^i\}. \tag{1.3}$$

If $K =$ const, then F is said to be of constant flag curvature. We prove the following rigidity theorem on isotropic Berwald manifolds.

THEOREM 1.1. *Let (M, F) be an isotropic Berwald manifold with dimension greater than two. Suppose that F is of scalar flag curvature. Then F is a Randers metric.*

Two regular metrics on a manifold are said to be pointwise projectively related if they have the same geodesics as point sets. It is well known that two Finsler metrics F and $\bar{F}$ are projectively equivalent if and only if $G^i = \bar{G}^i + Py^i$, where G^i and $\bar{G}^i$ are the the spray coefficients of F and $\bar{F}$, respectively, and $P = P(x, y)$ is positively y-homogeneous of degree one.

THEOREM 1.2. *Let F be an isotropic Berwald metric which is pointwise projectively related to a Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$. Then $\bar{F}$ has isotropic S-curvature if and only if it has isotropic Berwald curvature.*

The class of constant isotropic Berwald metrics, which includes Funk metrics, is a rich class of Finsler metrics. Hence, it is of interest to study constant isotropic Berwald metrics.

THEOREM 1.3. *Let (M, F) be a constant isotropic Berwald manifold. Then F is an R-quadratic metric if and only if F is a stretch metric.*

We will prove that on a complete non-Riemannian generalized Landsberg manifold there is no isotropic Berwald metric but the trivial one.

THEOREM 1.4. *Let (M, F) be a complete generalized Landsberg manifold. Suppose that F has isotropic Berwald curvature. Then F reduces to a Berwald metric.*

Finally, we study C-conformal transformations of isotropic Berwald metrics and prove the following.

Theorem 1.5. *Let F and $\bar{F}$ be two isotropic Berwald metrics. Suppose that there is a C-conformal transformation between them. Then F and $\bar{F}$ reduce to a Berwald metric.*

2. Preliminaries. Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \bigcup_{x\in M} T_xM$ the tangent bundle of M, and by $TM_0 = TM\backslash\{0\}$ the slit tangent bundle of M. A *Finsler metric* on M is a function $F : TM \to [0,\infty)$ which has the following properties: (i) F is C^∞ on TM_0; (ii) F is positively 1-homogeneous on the fibers of the tangent bundle of M, and (iii) for each $y \in T_xM$, the quadratic form g_y on T_xM is positive definite, where

$$g_y(u,v) := \frac{1}{2}\,\frac{\partial^2}{\partial s\partial t}[F^2(y+su+tv)]\Big|_{s,t=0}, \qquad u,v \in T_xM.$$

Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric, and $\beta = b_i(x)y^i$ be a 1-form on M with $\|\beta\| = \sqrt{a^{ij}b_ib_j} < 1$. The Finsler metric $F = \alpha + \beta$ is called a *Randers metric*; it has important applications both in mathematics and physics [Ra].

Let $x \in M$ and $F_x := F|_{T_xM}$. We define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\,\frac{d}{dt}[g_{y+tw}(u,v)]\Big|_{t=0}, \qquad u,v,w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the *Cartan torsion*. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian [Sh1]. For $y \in T_xM_0$, define the *mean Cartan torsion* $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$. By Deicke's theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$ [Sh1].

Let (M,F) be a Finsler manifold. For $y \in T_xM_0$, define the *Matsumoto torsion* $\mathbf{M}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{M}_y(u,v,w) := M_{ijk}(y)u^iv^jw^k$, where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\},$$

and $h_{ij} := g_{ij} - \frac{1}{F^2}g_{ip}y^pg_{jq}y^q$ is the angular metric. A Finsler metric F is said to be *C-reducible* if $\mathbf{M}_y = 0$ [M]. Matsumoto proved that every Randers metric satisfies $\mathbf{M}_y = 0$. Later on, Matsumoto–Hōjō also proved the converse.

Lemma 2.1 ([MH]). *A Finsler metric F on a manifold of dimension greater than two is a Randers metric if and only if its Matsumoto torsion vanishes.*

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the *Landsberg curvature* $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ defined by $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^iv^jw^k$, where $L_{ijk} := C_{ijk|s}y^s$. A Finsler metric is called a *Landsberg metric* if $\mathbf{L} = 0$. Using the notion of Landsberg curvature, Berwald defined the *stretch curvature* $\mathbf{\Sigma}_y : T_xM \otimes T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{\Sigma}_y(u, v, w, z) := \Sigma_{ijkl}u^iv^jw^kz^l$, where $\Sigma_{ijkl} := 2(L_{ijk|l} - L_{ijl|k})$. A Finsler metric satisfying $\mathbf{\Sigma} = 0$ is called a *stretch metric* [Be].

For $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}u^jv^kw^l \frac{\partial}{\partial x^i}\big|_x$ and $\mathbf{E}_y(u, v) := E_{jk}u^jv^k$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \qquad E_{jk} := \frac{1}{2}B^m{}_{jkm}.$$

$\mathbf{B}$ and $\mathbf{E}$ are called the *Berwald curvature* and *mean Berwald curvature*, respectively. Then F is called a *Berwald* or *weakly Berwald metric* if $\mathbf{B} = 0$ or $\mathbf{E} = 0$ [Sh1], [TP] .

Define $\mathbf{D}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by

$$\mathbf{D}_y(u, v, w) := D^i{}_{jkl}u^iv^jw^k \frac{\partial}{\partial x^i}\bigg|_x$$

where

$$D^i{}_{jkl} := B^i{}_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk,l}y^i\}.$$

We call $\mathbf{D} := \{\mathbf{D}_y\}_{y\in TM_0}$ the *Douglas curvature*. A Finsler metric with $\mathbf{D} = 0$ is called a *Douglas metric*. The notion of Douglas metric was first proposed by Bácsó–Matsumoto as a generalization of Berwald metric [BM1], [BM2].

Let

$$\tau(x, y) := \ln\left[\frac{\sqrt{\det g_{ij}(x, y)}}{\mathrm{Vol}(\mathrm{B}^n(1))} \cdot \mathrm{Vol}\left\{(y^i) \in \mathbb{R}^n \;\middle|\; F\left(y^i \frac{\partial}{\partial x^i}\bigg|_x\right) < 1\right\}\right].$$

Then $\tau = \tau(x, y)$ is a scalar function on TM_0, called the *distortion* [Sh1]. For a vector $\mathbf{y} \in T_xM$, let $c(t), -\epsilon < t < \epsilon$, denote the geodesic with $c(0) = x$ and $\dot{c}(0) = \mathbf{y}$. Define

$$\mathbf{S}(\mathbf{y}) := \frac{d}{dt}[\tau(\dot{c}(t))]\bigg|_{t=0}.$$

We call $\mathbf{S}$ the *S-curvature*. This quantity was first introduced by Shen for a volume comparison theorem [Sh1], [Sh2].

Lemma 2.2 (Rapcsák [Rap]). *Let $F(x, y)$ be a Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$. Then F is projectively flat on $\mathcal{U}$ if and only if $F_{x^ky^l}y^k = F_{x^l}$. In this case, the projective factor $P(x, y)$ is given by*

$$P = \frac{F_{x^k}y^k}{2F}. \tag{2.1}$$

Much earlier, in [Ha], G. Hamel proved that a Finsler metric F on $\mathcal{U} \subset \mathbb{R}^n$ is projectively flat if and only if $F_{x^k} = F_{x^m y^k} y^m$. Let F be a projectively flat Finsler metric on $\mathcal{U} \subset \mathbb{R}^n$ and $P(x,y)$ be its projective factor. Put

$$\Xi := P^2 - P_{x^k} y^k. \tag{2.2}$$

Plugging $G^i = Py^i$ into (1.2) yields $R^i{}_k = \Xi \delta^i_k + \tau_k y^i$, where $\tau_k = 3(P_{x^k} - PP_{y^k}) + \Xi_{y^k}$. It is well known that $g_{ji} R^i{}_k = g_{ki} R^i{}_j$. Then (1.3) holds with

$$\mathbf{K} = \frac{\Xi}{F^2} = \frac{P^2 - P_{x^k} y^k}{F^2}. \tag{2.3}$$

There are many connections in Finsler geometry [BT1], [BT2], [TAE], [TN]. In this paper, we use the Berwald connection, and the h- and v-covariant derivatives of a Finsler tensor field are denoted by " | " and " , " respectively.

3. Proof of Theorem 1.1.

First, we recall the following.

Lemma 3.1 ([NBT]). *For the Berwald connection, the following Bianchi identities hold:*

$$R^i{}_{jkl|m} + R^i{}_{jlm|k} + R^i{}_{jmk|l} = B^i{}_{jku} R^u{}_{lm} + B^i{}_{jlu} R^u{}_{km} + B^i{}_{klu} R^u{}_{jm}, \tag{3.1}$$

$$B^i{}_{jml|k} - B^i{}_{jkm|l} = R^i{}_{jkl,m}, \tag{3.2}$$

$$B^i{}_{jkl,m} = B^i{}_{jkm,l}, \tag{3.3}$$

where $R^i{}_{jkl}$ is the Riemannian curvature of the Berwald connection and $R^i{}_{kl} := \ell^j R^i{}_{jkl}$.

Here, we deal with isotropic Berwald manifolds of scalar flag curvature and prove the following.

Lemma 3.2. *Let (M,F) be an isotropic Berwald manifold with scalar flag curvature K. Then*

$$KC_{jlm} = a_j h_{lm} + a_l h_{mj} + a_m h_{jl} + a_0 F F_{y^j y^l y^m}, \tag{3.4}$$

where $a_j = -\frac{1}{3} K_{,j} + \frac{1}{2} c_0 F^{-2} F_j$, $K_{,j} = \partial K / \partial y^j$, $c_0 = c_{|i} y^i$ and $a_0 = a_i y^i$.

Proof. We have

$$R^i{}_{jkl} = \frac{1}{3} \left\{ \frac{\partial^2 R^i{}_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i{}_l}{\partial y^j \partial y^k} \right\}. \tag{3.5}$$

By assumption, F is of scalar curvature $K = K(x,y)$, which is equivalent to

$$R^i{}_k = KF^2 h^i_k. \tag{3.6}$$

Plugging (3.6) into (3.5) gives

$$R^i{}_{jkl} = \tfrac{1}{3}F^2\{K_{,j,l}h^i_k - K_{,j,k}h^i_l\} + K_{,j}\{FF_{y^l}h^i_k - FF_{y^k}h^i_l\} + \tfrac{1}{3}K_{,k}\{2FF_{y^j}\delta^i_l - g_{jl}y^i - FF_{y^l}\delta^i_j\} + K\{g_{jl}\delta^i_k - g_{jk}\delta^i_l\} + \tfrac{1}{3}K_{,l}\{2FF_{y^j}\delta^i_k - g_{jk}y^i - FF_{y^k}\delta^i_j\}. \tag{3.7}$$

Differentiating (3.7) with respect to y^m gives a formula for $R^i{}_{jkl,m}$ expressed in terms of K and its derivatives. Contracting (3.2) with y^k, we obtain

$$B^i{}_{jml|k}y^k = 2KC_{jlm}y^i - \tfrac{1}{3}K_{,j}\{FF_{y^l}\delta^i_m + FF_{y^m}\delta^i_l - 2g_{lm}y^i\} - \tfrac{1}{3}K_{,l}\{FF_{y^j}\delta^i_m + FF_{y^m}\delta^i_j - 2g_{jm}y^i\} - \tfrac{1}{3}K_{,m}\{FF_{y^j}\delta^i_l + FF_{y^l}\delta^i_j - 2g_{jl}y^i\} - \tfrac{1}{3}F^2\{K_{,j,m}h^i_l + K_{,j,l}h^i_m + K_{,l,m}h^i_j\}. \tag{3.8}$$

Since F has isotropic Berwald curvature, we have

$$B^i{}_{jml|k}y^k = c_0\{F_{y^jy^m}\delta^i{}_l + F_{y^my^l}\delta^i{}_j + F_{y^ly^j}\delta^i{}_m + F_{y^jy^my^l}y^i\}. \tag{3.9}$$

By (3.8) and (3.9), it follows that

$$2KFC_{jlm} = -\tfrac{2}{3}F^2\{F_{y^ly^m}K_{,j} + K_{,l}F_{y^jy^m} + K_{,m}F_{y^ly^j}\} + c_0\{F_{y^jy^m}F_{y^l} + F_{y^my^l}F_{y^j} + F_{y^ly^j}F_{y^m} + FF_{y^jy^ly^m}\}. \tag{3.10}$$

This implies that

$$KC_{jlm} = a_jh_{lm} + a_lh_{mj} + a_mh_{jl} + \tfrac{1}{2}c_0F_{y^jy^ly^m}, \tag{3.11}$$

where $a_j = -\frac{1}{3}K_{,j} + \frac{1}{2}c_0F^{-2}F_{y^j}$. By contracting (3.11) with y^jg^{lm}, we conclude that $c_0 = 2Fa_0$. This completes the proof. ■

Proof of Theorem 1.1. Every isotropic Berwald metric is a Douglas metric [CS]. By assumption, F is of scalar flag curvature. Therefore, F is a projectively flat Finsler metric. Let P be the projective factor of F. Contracting i and l in (1.1), we get

$$E_{jk} = \tfrac{1}{2}(n+1)cF_{y^jy^k}. \tag{3.12}$$

On the other hand, for a projectively flat Finsler metric F, we have

$$B^i{}_{jkl} = P_{y^jy^ky^l}y^i + P_{y^jy^k}\delta^i_l + P_{y^jy^l}\delta^i_k + P_{y^ly^k}\delta^i_j,$$

which implies that

$$E_{jk} = \tfrac{1}{2}(n+1)P_{y^jy^k}. \tag{3.13}$$

Comparing (3.12) and (3.13), we have

$$P = cF + q, \tag{3.14}$$

where $q = q_i(x)y^i$ is a 1-form on M. By (2.1) and (3.14), we get

$$F_{x^i}y^i = 2FP = 2F\{cF + q_iy^i\}. \tag{3.15}$$

Plugging (3.14) into (2.3) and using (3.15), one can obtain

$$(3.16)\qquad K = \frac{(cF+q)^2 - (c_0 F + cF_{x^i}y^i + (q_i)_{x^j}y^iy^j)}{F^2} = \frac{-2c^2F^2 - 2c_0F + (2q_iq_j - (q_i)_{x^j} - (q_j)_{x^i})y^iy^j}{2F^2}.$$

Since F is an isotropic mean Berwald metric, i.e., the **E**-curvature of F is

$$\mathbf{E} = \frac{n+1}{2}cF^{-1}\mathbf{h},$$

applying Theorem 1.1 in [NBT] implies that the flag curvature is

$$(3.17)\qquad K = 3c_0/F + \sigma,$$

where $\sigma = \sigma(x)$ is a scalar function on M. Inserting (3.16) into (3.17), we obtain the quadratic equation

$$(3.18)\qquad 2(\sigma + c^2)F^2 + 8c_0F - (2q_iq_j - (q_i)_{x^j} - (q_j)_{x^i})y^iy^j = 0.$$

Let $K \neq -c^2 + c_0/F$. By (3.17), this assumption is the same as

$$(3.19)\qquad \sigma + c^2 + 2c_0/F \neq 0.$$

From (3.18), (3.19) and regularity of F, we conclude that

$$\sigma + c^2 \neq 0.$$

Solving (3.18) for F, we get

$$(3.20)\qquad F = \frac{\sqrt{2(\sigma + c^2)(2q_iq_j - (q_i)_{x^j} - (q_j)_{x^i})y^iy^j + 16c_0^2} - 4c_0}{2(\sigma + c^2)}.$$

This means that F is a Randers metric.

Now suppose that $K = -c^2 + c_0/F$. Then by (3.17), we get

$$(3.21)\qquad \sigma + c^2 + 2c_0/F \equiv 0.$$

By (3.21), it follows that $c =$ const and so $\sigma = -c^2$ is a constant and $K = -c^2$. By Lemma 3.2, we have

$$(3.22)\qquad C_{jlm} = b_jh_{lm} + b_lh_{mj} + b_mh_{jl},$$

where $b_j := -\frac{1}{3K}K_{,j}$. Contracting (3.22) with g^{jl} yields

$$(3.23)\qquad b_m = \frac{1}{n+1}I_m.$$

Substituting (3.23) into (3.22) implies that F is C-reducible. By Lemma 2.1, F is a Randers metric of constant flag curvature $K = -c^2$. If $c = 0$, then by (1.1), F is a Berwald metric with $K = 0$. It is well known that every Berwald metric with $K = 0$ is locally Minkowskian. ∎

4. Proof of Theorem 1.2

Proposition 4.1. *Let F and $\bar{F}$ be two Finsler metrics on a manifold M such that F is pointwise projectively related to $\bar{F}$. Suppose that F has isotropic Berwald curvature. Then the following are equivalent:*

- *$\bar{F}$ has isotropic mean Berwald curvature,*
- *$\bar{F}$ has isotropic Berwald curvature.*

Proof. By assumption F has isotropic Berwald curvature

$$B^i{}_{jkl} = c\{F_{y^jy^k}\delta^i{}_l + F_{y^ky^l}\delta^i{}_j + F_{y^ly^j}\delta^i{}_k + F_{y^jy^ky^l}y^i\}, \tag{4.1}$$

where $c = c(x)$ is a scalar function on M. Hence F has isotropic mean Berwald curvature. The same is true for $\bar{F}$. Therefore, it suffices to prove the converse. Suppose that $\bar{F}$ has isotropic mean Berwald curvature

$$\bar{E}_{ij} = \frac{n+1}{2}\bar{c}\bar{F}_{y^iy^j}, \tag{4.2}$$

where $\bar{c} = \bar{c}(x)$ is a scalar function on M. By assumption we have

$$\bar{c}\bar{F}_{y^iy^j} = cF_{y^iy^j} + P_{y^iy^j}. \tag{4.3}$$

By (4.3) we get

$$\bar{c}\bar{F}_{y^iy^jy^k} = cF_{y^iy^jy^k} + P_{y^iy^jy^k}. \tag{4.4}$$

On the other hand,

$$\bar{B}^i{}_{jkl} = B^i{}_{jkl} + \{P_{y^jy^k}\delta^i_l + P_{y^ky^l}\delta^i_j + P_{y^ly^j}\delta^i_k + P_{y^jy^ky^l}y^i\}. \tag{4.5}$$

From (4.1) and (4.5), it follows that

$$\begin{aligned}\bar{B}^i{}_{jkl} = {} & c\{F_{y^jy^k}\delta^i{}_l + F_{y^ky^l}\delta^i{}_j + F_{y^ly^j}\delta^i{}_k + F_{y^jy^ky^l}y^i\} \\ & + \{P_{y^jy^k}\delta^i_l + P_{y^ky^l}\delta^i_j + P_{y^ly^j}\delta^i_k + P_{y^jy^ky^l}y^i\}.\end{aligned} \tag{4.6}$$

Putting (4.2)–(4.4) into (4.6) yields

$$\bar{B}^i{}_{jkl} = \bar{c}\{\bar{F}_{y^jy^k}\delta^i{}_l + \bar{F}_{y^ky^l}\delta^i{}_j + \bar{F}_{y^ly^j}\delta^i{}_k + \bar{F}_{y^jy^ky^l}y^i\}. \tag{4.7}$$

This means that $\bar{F}$ has isotropic Berwald curvature. ∎

Lemma 4.2. *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold M. Then the following are equivalent:*

- *F has isotropic S-curvature $\mathbf{S} = (n+1)cF$,*
- *F has isotropic mean Berwald curvature $\mathbf{E} = \frac{n+1}{2}cF^{-1}\mathbf{h}$,*

where $c = c(x)$ is a scalar function on M.

Proof of Theorem 1.2. Apply Proposition 4.1 and Lemma 4.2. ∎

5. Proof of Theorem 1.3. In [Sh3], Shen introduces the notion of R-quadratic Finsler metrics as a new family of Finsler metrics including Berwald metrics and R-flat metrics. A Finsler space is said to be *R-quadratic* if its Riemann curvature R_y is quadratic in $y \in T_xM$ [NBT]. Indeed, a Finsler metric is R-quadratic if and only if the h-curvature of the Berwald connection depends on position only in the sense of Bácsó–Matsumoto [BM3]. In [Sh3], Shen proves that every compact R-quadratic manifold is a Landsberg manifold.

Lemma 5.1. *Every R-quadratic Finsler metric is a stretch metric.*

Proof. The following Bianchi identity holds:

$$R^i{}_{jkl,m} = B^i{}_{jml|k} - B^i{}_{jkm|l}, \tag{5.1}$$

Contracting (5.1) with y_i yields

$$\begin{aligned} y_iR^i{}_{jkl,m} &= y_iB^i{}_{jml|k} - y_iB^i{}_{jkm|l} = (y_iB^i{}_{jml})_{|k} - (y_iB^i{}_{jkm})_{|l} \\ &= -2L_{jml|k} + 2L_{jkm|l} = \Sigma_{jkml}. \end{aligned} \tag{5.2}$$

Therefore, every R-quadratic Finsler metric is a stretch metric. ∎

It is interesting to find conditions under which the notions of R-quadratic curvature and stretch curvature coincide. In [Sh1], Shen finds a new non-Riemannian quantity for Finsler metrics, called $\bar{\mathbf{E}}$-*curvature*, which is closely related to $\mathbf{E}$-curvature. For any tangent vector $y \in T_xM_0$, define $\bar{\mathbf{E}}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\bar{\mathbf{E}}_y(u,v,w) := \bar{E}_{jkl}(y)u^iv^jw^k$, where $\bar{E}_{ijk} := E_{ij|k}$. It is easy to see that if $\bar{\mathbf{E}} = 0$, then $\mathbf{E}$-curvature is covariantly constant along all horizontal directions on TM_0 [Sh1].

Proposition 5.2. *Let (M,F) be a Douglas manifold. Suppose that $\bar{\mathbf{E}} = 0$. Then F is an R-quadratic metric if and only if F is a stretch metric.*

Proof. By Lemma 5.1, it is sufficient to prove the converse implication. Let F be a Douglas metric, i.e.,

$$B^i{}_{jkl} = \frac{2}{n+1}\{E_{jk}\delta^i{}_l + E_{kl}\delta^i{}_j + E_{lj}\delta^i{}_k + E_{jk,l}y^i\}. \tag{5.3}$$

By contracting (5.3) with h^m_i and using $y_iB^i{}_{jkl} = -2L_{jkl}$, we get

$$B^m_{jkl} = -\frac{2}{F^2}y^mL_{jkl} + \frac{2}{n+1}\{E_{jk}h^m_l + E_{kl}h^m_j + E_{lj}h^m_k\}. \tag{5.4}$$

Taking a horizontal derivative of (5.4) yields

$$B^m_{jkl|h} = -\frac{2}{F^2}y^mL_{jkl|h} + \frac{2}{n+1}\{E_{jk|h}h^m_l + E_{kl|h}h^m_j + E_{lj|h}h^m_k\}. \tag{5.5}$$

Similarly, we have

$$B^m_{jkh|l} = -\frac{2}{F^2}y^mL_{jkh|l} + \frac{2}{n+1}\{E_{jk|l}h^m_h + E_{kh|l}h^m_j + E_{hj|l}h^m_k\}. \tag{5.6}$$

Subtracting (5.6) from (5.5) implies that

$$B^m{}_{jkl|h} - B^m{}_{jkh|l} = -\frac{1}{F^2}y^m \Sigma_{jklh} + \frac{2}{n+1}\{E_{jk|h}h^m{}_l - E_{jk|l}h^m{}_h\} + \frac{2}{n+1}\{(E_{kl|h} - E_{kh|l})h^m{}_j + (E_{lj|h} - E_{hj|l})h^m{}_k\}. \tag{5.7}$$

Since $E_{ij|k} = 0$ and $\Sigma_{ijkl} = 0$, by (5.7) we conclude that

$$B^m{}_{jkl|h} - B^m{}_{jkh|l} = 0. \tag{5.8}$$

This means that F is R-quadratic. ∎

Proof of Theorem 1.3. In [CS], it is proved that every isotropic Berwald metric is a Douglas metric. By assumption F has constant isotropic Berwald curvature

$$B^i{}_{jkl} = c\{F_{y^jy^k}\delta^i{}_l + F_{y^ky^l}\delta^i{}_j + F_{y^ly^j}\delta^i{}_k + F_{y^jy^ky^l}y^i\} \tag{5.9}$$

for some real constant $c \in \mathbb{R}$. It follows that $E_{ij|k} = 0$. Hence, Proposition 5.2 completes the proof. ∎

6. Proof of Theorem 1.4. One can see that a Finsler metric is a Landsberg metric if and only if the Berwald connection coincides with the Chern connection. With this characterization of the Landsberg manifolds in mind, we may introduce a new class of Finsler manifolds, as follows. We have

$$R^i{}_{jkl} = H^i{}_{jkl} + [L^i{}_{jl|k} - L^i{}_{jk|l} + L^i{}_{sk}L^s{}_{jl} - L^i{}_{sl}L^s{}_{jk}], \tag{6.1}$$

where R and H denote the Riemannian curvatures of Berwald and Chern connections, respectively. We say that a Finsler metric F is a *generalized Landsberg metric* if $R = H$. By definition, we then have

$$L^i{}_{jl|k} - L^i{}_{jk|l} + L^i{}_{sk}L^s{}_{jl} - L^i{}_{sl}L^s{}_{jk} = 0. \tag{6.2}$$

It is easy to see that every Landsberg manifold is a generalized Landsberg manifold.

LEMMA 6.1 ([TP]). *Let (M, F) be a Finsler manifold. Then F is a generalized Landsberg metric if and only if*

$$L_{isk}L^s{}_{jl} - L_{isl}L^s{}_{jk} = 0, \tag{6.3}$$

$$L_{ijl|k} - L_{ijk|l} = 0. \tag{6.4}$$

Let (M, F) be a Landsberg manifold. Suppose that F has isotropic Berwald curvature (1.1). Then F has isotropic Landsberg curvature $\mathbf{L} + cF\mathbf{C} = 0$, which implies that $\mathbf{C} = 0$ or $c = 0$. In each case, F reduces to a Berwald metric. Summarizing we have the following.

COROLLARY 6.2. *Let (M, F) be a Landsberg manifold. Suppose that F has isotropic Berwald curvature. Then F reduces to a Berwald metric.*

It is interesting to find conditions under which a generalized Landsberg metric reduces to a Berwald metric. In Theorem 1.4, we prove that every complete generalized Landsberg manifold with isotropic Berwald curvature is a Berwald manifold.

Proof of Theorem 1.4. Contracting (1.1) with y_i and using

$$y_i B^i{}_{jkl} = -2L_{jkl}$$

imply that F is of isotropic Landsberg curvature

$$L_{ijk} + cFC_{ijk} = 0,$$

which yields

$$L_{ijk|l}y^l = (c^2F^2 - c_0F)C_{ijk}. \tag{6.5}$$

Contracting (6.4) with y^l implies that

$$L_{ijk|l}y^l = 0. \tag{6.6}$$

By (6.5) and (6.6), we have

$$(c^2F^2 - c_0F)C_{ijk} = 0.$$

If $C_{ijk} = 0$, then F is a Riemannian metric which is a special Berwald metric. Let F be a non-Riemannian generalized Landsberg metric. Then

$$c^2F - c_0 = 0. \tag{6.7}$$

Considering this equation on the indicatrix, we get

$$c(t) = -\frac{1}{t+b}, \tag{6.8}$$

where b is a constant real number. Assume that (M, F) is complete. Then, letting $t \to \pm\infty$, we conclude that $c = 0$, which implies that F is a Berwald metric. ■

7. Proof of Theorem 1.5. Besides Randers changes, we have another class of special transformations, named C-conformal transformations. The notion of C-conformal transformation and its properties were studied by Hashiguchi [H]. A C-conformal transformation is a conformal transformation satisfying a condition on the Cartan tensor and the conformal factor.

Two Finsler metrics F and $\bar{F}$ on M are called *conformal* if $\bar{g}_{ij} = \varphi g_{ij}$, where φ is a positive scalar function on TM. Indeed, by Knebelman's theorem φ depends only on position hence it can be considered as a function on M. Thus we can assume $\varphi = e^{2\alpha}$, where α is a scalar function on M. If φ is a constant, F and $\bar{F}$ are called *homothetic*. Put

$$\alpha_i = \frac{\partial \alpha}{\partial x^i}, \quad C^i_j := C^{ir}_j \alpha_r, \quad \alpha_0 = \alpha_i y^i.$$

Then

$$\bar{F} = e^{\alpha}F \quad \text{and} \quad \bar{g}^{ij} = e^{-2\alpha}g^{ij}.$$

Two Finsler metrics F and $\bar{F}$ on M are called *C-conformal* if they are not homothetic and the equations $C^i_j = 0$ hold [H].

Finally, we study C-conformal transformations of isotropic Berwald curvature metrics and prove Theorem 1.5.

Proof of Theorem 1.5. Let F and $\bar{F}$ be two isotropic Berwald metrics,

$$B^i{}_{jkl} = c\{E_{ij}\delta^i{}_l + E_{kl}\delta^i{}_j + E_{jl}\delta^i{}_k + E_{jk,l}y^i\}, \tag{7.1}$$

$$\bar{B}^i{}_{jkl} = \bar{c}\{\bar{E}_{ij}\delta^i{}_l + \bar{E}_{kl}\delta^i{}_j + \bar{E}_{jl}\delta^i{}_k + \bar{E}_{jk,l}y^i\}, \tag{7.2}$$

where $c = c(x)$ and $\bar{c} = \bar{c}(x)$ are scalar functions on M. Since there exists a C-conformal change between F and $\bar{F}$, we have

$$\bar{B}^i{}_{jkl} = B^i{}_{jkl} - C_{jkl}\alpha^i, \tag{7.3}$$

where $\alpha^i = g^{ij}\alpha_j$. Contracting i and j in (7.3) yields

$$\bar{E}_{kl} = E_{kl}. \tag{7.4}$$

By (7.1)–(7.4) we have

$$(c - \bar{c})\{E_{ij}\delta^i{}_l + E_{kl}\delta^i{}_j + E_{jl}\delta^i{}_k + E_{jk,l}y^i\} = C_{jkl}\alpha^i. \tag{7.5}$$

Contracting i and l in (7.5) implies that

$$(n+1)(c - \bar{c})E_{ij} = 0. \tag{7.6}$$

If $c = \bar{c}$, then by (7.5) we conclude that $C_{jkl}\alpha^i = 0$, which implies that $C_{jkl} = 0$ and F is Riemannian. If $E_{ij} = 0$, then by (7.4) we have $\bar{E}_{ij} = 0$. Thus by (7.1) and (7.2), F and $\bar{F}$ reduce to Berwald metrics. ∎

We know that every Funk metric has isotropic Berwald curvature. Then by Theorem 1.5, we get the following.

Corollary 7.1. *There is no C-conformal change between two Funk metrics.*

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