ANNALES
POLONICI MATHEMATICI
103.2 (2012)

Comparison of explicit and implicit difference schemes for parabolic functional differential equations

by Zdzisław Kamont and Karolina Kropielnicka (Gdańsk)

Abstract. Initial-boundary value problems of Dirichlet type for parabolic functional differential equations are considered. Explicit difference schemes of Euler type and implicit difference methods are investigated. The following theoretical aspects of the methods are presented. Sufficient conditions for the convergence of approximate solutions are given and comparisons of the methods are presented. It is proved that the assumptions on the regularity of the given functions are the same for both methods. It is shown that the conditions on the mesh for explicit difference schemes are more restrictive than the suitable assumptions for implicit methods. There are implicit difference schemes which are convergent while the corresponding explicit difference methods are not convergent. Error estimates for both methods are constructed.

1. Introduction. Difference methods for parabolic functional differential equations (with initial-boundary conditions) are obtained by replacing partial derivatives with difference operators. Moreover, since differential equations contain functional variables which are elements of the set of continuous functions defined on a subset of a finite-dimensional space, some interpolating operators are needed. This leads to functional difference equations of Volterra type. The stability of functional difference schemes is investigated by using comparison techniques.

In recent years, a number of papers concerning explicit difference methods for parabolic problems have been published. Difference approximations of nonlinear equations with initial-boundary conditions of Dirichlet type were considered in [8], [10], [22]. Numerical treatment of functional differential equations with initial-boundary conditions of Neumann type can be found in [3], [12]. The convergence results for a general class of difference schemes related to parabolic problems with solutions defined on unbounded domains can be found in [16], [29]. The papers [4], [6], [9], [13], [14], [17] concern

2010 *Mathematics Subject Classification*: Primary 35R10; Secondary 65M12.
Key words and phrases: functional differential equations, stability and convergence, comparison methods, differential and difference inequalities.

DOI: 10.4064/ap103-2-3

implicit difference schemes for parabolic problems. Error estimates implying the convergence of implicit difference schemes are obtained in those papers by using difference inequalities or simple theorems on recurrent inequalities.

Monotone iterative methods and implicit difference schemes for computing approximate solutions to parabolic equations with time delay were analyzed in [15], [18], [19], [30].

All the above results on numerical methods for parabolic functional differential equations have the following property: the authors have assumed that solutions of initial-boundary value problems are defined on intervals $[0,a] \times [-c,c] \subset \mathbb{R}^{1+n}$. It is clear that the sets of the form $[0,a] \times Q$ where $Q \subset \mathbb{R}^n$ is a bounded set are natural domains on which solutions of mixed problems for parabolic functional differential equations are considered. We start investigations of difference schemes for nonlinear parabolic functional equations with solutions defined on $[0,a] \times Q$.

Note that cylindrical domains appear in theoretical results on the existence and uniqueness of solutions to parabolic functional differential problems ([11], [20], [26], [27]).

In this paper we investigate theoretical questions for explicit and implicit difference schemes generated by initial-boundary value problems of Dirichlet type for functional differential equations with solutions considered on cylindrical domains.

Now we formulate our functional differential problems. For any metric spaces X and Y we denote by $\mathbf{C}(X,Y)$ the class of all continuous functions from X into Y. We use vectorial inequalities to mean that the same inequalities hold between their corresponding components.

Let $Q \subset \mathbb{R}^n$ be a bounded domain with the boundary ∂Q. Write $E = [0,a] \times \overline{Q}$, $E_0 = [-b_0, 0] \times \overline{Q}$ where $\overline{Q}$ is the closure of Q. For each $(t,x) \in E$ we define

$$D[t,x] = \{(\tau,y) \in \mathbb{R}^{1+n} : \tau \le 0,\ (t+\tau, x+y) \in E_0 \cup E\}.$$

For a function $z : E_0 \cup E \to \mathbb{R}$ and $(t,x) \in E$ we define a function $z_{(t,x)} : D[t,x] \to \mathbb{R}$ by $z_{(t,x)}(\tau,y) = z(t+\tau, x+y)$ for $(\tau,y) \in D[t,x]$. Thus $z_{(t,x)}$ is the restriction of z to the set $(E_0 \cup E) \cap ([-b_0,t] \times \mathbb{R}^n)$, shifted to the set $D[t,x]$.

There is $[c,d] \subset \mathbb{R}^n$ such that $D[t,x] \subset [-b_0 - a, 0] \times [c,d]$ for $(t,x) \in E$. Write $I = [-b_0 - a, 0]$, $B = I \times [c,d]$. Let $M_{n\times n}$ be the class of all $n \times n$ symmetric matrices with real elements. Set $\Omega = E \times \mathbf{C}(B,\mathbb{R}) \times \mathbb{R}^n \times M_{n\times n}$, $\partial_0 E = [0,a] \times \partial Q$ and suppose that $F : \Omega \to \mathbb{R}$, $\varphi : E_0 \cup \partial_0 E \to \mathbb{R}$ are given functions. Let z be an unknown function of the variables (t,x), $x = (x_1,\dots,x_n)$. We consider the problem consisting of the functional differential equation

$$\partial_t z(t,x) = F(t,x,z_{(t,x)}, \partial_x z(t,x), \partial_{xx} z(t,x)) \tag{1.1}$$

and the initial-boundary condition

$$z(t,x) = \varphi(t,x) \quad \text{on } E_0 \cup \partial_0 E \tag{1.2}$$

where $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$, $\partial_{xx} = [\partial_{x_i x_j} z]_{i,j=1}^n$. We will say that F satisfies *condition* (V) if for each $(t,x,q,s) \in E \times \mathbb{R}^n \times M_{n\times n}$ and for $w, \tilde{w} \in \mathbf{C}(B,\mathbb{R})$ such that $w(\tau,y) = \tilde{w}(\tau,y)$ for $(\tau,y) \in D[t,x]$ we have $F(t,x,w,q,s) = F(t,x,\tilde{w},q,s)$. Condition (V) means that the value of F at $(t,x,w,q,s) \in \Omega$ depends on (t,x,q,s) and on the restriction of w to the set $D[t,x]$ only. We assume that F satisfies condition (V) and we consider classical solutions to (1.1), (1.2).

We prove that there are explicit and implicit difference schemes for (1.1), (1.2) which are convergent, and we compare them.

Two types of of assumptions are needed in theorems on the convergence of functional difference problems generated by (1.1), (1.2). The first type conditions concern the regularity of F. Assumptions on the regularity of the given functions are the same for explicit and for implicit difference methods. We prove that error estimates are the same for both methods.

The second type conditions concern the mesh. We show that we need strong assumptions on the mesh for explicit difference methods and we do not need assumptions on the mesh for implicit difference schemes. We show that there are implicit difference methods which are convergent, while the corresponding explicit difference schemes are not. We present error estimates for both methods.

The authors of [1], [3], [4], [8], [10], [12], [13], [16], [22] have assumed that the given functions satisfy the Lipschitz condition or satisfy nonlinear estimates of Perron type with respect to function variables and that these conditions are global with respect to function variables. Our assumptions are more general. We assume that nonlinear estimates of Perron type are local with respect to function variables. It is clear that there are differential equations with deviated variables and differential integral equations such that local estimates of Perron type hold and global inequalities are not satisfied.

Relations between explicit and implicit difference methods for quasilinear functional differential equations are given in [5]. The present paper is a continuation of [5].

Sufficient conditions for the existence and uniqueness of classical or generalized solutions of parabolic functional differential equations can be found in [2], [11], [21], [23]–[26], [31].

Differential equations with deviated arguments and differential integral problems can be derived from (1.1), (1.2) by specializing given operators. Information on applications of functional differential equations can be found in [31].

2. Discretization of functional differential equations. For any spaces X and Y we denote by $\mathbf{F}(X,Y)$ the class of all functions defined on X and taking values in Y. We will denote by $\mathbb{N}$ and $\mathbb{Z}$ the sets of natural numbers and integers respectively.

We define a mesh on $E_0 \cup E$ in the following way. Let (h_0, h'), $h' = (h_1, \ldots, h_n)$, stand for steps of the mesh with respect to t and x respectively. Set $h = (h_0, h')$ and $t^{(r)} = rh_0$, $r \in \mathbb{Z}$. For $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ we put $x^{(m)} = (m_1 h_1, \ldots, m_n h_n)$ and

$$\mathbb{R}^n_h = \{x^{(m)} : m \in \mathbb{Z}^n\}, \quad Q_h = Q \cap \mathbb{R}^n_h, \quad \overline{Q}_h = \overline{Q} \cap \mathbb{R}^n_h.$$

Write $J = \{(i,j) : i,j = 1, \ldots, n,\ i \neq j\}$ and $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^n$ with 1 in the ith position, $1 \le i \le n$. Suppose that we have defined the sets J_+, $J_- \subset J$ such that $J_+ \cup J_- = J$, $J_+ \cap J_- = \emptyset$. We assume that $(j,i) \in J_+$ if $(i,j) \in J_+$. In particular, it may happen that $J_+ = \emptyset$ or $J_- = \emptyset$. Relations between J_+, J_- and equation (1.1) are given in Section 3. For $x^{(m)} \in \mathbb{R}^n_h$ we put

$$\begin{aligned}
\theta_1^{(m)} &= \{x^{(m+e_i)} : i = 1, \ldots, n\} \cup \{x^{(m-e_i)} : i = 1, \ldots, n\},\\
\theta_2^{(m)} &= \{x^{(m+e_i+e_j)} : (i,j) \in J\} \cup \{x^{(m-e_i-e_j)} : (i,j) \in J\}\\
&\quad \cup \{x^{(m+e_i-e_j)} : (i,j) \in J\}
\end{aligned}$$

and $\theta^{(m)} = \theta_1^{(m)} \cup \theta_2^{(m)}$. Write

$$\operatorname{Int} Q_h = \{x^{(m)} \in Q_h : \theta^{(m)} \subset \overline{Q}_h\}, \quad \partial_0 Q_h = Q_h \setminus \operatorname{Int} Q_h.$$

We will approximate the partial derivatives $\partial_x = (\partial_{x_1}, \ldots, \partial_{x_n})$ and

$$\partial_{xx} = [\partial_{x_i x_j}]_{i,j=1}^n$$

with difference operators $\delta = (\delta_1, \ldots, \delta_n)$ and

$$\delta^{(2)} = [\delta_{ij}]_{i,j=1}^n$$

respectively. We will calculate the difference expressions $\delta z(t^{(r)}, x^{(m)})$ and $\delta^{(2)} z(t^{(r)}, x^{(m)})$ for each point $(t^{(r)}, x^{(m)}) \in [0,a] \times Q_h$. Then we need additional mesh points on the set ∂Q. For each $x^{(m)} \in \partial_0 Q_h$ and $i,j = 1, \ldots, n$, $i \neq j$, we define

$$\begin{aligned}
\lambda_{i_+}^{(m)} &= \max\{\lambda \in (0,1] : x^{(m)} + \lambda h_i e_i \in \overline{Q}\},\\
\lambda_{i_-}^{(m)} &= \max\{\lambda \in (0,1] : x^{(m)} - \lambda h_i e_i \in \overline{Q}\},\\
\lambda_{i_+ j_+}^{(m)} &= \max\{\lambda \in (0,1] : x^{(m)} + \lambda(h_i e_i + h_j e_j) \in \overline{Q}\},\\
\lambda_{i_- j_+}^{(m)} &= \max\{\lambda \in (0,1] : x^{(m)} + \lambda(-h_i e_i + h_j e_j) \in \overline{Q}\},\\
\lambda_{i_+ j_-}^{(m)} &= \max\{\lambda \in (0,1] : x^{(m)} + \lambda(h_i e_i - h_j e_j) \in \overline{Q}\},\\
\lambda_{i_- j_-}^{(m)} &= \max\{\lambda \in (0,1] : x^{(m)} - \lambda(h_i e_i + h_j e_j) \in \overline{Q}\}.
\end{aligned}$$

It is clear that these numbers depend on $x^{(m)} \in \partial_0 Q_h$. For simplicity of notation we write $\lambda_{i_+}, \lambda_{i_-}, \lambda_{i_+ j_+}, \lambda_{i_+ j_-}, \lambda_{i_- j_+}, \lambda_{i_- j_-}$ instead of $\lambda^{(m)}_{i_+}$, etc. Set

$$\begin{aligned} S_h^{(1)} = \{x \in \partial Q :\ & \text{there are } x^{(m)} \in \partial_0 Q_h \text{ and } i \in \{1,\dots,n\} \text{ such that} \\ & x = x^{(m)} + \lambda_{i_+} h_i e_i \text{ or } x = x^{(m)} - \lambda_{i_-} h_i e_i\}, \\ S_h^{(2)} = \{x \in \partial Q :\ & \text{there are } x^{(m)} \in \partial_0 Q_h \text{ and } (i,j) \in J \text{ such that} \\ & x = x^{(m)} + \lambda_{i_+ j_+}(h_i e_i + h_j e_j) \text{ or} \\ & x = x^{(m)} + \lambda_{i_+ j_-}(h_i e_i - h_j e_j) \text{ or} \\ & x = x^{(m)} - \lambda_{i_- j_-}(h_i e_i + h_j e_j)\} \end{aligned}$$

and $S_h = S_h^{(1)} \cup S_h^{(2)}$. Write $X_h = Q_h \cup S_h$. Then X_h is the set of all mesh points in $\overline{Q}$.

Denote by Δ the set of all $h = (h_0, h')$ such that there are $K_0 \in \mathbb{C}$ and $c_0 > 0$ satisfying $K_0 h_0 = b_0$ and $h_i h_j^{-1} \le c_0$ for $(i,j) \in J$. Let $K \in \mathbb{N}$ be defined by $K h_0 \le a < (K+1) h_0$. Write

$$\begin{aligned} E_h &= \{(t^{(r)}, x) : 0 \le r \le K,\ x \in X_h\}, \\ E_{0.h} &= \{(t^{(r)}, x) : -K_0 \le r \le 0,\ x \in X_h\}, \\ \partial_0 E_h &= \{(t^{(r)}, x) : 0 \le r \le K,\ x \in S_h\}, \\ E'_h &= \{(t^{(r)}, x) : 0 \le r \le K-1,\ x \in Q_h\}, \\ \theta_h &= \{t^{(r)} : 0 \le r \le K\} \end{aligned}$$

where $h \in \Delta$. For functions $z : E_{0.h} \cup E_h \to \mathbb{R}$, $\chi : X_h \to \mathbb{R}$, $\omega : \theta_h \to \mathbb{R}$ we write $z^{(r,m)} = z(t^{(r)}, x^{(m)})$ for $(t^{(r)}, x^{(m)}) \in \theta_h \times Q_h$ and $\chi^{(m)} = \chi(x^{(m)})$ for $x^{(m)} \in Q_h$ and $\omega^{(r)} = \omega(t^{(r)})$ for $t^{(r)} \in \theta_h$.

If $\mu, \nu \in [-1, 1]$ and $(t^{(r)}, x^{(m)} + \mu h_i e_i + \nu h_j e_j) \in [-b_0, a] \times X_h$ then we put $z^{(r, m+\mu e_i + \nu e_j)} = z(t^{(r)}, x^{(m)} + \mu h_i e_i + \nu h_j e_j)$ and $\chi^{(m+\mu e_i+\nu e_j)} = \chi(x^{(m+\mu h_i e_i + \nu h_j e_j)})$.

Solutions of difference functional equations are elements of the space $\mathbf{F}(E_{0.h} \cup E_h, \mathbb{R})$. Equation (1.1) contains the function variable $z_{(t,x)}$ which is an element of the space $\mathbf{C}(D[t,x], \mathbb{R})$. Thus we need an interpolating operator $T_h : \mathbf{F}(E_{0.h} \cup E_h, \mathbb{R}) \to \mathbf{C}(E_0 \cup E, \mathbb{R})$. In Section 3 we adopt additional assumptions on T_h. For $z \in \mathbf{F}(E_{0.h} \cup E_h, \mathbb{R})$ and $(t^{(r)}, x^{(m)}) \in \theta_h \times Q_h$ we write $(T_h z)_{[r,m]}$ instead of $(T_h z)_{(t^{(r)}, x^{(m)})}$. Set

$$\mathbb{F}_{\mathrm{ex}.h}[z]^{(r,m)} = F(t^{(r)}, x^{(m)}, (T_h z)_{[r,m]}, \delta z^{(r,m)}, \delta^{(2)} z^{(r,m)})$$

and

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}[z^{(r+1,m)} - z^{(r,m)}].$$

We will approximate classical solutions of (1.1), (1.2) with solutions of the difference functional equation

$$\delta_0 z^{(r,m)} = \mathbb{F}_{\mathrm{ex}.h}[z]^{(r,m)} \tag{2.1}$$

with the initial-boundary condition

$$z(t^{(r)}, x) = \varphi_h(t,x)) \quad \text{on } E_{0.h} \cup \partial_0 E_h \tag{2.2}$$

where $\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}$ is a given function. Problem (2.1), (2.2) is considered as an explicit difference scheme of Euler type for (1.1), (1.2).

Set

$$\mathbb{F}_{\mathrm{im}.h}[z]^{(r,m)} = F(t^{(r)}, x^{(m)}, (T_h z)_{[r,m]}, \delta z^{(r+1,m)}, \delta^{(2)} z^{(r+1,m)}).$$

The functional difference equation

$$\delta_0 z^{(r,m)} = \mathbb{F}_{\mathrm{im}.h}[z]^{(r,m)} \tag{2.3}$$

with the initial-boundary condition (2.2) is considered as an implicit difference scheme for (1.1), (1.2).

The above numerical methods have the following properties: the difference operators δz and $\delta^{(2)} z$ are calculated at the point $(t^{(r)}, x^{(m)})$ in (2.1) and at $(t^{(r+1)}, x^{(m)})$ in (2.3). The function variable $(T_h)_{[r,m]}$ appears in a classical sense in both methods.

The definition of the difference operators

$$\delta z = (\delta_1 z, \ldots, \delta_n z), \qquad \delta^{(2)} z = [\delta_{ij} z]_{i,j=1}^{n} \tag{2.4}$$

falls naturally into two steps. In the first step we assume that $x^{(m)} \in \operatorname{Int} Q_h$. Then we consider the case when $x^{(m)} \in \partial_0 Q_h$. For $(t^{(r)}, x^{(m)}) \in \theta_h \times \operatorname{Int} Q_h$ we write

$$\delta_i^+ z^{(r,m)} = \frac{1}{h_i}[z^{(r,m+e_i)} - z^{(r,m)}],$$
$$\delta_i^- z^{(r,m)} = \frac{1}{h_i}[z^{(r,m)} - z^{(r,m-e_i)}], \qquad 1 \le i \le n,$$

and

$$\delta_i z^{(r,m)} = \tfrac{1}{2}[\delta_i^+ z^{(r,m)} + \delta_i^- z^{(r,m)}],\ \delta_{ii} z^{(r,m)} = \delta_i^+ \delta_i^- z^{(r,m)}),\ 1 \le i \le n. \tag{2.5}$$

The difference expressions $\delta_{ij} z^{(r,m)}$ for $(i,j) \in J$ are defined by

$$\delta_{ij} z^{(r,m)} = \tfrac{1}{2}[\delta_i^+ \delta_j^- z^{(r,m)} + \delta_i^- \delta_j^+ z^{(r,m)}] \quad \text{for } (i,j) \in J_-, \tag{2.6}$$

$$\delta_{ij} z^{(r,m)} = \tfrac{1}{2}[\delta_i^+ \delta_j^+ z^{(r,m)} + \delta_i^- \delta_j^- z^{(r,m)}] \quad \text{for } (i,j) \in J_+. \tag{2.7}$$

We now define the difference operators (2.4) for $(t^{(r)}, x^{(m)}) \in \theta_h \times \partial_0 Q_h$. We put first, for $1 \le i \le n$,

$$\delta_i z^{(r,m)} = \frac{1}{h_i(\lambda_{i_+} + \lambda_{i_-})}[z^{(r,m+\lambda_{i_+} e_i)} - z^{(r,m-\lambda_{i_-} e_i)}], \tag{2.8}$$

$$\delta_{ii} z^{(r,m)} = \frac{2}{h_i^2}\Big[\frac{1}{\lambda_{i_+}(\lambda_{i_+}+\lambda_{i_-})} z^{(r,m+\lambda_{i_+}e_i)} - \frac{1}{\lambda_{i_+}\lambda_{i_-}} z^{(r,m)} + \frac{1}{\lambda_{i_-}(\lambda_{i_+}+\lambda_{i_-})} z^{(r,m-\lambda_{i_-}e_i)}\Big]. \tag{2.9}$$

For $(i,j) \in J_-$ we write

$$\delta_{ij} z^{(r,m)} = \frac{1}{h_i h_j}\{A_-^{(m)} z^{(r,m)} + B_-^{(m)} z^{(r,m+\lambda_{i+}e_i)} + C_-^{(m)} z^{(r,m-\lambda_{i_-}e_i)} + D_-^{(m)} z^{(r,m+\lambda_{j_+}e_j)} + E_-^{(m)} z^{(r,m-\lambda_{j_-}e_j)} + F_-^{(m)} z^{(r,m-\lambda_{i_-j_+}(e_i-e_j))} + G_-^{(m)} z^{(r,m+\lambda_{i_+j_-}(e_i-e_j))}\} \tag{2.10}$$

where

$$A_-^{(m)} = \frac{1}{\lambda_{i_+j_-}\lambda_{i_-j_+}} - \frac{1}{\lambda_{j_+}\lambda_{j_-}} - \frac{1}{\lambda_{i_+}\lambda_{i_-}},$$

$$B_-^{(m)} = \frac{1}{\lambda_{i_+}(\lambda_{i_+}+\lambda_{i_-})}, \qquad C_-^{(m)} = \frac{1}{\lambda_{i-}(\lambda_{i_+}+\lambda_{i_-})},$$

$$D_-^{(m)} = \frac{1}{\lambda_{j_+}(\lambda_{j_+}+\lambda_{j_-})}, \qquad E_-^{(m)} = \frac{1}{\lambda_{j_-}(\lambda_{j_+}+\lambda_{j_-})},$$

$$F_-^{(m)} = \frac{-1}{\lambda_{i_-j_+}(\lambda_{i_-j_+}+\lambda_{i_+j_-})}, \qquad G_-^{(m)} = \frac{-1}{\lambda_{i_+j_-}(\lambda_{i_+j_-}+\lambda_{i_-j_+})}.$$

For $(i,j) \in J_+$ we write

$$\delta_{ij} z^{(r,m)} = \frac{1}{h_i h_j}\{A_+^{(m)} z^{(r,m)} + B_+^{(m)} z^{(r,m+\lambda_{i+}e_i)} + C_+^{(m)} z^{(r,m-\lambda_{i_-}e_i)} + D_+^{(m)} z^{(r,m+\lambda_{j_+}e_j)} + E_+^{(m)} z^{(r,m-\lambda_{j_-}e_j)} + F_+^{(m)} z^{(r,m+\lambda_{i_+j_+}(e_i+e_j))} + G_+^{(m)} z^{(r,m-\lambda_{i_-j_-}(e_i+e_j))}\} \tag{2.11}$$

where

$$A_+^{(m)} = \frac{-1}{\lambda_{i_+j_+}\lambda_{i_-j_-}} + \frac{1}{\lambda_{i_+}\lambda_{i_-}} + \frac{1}{\lambda_{j_-}\lambda_{j_+}},$$

$$B_+^{(m)} = \frac{-1}{\lambda_{i_+}(\lambda_{i_+}+\lambda_{i_-})}, \qquad C_+^{(m)} = \frac{-1}{\lambda_{i-}(\lambda_{i_+}+\lambda_{i_-})},$$

$$D_+^{(m)} = \frac{-1}{\lambda_{j_+}(\lambda_{j_+}+\lambda_{j_-})}, \qquad E_+^{(m)} = \frac{-1}{\lambda_{j_-}(\lambda_{j_+}+\lambda_{j_-})},$$

$$F_+^{(m)} = \frac{1}{\lambda_{i_+j_+}(\lambda_{i_+j_+}+\lambda_{i_-j_-})}, \qquad G_+^{(m)} = \frac{1}{\lambda_{i_-j_-}(\lambda_{i_-j_-}+\lambda_{i_+j_+})}.$$

The vector $\delta z^{(r,m)}$ and the matrix $\delta^{(2)} z^{(r,m)}$ are defined by (2.5)–(2.11). The vector $\delta z^{(r+1,m)}$ and the matrix $\delta^{(2)} z^{(r+1,m)}$ appear in (2.3).

In the same way we define the difference expressions

$$\delta\chi^{(m)} = (\delta_1\chi^{(m)}, \ldots, \delta_n\chi^{(m)}) \quad \text{and} \quad \delta^{(2)}\chi^{(m)} = [\delta_{ij}\chi^{(m)}]_{i,j=1}^n$$

where $\chi : X_h \to \mathbb{R}$.

The above definitions have the following properties. Put $\lambda_{i_+} = \lambda_{i_-} = 1$ for $i = 1, \ldots, n$ and $\lambda_{i_+ j_-} = \lambda_{i_- j_+} = 1$ for $(i,j) \in J_-$ and $\lambda_{i_+ j_+} = \lambda_{i_- j_-} = 1$ for $(i,j) \in J_+$. Then the definitions (2.8)–(2.11) are equivalent to (2.5)–(2.7) respectively. It follows that we can use formulas (2.8)–(2.11) for all points $(t^{(r)}, x^{(m)}) \in \theta_h \times Q_h$. Note that the numbers $z^{(r+1,m+\mu h_i+\nu h_j)}$ where μ, $\nu \in [-1,1]$ appear in (2.3). It follows that (2.3), (2.2) is an implicit difference method for (1.1), (1.2).

We will prove that under natural assumptions on the given functions and on the mesh there exists exactly one solution $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ to (2.1), (2.2) and there is exactly one solution $v_h : E_{0.h} \cup E_h \to \mathbb{R}$ to (2.3), (2.2). Solutions of the above difference functional problems are approximate solutions to (1.1), (1.2). We give sufficient conditions for the convergence of the sequences of approximate solutions to a classical solution to (1.1), (1.2).

3. Solutions of functional differential and difference problems. We first construct estimates for solutions to (1.1), (1.2). A function $z : E_0 \cup E \to \mathbb{R}$ will be called *of class* $\mathbf{C}^{1.2}$ if $z \in \mathbf{C}(E_0 \cup E, \mathbb{R})$ and $z(\cdot, x) : [-b_0, a] \to \mathbb{R}$ is of class C^1 for $x \in \overline{Q}$ and $z(t, \cdot) : \overline{Q} \to \mathbb{R}$ is of class C^2 for $t \in [-b_0, a]$. For $z \in \mathbf{C}(E_0 \cup E, \mathbb{R})$, $u \in \mathbf{F}(E_{0.h} \cup E_h)$ we define the seminorms

$$\|z\|_t = \max\{|z(\tau, x)| : (\tau, x) \in E_0 \cup E,\ \tau \le t\}, \qquad 0 \le t \le a,$$

$$\|z\|_{h.r} = \max\{|u(\tau, x)| : (\tau, x) \in E_{0.h} \cup E_h,\ \tau \le t^{(r)}\}, \quad 0 \le r \le K,$$

For $w \in \mathbf{C}(B, \mathbb{R})$ we put $\|w\|_B = \max\{|w(\tau, y)| : (\tau, y) \in B\}$.

Assumption $H_0[F, \varphi]$. The function $F : \Omega \to \mathbb{R}$ of the variables (t, x, w, q, s), where $q = (q_1, \ldots, q_n)$, $s = [s_{ij}]_{i,j=1,\ldots,n}$, satisfies the conditions:

1) F is continuous and satisfies condition (V),
2) the partial derivatives

$$\partial_q F = (\partial_{q_1} F, \ldots, \partial_{q_n} F), \qquad \partial_s F = [\partial_{s_{ij}} F]_{i,j=1}^n$$

exist on Ω and the functions $\partial_q F : \Omega \to \mathbb{R}^n$, $\partial_s F : \Omega \to M_{n\times n}$ are continuous and bounded,

3) the matrix $\partial_s F$ is symmetric and

$$\sum_{i,j=1}^n \partial_{s_{ij}} F(P) y_i y_j \ge 0 \quad \text{for } y = (y_1, \ldots, y_n) \in \mathbb{R}^n, \tag{3.1}$$

where $P = (t, x, w, q, s) \in \Omega$,

4) there is $\varrho : [0, a] \times \mathbb{R}_+$ such that

(i) ϱ is continuous and it is nondecreasing with respect to both variables and for each $\eta \in \mathbb{R}_+$ the maximal solution of the Cauchy problem

$$\omega'(t) = \varrho(t, \omega(t)), \qquad \omega(0) = \eta, \tag{3.2}$$

is defined on $[0, a]$,

(ii) the estimate

$$|F(t, x, w, 0_{[n]}, 0_{[n\times n]})| \le \varrho(t, \|w\|_B)$$

is satisfied for $(t, x, w) \in E \times \mathbf{C}(B, \mathbb{R})$, where $0_{[n]} = (0, \ldots, 0) \in \mathbb{R}^n$ and $0_{[n\times n]} \in M_{n\times n}$ is the zero matrix,

5) $\varphi \in \mathbf{C}(E_0 \cup \partial_0 E, \mathbb{R})$ and $\tilde\eta \in \mathbb{R}_+$ is defined by the relations

$$|\varphi(t, x)| \le \tilde\eta \text{ on } E_0 \quad \text{and} \quad |\varphi(t, x)| \le \omega(t, \tilde\eta) \text{ on } \partial_0 E \tag{3.3}$$

where $\omega(\,\cdot\,, \tilde\eta)$ is the maximal solution to (3.2) with $\eta = \tilde\eta$.

Lemma 3.1. *If Assumption $H_0[F, \varphi]$ is satisfied and $\tilde z : E_0 \cup E \to \mathbb{R}$ is a solution to* (1.1), (1.2) *and $\tilde z$ is of class $\mathbf{C}^{1.2}$ then*

$$|\tilde z(t, x)| \le \omega(t, \tilde\eta) \quad \text{on } E. \tag{3.4}$$

Proof. For $\varepsilon > 0$ we denote by $\omega(\,\cdot\,, \tilde\eta, \varepsilon)$ the maximal solution of the Cauchy problem

$$\omega'(t) = \varrho(t, \omega(t)) + \varepsilon, \qquad \omega(0) = \tilde\eta + \varepsilon.$$

There exists $\tilde\varepsilon > 0$ such that for $0 < \varepsilon < \tilde\varepsilon$ the function $\omega(\,\cdot\,, \tilde\eta, \varepsilon)$ is defined on $[0, a]$ and

$$\lim_{\varepsilon\to 0} \omega(t, \tilde\eta, \varepsilon) = \omega(t, \tilde\eta) \quad \text{uniformly on } [0, a].$$

Write $\zeta(t) = \|\tilde z\|_t$ for $t \in [0, a]$. We now prove that

$$\zeta(t) < \omega(t, \tilde\eta, \varepsilon) \quad \text{for } t \in [0, a]. \tag{3.5}$$

Suppose for contradiction that (3.5) fails to be true. Then the set

$$\Sigma_+ = \{t \in [0, a] : \zeta \ge \omega(t, \tilde\eta, \varepsilon)\}$$

is not empty. Write $\tilde t = \min \Sigma_+$. From (3.3) it follows that $\tilde t > 0$ and there exists $\tilde x \in \overline{Q}$ such that $\omega(\tilde t, \tilde\eta, \varepsilon) = \zeta(\tilde t) = |\tilde z(\tilde t, \tilde x)|$. The condition $|\tilde z(t, x)| < \omega(t, \tilde\eta, \varepsilon)$ for $(t, x) \in \partial_0 E$ implies that $\tilde x \in Q$. Two cases are possible: either (i) $\tilde z(\tilde t, \tilde x) = \omega(\tilde t, \tilde\eta, \varepsilon)$ or (ii) $\tilde z(\tilde t, \tilde x) = -\omega(\tilde t, \tilde\eta, \varepsilon)$. In the first case,

$$D_-\zeta(\tilde t) \ge \omega'(\tilde t, \tilde\eta, \varepsilon) \tag{3.6}$$

where D_- is the left-hand lower Dini derivative. Write

$$A(t, x) = F(t, x, \tilde z_{(t,x)}, \partial_x \tilde z(t, x), \partial_{xx} \tilde z(t, x)) - F(t, x, \tilde z_{(t,x)}, 0_{[n]}, 0_{[n\times n]}),$$
$$B(t, x) = F(t, x, \tilde z_{(t,x)}, 0_{[n]}, 0_{[n\times n]}).$$

Then

$$A(t,x) = \sum_{i=1}^{n} \int_0^1 \partial_{q_i} F(P(\tau,t,x))\,d\tau\,\partial_{x_i}\tilde{z}(t,x) + \sum_{i,j=1}^{n} \int_0^1 \partial_{s_{ij}} F(P(\tau,t,x))\,d\tau\,\partial_{x_i x_j}\tilde{z}(t,x)$$

where $P(\tau,t,x)$, $0 \le \tau \le 1$, are intermediate points defined by the Hadamard mean value theorem. Since $\tilde{x} \in Q$, we have $\partial_x \tilde{z}(\tilde{t},\tilde{x}) = 0_{[n]}$ and

$$\sum_{i,j=1}^{n} \partial_{x_i x_j}\tilde{z}(\tilde{t},\tilde{x}) y_i y_j \le 0 \quad \text{for } y = (y_1,\ldots,y_n) \in \mathbb{R}^n.$$

The above relations and (3.1) imply

$$\sum_{i,j=1}^{n} \int_0^1 \partial_{s_{ij}} F(P(\tau,\tilde{t},\tilde{x}))\,d\tau\,\partial_{x_i x_j}\tilde{z}(\tilde{t},\tilde{x}) \le 0$$

and consequently $A(\tilde{t},\tilde{x}) \le 0$. Then

$$D_-\zeta(\tilde{t}) \le \partial_t \tilde{z}(\tilde{t},\tilde{x}) = A(\tilde{t},\tilde{x}) + B(\tilde{t},\tilde{x}) \le \varrho(\tilde{t},\omega(\tilde{t},\tilde{\eta},\varepsilon)) < \omega'(\tilde{t},\tilde{\eta},\varepsilon),$$

which contradicts (3.6). The case $\tilde{z}(\tilde{t},\tilde{x}) = -\omega(\tilde{t},\tilde{\eta},\varepsilon)$ can be treated in a similar way. Hence, Σ_+ is empty and inequality (3.5) is proved. Letting ε tend to 0 in (3.5) we obtain (3.4). ∎

Assumption $H[F,\varphi,\varphi_h]$. The functions $F : \Omega \to \mathbb{R}$ and $\varphi : E_0 \cup \partial_0 E \to \mathbb{R}$ satisfy Assumption $H_0[F,\varphi]$ and

1) for $P = (t,x,w,q,s) \in \Omega$ we have

(3.7) $$\partial_{s_{ij}} F(P) \ge 0 \text{ for } (i,j) \in J_+, \quad \partial_{s_{ij}} F(P) \le 0 \text{ for } (i,j) \in J_-,$$

2) the steps of the mesh satisfy the conditions

(3.8) $$\frac{1}{h_i}\partial_{s_{ii}} F(P) - \sum_{\substack{j=1\\ j\ne i}}^{n} \frac{1}{h_j}|\partial_{s_{ij}} F(P)| - \frac{1}{2}|\partial_{q_i} F(P)| \ge 0,$$

where $P \in \Omega$, $i = 1,\ldots,n$,

3) there is $\alpha_0 : \Delta \to \mathbb{R}_+$ such that

$$|\varphi(t,x) - \varphi_h(t,x)| \le \alpha_0(h) \text{ on } E_0 \cup \partial_0 E \quad \text{and} \quad \lim_{h\to 0} \alpha_0(h) = 0,$$

4) the constant $\bar{\eta} \in \mathbb{R}_+$ is defined by the relations

(3.9) $$|\varphi_h(t,x)| \le \bar{\eta} \text{ on } E_{0.h} \quad \text{and} \quad |\varphi_h(t,x)| \le \omega_h(t,\bar{\eta}) \text{ on } \partial_0 E_h$$

where $\omega(\,\cdot\,,\bar{\eta})$ is the maximal solution to (3.2) with $\eta = \bar{\eta}$.

Remark 3.2. We have assumed that the matrix $\partial_s F$ satisfies the condition: for each $(i,j) \in J$ we have

$$\partial_{s_{ij}} F(P) \geq 0 \text{ on } \Omega \quad \text{or} \quad \partial_{s_{ij}} F(P) \leq 0 \text{ on } \Omega.$$

Conditions (3.7) can be considered as definitions of J_+ and J_-.

Remark 3.3. Suppose that there is $\tilde{c} > 0$ such that

$$\partial_{s_{ii}} F(P) - \sum_{\substack{j=1 \\ j \neq i}}^{n} |\partial_{s_{ij}} F(P)| \geq \tilde{c}, \qquad P \in \Omega, \ i = 1, \dots n.$$

Then condition (3.1) is satisfied (see [28]) and there is $\varepsilon_0 > 0$ such that for $\|h\| < \varepsilon_0$ and for $h_1 = \dots = h_n$ inequalities (3.8) hold.

Assumption $H[T_h]$. The operator $T_h : \mathbf{F}(E_{0.h} \cup E_h, \mathbb{R}) \to \mathbf{C}(E_0 \cup E, \mathbb{R})$ satisfies the conditions:

1) for $z, \bar{z} \in \mathbf{F}(E_{0.h} \cup E_h, \mathbb{R})$ we have
$$\|T_h[z] - T_h[\bar{z}]\|_{t^{(r)}} \leq \|z - \bar{z}\|_{h.r}, \qquad 0 \leq r \leq K,$$
2) if $z : E_0 \cup E \to \mathbb{R}_+$ is of class $\mathbf{C}^{1.2}$ then there is $\gamma_\star : \Delta \to \mathbb{R}_+$ such that
$$\|T_h[z_h] - z\|_t \leq \gamma_\star(h), \qquad 0 \leq t \leq a, \qquad \lim_{h \to 0} \gamma_\star(h) = 0,$$
where z_h is the restriction of z to $E_{0.h} \cup E_h$,
3) if $\mathbf{0}_h \in \mathbf{F}(E_{0.h} \cup E_h, \mathbb{R})$ is given by $\mathbf{0}_h(t,x) = 0$ for $(t,x) \in E_{0.h} \cup E_h$ then $T_h[\mathbf{0}_h](t,x) = 0$ for $(t,x) \in E_0 \cup E$.

Remark 3.4. If $Q = (-c, c)$ where $(-c, c) \subset \mathbb{R}^n$, $c = (c_1, \dots, c_n)$, $c_i > 0$ for $1 \leq i \leq n$, then the interpolating operator T_h given in [7, Chapter VI] satisfies Assumption $H[T_h]$. The construction of T_h presented in [7] can be extended to the set $E_0 \cup E$ considered in this paper.

Suppose that Assumption $H[F, \varphi, \varphi_h]$ is satisfied. For $P \in \Omega$ we put

$$\begin{aligned} X_0(P) = & -2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} \frac{1}{\lambda_{i_-} \lambda_{i_+}} \partial_{s_{ii}} F(P) \\ & + 2h_0 \sum_{(i,j) \in J} \frac{1}{h_i h_j} \frac{1}{\lambda_{i_-} \lambda_{i_+}} |\partial_{s_{ij}} F(P)| \\ & + h_0 \sum_{(i,j) \in J_-} \frac{1}{h_i h_j} \frac{1}{\lambda_{i_- j_+} \lambda_{i_+ j_-}} \partial_{s_{ij}} F(P) \\ & - h_0 \sum_{(i,j) \in J_+} \frac{1}{h_i h_j} \frac{1}{\lambda_{i_- j_-} \lambda_{i_+ j_+}} \partial_{s_{ij}} F(P), \end{aligned} \tag{3.10}$$

and

$$X_+^{(i)}(P) = \frac{2h_0}{\lambda_{i_+}(\lambda_{i_-} + \lambda_{i_+})}\Big[\frac{\lambda_{i_+}}{2h_i}\partial_{q_i}F(P) + \frac{1}{h_i^2}\partial_{s_{ii}}F(P) - \sum_{\substack{j=1\\ j\neq i}}^n \frac{1}{h_i h_j}|\partial_{s_{ij}}F(P)|\Big],$$

$$X_-^{(i)}(P) = \frac{2h_0}{\lambda_{i_-}(\lambda_{i_-} + \lambda_{i_+})}\Big[\frac{-\lambda_{i_-}}{2h_i}\,\partial_{q_i}F(P) + \frac{1}{h_i^2}\partial_{s_{ii}}F(P) - \sum_{\substack{j=1\\ j\neq i}}^n \frac{1}{h_i h_j}|\partial_{s_{ij}}F(P)|\Big],$$

where $i = 1, \ldots, n$ and

$$Y_+^{(i,j)}(P) = \frac{h_0}{h_i h_j}\,\frac{1}{\lambda_{i_+ j_+}(\lambda_{i_- j_-} + \lambda_{i_+ j_+})}\partial_{s_{ij}}F(P) \quad \text{for } (i,j) \in J_+,$$

$$Z_+^{(i,j)}(P) = \frac{h_0}{h_i h_j}\,\frac{1}{\lambda_{i_- j_-}(\lambda_{i_- j_-} + \lambda_{i_+ j_+})}\partial_{s_{ij}}F(P) \quad \text{for } (i,j) \in J_+,$$

$$Y_-^{(i,j)}(P) = \frac{-h_0}{h_i h_j}\,\frac{1}{\lambda_{i_- j_+}(\lambda_{i_- j_+} + \lambda_{i_+ j_-})}\partial_{s_{ij}}F(P) \quad \text{for } (i,j) \in J_-,$$

$$Z_-^{(i,j)}(P) = \frac{-h_0}{h_i h_j}\,\frac{1}{\lambda_{i_+ j_-}(\lambda_{i_- j_+} + \lambda_{i_+ j_-})}\partial_{s_{ij}}F(P) \quad \text{for } (i,j) \in J_-.$$

Suppose that $\chi : X_h \to \mathbb{R}$ and $x^{(m)} \in Q_h$. Write

(3.11)

$$\begin{aligned}\Theta_h^{(m)}[\chi, P] &= \chi^{(m)} X_0(P) + \sum_{i=1}^n [\chi^{(m+\lambda_{i+}e_i)} X_+^{(i)}(P) + \chi^{(m-\lambda_{i-}e_i)} X_-^{(i)}(P)] \\ &+ \sum_{(i,j)\in J_+} [\chi^{(m+\lambda_{i+j+}(e_i+e_j))} Y_+^{(i,j)}(P) + \chi^{(m-\lambda_{i-j-}(e_i+e_j))} Z_+^{(i,j)}(P)] \\ &+ \sum_{(i,j)\in J_-} [\chi^{(m-\lambda_{i-j+}(e_i-e_j))} Y_-^{(i,j)}(P) + \chi^{(m+\lambda_{i+j-}(e_i-e_j))} Z_-^{(i,j)}(P)]\end{aligned}$$

where $P = (t, x, w, q, s) \in \Omega$. Important properties of difference schemes are given in the next lemma.

LEMMA 3.5. *Suppose that Assumption $H_0[F, \varphi]$ and conditions* 1), 2) *of Assumption $H[F, \varphi, \varphi_h]$ are satisfied and $\chi : X_h \to \mathbb{R}$, $x^{(m)} \in Q_h$ and*

$$G_h^{(m)}[\chi, P] = h_0 \sum_{i=1}^n \partial_{q_i}F(P)\,\delta_i\chi^{(m)} + h_0 \sum_{i,j=1}^n \partial_{s_{ij}}F(P)\delta_{ij}\chi^{(m)}, \qquad P = (t, x, w, q, s) \in \Omega,$$

where $\delta\chi$ *and* $\delta^{(2)}\chi$ *are given by* (2.8)–(2.11). *Then*

$$G_h^{(m)}[\chi,P] = \Theta_h^{(m)}[\chi,P] \tag{3.12}$$

and

$$X_+^{(i)}(P) \ge 0, \quad X_-^{(i)}(P) \ge 0 \quad \text{for } i=1,\ldots,n, \tag{3.13}$$

$$Y_+^{(i,j)}(P) \ge 0, \quad Z_+^{(i,j)}(P) \ge 0 \quad \text{for } (i,j)\in J_+, \tag{3.14}$$

$$Y_-^{(i,j)}(P) \ge 0, \quad Z_-^{(i,j)}(P) \ge 0 \quad \text{for } (i,j)\in J_-, \tag{3.15}$$

and

$$X_0(P) + \sum_{i=1}^{n}[X_+^{(i)}(P) + X_-^{(i)}(P)] + \sum_{(i,j)\in J_+}[Y_+^{(i,j)}(P) + Z_+^{(i,j)}(P)] + \sum_{(i,j)\in J_-}[Y_-^{(i,j)}(P) + Z_-^{(i,j)}(P)] = 0. \tag{3.16}$$

Proof. An easy computation shows that (3.12) is a consequence of (2.8)–(2.11). From (3.7), (3.8) we obtain (3.13)–(3.15). Condition (3.16) follows from the formulas for $X_0(P)$, $X_+^{(i)}(P)$, $X_-^{(i)}(P)$, $Y_+^{(i,j)}(P)$, $Y_-^{(i,j)}(P)$, $Z_+^{(i,j)}(P)$, $Z_-^{(i,j)}(P)$. ■

THEOREM 3.6. *Suppose that Assumptions* $H[T_h]$ *and* $H[F,\varphi,\varphi_h]$ *are satisfied.*

I. *There exists exactly one solution* $v_h : E_{0.h}\cup E_h \to \mathbb{R}$ *to* (2.3), (2.2), *and*

$$|v_h(t,x)| \le \omega(t,\bar\eta) \quad \text{for } (t,x)\in E_h, \tag{3.17}$$

where $\omega(\,\cdot\,,\bar\eta)$ *is the maximal solution of the Cauchy problem* (3.2) *with* $\eta=\bar\eta$ *and* $\bar\eta$ *is defined by* (3.9).

II. *Assume additionally that the steps of the mesh satisfy the condition*

$$1 + X_0(P) \ge 0, \quad P=(t,x,w,q,s)\in\Omega, \tag{3.18}$$

where $X_0(P)$ *is given by* (3.10). *Then there is exactly one solution* $u_h : E_{0.h}\cup E_h \to \mathbb{R}$ *to* (2.1), (2.2), *and*

$$|u_h(t,x)| \le \omega(t,\bar\eta) \quad \text{for } (t,x)\in E_h. \tag{3.19}$$

Proof. The proof will be divided into two parts.

I. We prove that there exists exactly one solution $v_h : E_{0.h}\cup E_h \to \mathbb{R}$ to (2.3), (2.2). Suppose that $0\le r<K$ is fixed and that the solution v_h to (2.3), (2.2) is given on $(E_{0.h}\cup E_h)\cap([-b_0,t^{(r)}]\times\mathbb{R}^n)$. We prove that the values $v_h(t^{(r+1)},x)$, $x\in X_h$, exist and they are unique. It is sufficient to show that there exists exactly one solution of the system of equations

$$z^{(r+1,m)} = v_h^{(r,m)} + h_0 F(t^{(r)}, x^{(m)}, (T_h v_h)_{[r,m]}, \delta z^{(r+1,m)}, \delta^{(2)} z^{(r+1,m)}) \tag{3.20}$$

where $x^{(m)} \in Q_h$, and

$$z(t^{(r+1)}, x) = \varphi_h(t^{(r+1)}, x) \text{ for } x \in S_h. \tag{3.21}$$

Set

$$\Sigma_h = \{\chi \in \mathbf{F}(X_h, \mathbb{R}) : \chi(x) = \varphi(t^{(r+1)}, x) \text{ for } x \in S_h\}.$$

For $\chi, \tilde{\chi} \in \Sigma_h$ we put

$$[|\chi - \tilde{\chi}|] = \max\{|(\chi - \tilde{\chi})(x)| : x \in X_h\}.$$

It follows from Assumption $H[F, \varphi, \varphi_h]$ that there is $A_h \in \mathbb{R}_+$ such that

$$A_h + X_0(P) \geq 0 \quad \text{for } P = (t, x, w, q, s) \in \Omega, \tag{3.22}$$

where $X_0(P)$ is given by (3.10). Let W_h be the operator defined on Σ_h by

$$W_h[\chi]^{(m)} = \frac{1}{1 + A_h}[A_h \chi^{(m)} + v_h^{(r,m)} + h_0 F(t^{(r)}, x^{(m)}, (T_h v_h)_{[r,m]}, \delta\chi^{(m)}, \delta^{(2)}\chi^{(m)})], \quad x^{(m)} \in Q_h,$$

where $\delta\chi^{(m)}$ and $\delta^{(2)}\chi^{(m)}$ are defined by (2.8)–(2.11) and

$$W_h[\chi](x) = \varphi_h(t^{(r+1)}, x) \quad \text{for } x \in S_h.$$

Then $W_h : \Sigma_h \to \Sigma_h$. It is clear that problem (3.20), (3.21) is equivalent to the equation

$$\chi = W_h[\chi]. \tag{3.23}$$

We prove that

$$[|W_h[\chi] - W_h[\tilde{\chi}]|] \leq \frac{A_h}{1 + A_h}[|\chi - \tilde{\chi}|] \quad \text{on } \Sigma_h. \tag{3.24}$$

If $x^{(m)} \in Q_h$ then there is $P \in \Omega$ such that

$$(1 + A_h)[W_h[\chi]^{(m)} - W_h[\tilde{\chi}]^{(m)}] = A_h(\chi - \tilde{\chi})^{(m)} + h_0 \sum_{i=1}^{n} \partial_{q_i} F(P)\, \delta_i(\chi - \tilde{\chi})^{(m)} + h_0 \sum_{i,j=1}^{n} \partial_{s_{ij}} F(P)\, \delta_{ij}(\chi - \tilde{\chi})^{(m)}.$$

We conclude from Lemma 3.5 that

$$\begin{aligned}(1 + A_h)[W_h[\chi]^{(m)} - W_h[\tilde{\chi}]^{(m)}] &= (A_h + X_0(P))(\chi - \tilde{\chi})^{(m)} \\ &+ \sum_{i=1}^{n} X_+^{(i)}(P)(\chi - \tilde{\chi})^{(m+\lambda_{i+}e_i)} + \sum_{i=1}^{n} X_-^{(i)}(P)(\chi - \tilde{\chi})^{(m-\lambda_{i-}e_i)} \\ &+ \sum_{(i,j)\in J_+} [Y_+^{(i,j)}(P)(\chi - \tilde{\chi})^{(m+\lambda_{i+j+}(e_i+e_j))} + Z_+^{(i,j)}(P)(\chi - \tilde{\chi})^{(m-\lambda_{i-j-}(e_i+e_j))}] \\ &+ \sum_{(i,j)\in J_-} [Y_-^{(i,j)}(P)(\chi - \tilde{\chi})^{(m-\lambda_{i-j+}(e_i-e_j))} + Z_-^{(i,j)}(P)(\chi - \tilde{\chi})^{(m-\lambda_{i+j-}(e_i-e_j))}].\end{aligned}$$

It follows from (3.13)–(3.16), (3.22) that

$$|W_h[\chi]^{(m)} - W_h[\tilde{\chi}]^{(m)}| \le \frac{A_h}{1+A_h}[|\chi - \tilde{\chi}|], \qquad x^{(m)} \in Q_h.$$

If $x \in S_h$ then $W_h[\chi](x) - W_h[\tilde{\chi}](x) = 0$. The above relations imply (3.24). The Banach fixed point theorem implies that there exists exactly one solution to (3.23). It follows that the values $v_h(t^{(r)}, x)$, $x \in X_h$, exist and they are unique. The function v_h is given on $E_{0.h}$. Then the proof of the existence and uniqueness of a solution to (2.3), (2.2) is completed by induction on r, $0 \le r \le K$.

Now we prove (3.17), Write $\tilde{\omega}_h^{(r)} = \|v_h\|_{h.r}$, $0 \le r \le K$. It follows easily that

$$\tilde{\omega}_h^{(r+1)} \le \omega_h^{(r)} + h_0 \varrho(t^{(r)}, \tilde{\omega}_h^{(r)}), \qquad 0 \le r \le K-1,$$

and $\tilde{\omega}_k^{(0)} \le \bar{\eta}$. The function $\omega(\,\cdot\,, \bar{\eta})$ satisfies the recurrent inequality

$$\omega(t^{(r+1)}, \bar{\eta}) \ge \omega(t^{(r)}, \bar{\eta}) + h_0 \varrho(t^{(r)}, \omega(t^{(r)}, \bar{\eta})), \qquad 0 \le r \le K-1. \tag{3.25}$$

From the initial inequality $\tilde{\omega}_h^{(0)} \le \omega(t^{(0)}, \bar{\eta})$ we conclude that $\tilde{\omega}_h^{(r)} \le \omega(t^{(r)}, \bar{\eta})$ for $0 \le r \le K$ and (3.17) follows.

II. It is clear that there exists exactly one solution to (2.1), (2.2). We prove (3.19). It follows from (2.1) that

$$u_h^{(r+1,m)} = h_0 F(t^{(r)}, x^{(m)}, (T_h u_h)_{[r,m]}, 0_{[n]}, 0_{[n\times n]}) + A^{(r,m)} \tag{3.26}$$

where

$$\begin{aligned} A^{(r,m)} = u_h^{(r,m)} + h_0[F(t^{(r)}, x^{(m)}, (T_h u_h)_{[r,m]}, \delta u_h^{(r,m)}, \delta^{(2)} u_h^{(r,m)}) \\ - F(t^{(r)}, x^{(m)}, (T_h u_h)_{[r,m]}, 0_{[n]}, 0_{[n\times n]})]. \end{aligned}$$

We conclude from Assumption $H[F, \varphi, \varphi_h]$ and from Lemma 3.5 that for each $(t^{(r)}, x^{(m)}) \in \theta_h \times Q_h$ there is $P \in \Omega$ such that

$$\begin{aligned} A^{(r,m)} &= u_h^{(r,m)} + h_0 \sum_{i=1}^{n} \partial_{q_i} F(P) \delta_i u_h^{(r,m)} + h_0 \sum_{i,j=1}^{n} \partial_{s_{ij}} F(P) \delta_{ij} u_h^{(r,m)} \\ &= u_h^{(r,m)} + \Theta_h^{(m)}[u_h(t^{(r)}, \,\cdot\,), P] = (1 + X_0(P)) u_h^{(r,m)} \\ &\quad + \sum_{i=1}^{n} [X_+^{(i)}(P)\, u_h^{(r,m+\lambda_{i+} e_i)} + X_-^{(i)}(P)\, u_h^{(r,m-\lambda_{i-} e_i)}] \\ &\quad + \sum_{(i,j)\in J_+} [Y_+^{(i,j)}(P) u_h^{(r,m+\lambda_{i+j+}(e_i+e_j))} + Z_+^{(i,j)}(P) u_h^{(r,m-\lambda_{i-j-}(e_i+e_j))} \\ &\quad + \sum_{i,j)\in J_-} [Y_-^{(i,j)}(P) u_h^{(r,m-\lambda_{i-j+}(e_i-e_j))} + Z_-^{(i,j)}(P) u_h^{(r,m+\lambda_{i+j-}(e_i-e_j))}]. \end{aligned} \tag{3.27}$$

Write $\omega_h^{(r)} = \|u_h\|_{h.r}$, $0 \le r \le K$. It follows from (3.12)–(3.16) and from (3.18), (3.26), (3.27) that

$$\omega_h^{(r+1)} \le \omega_h^{(r)} + h_0 \varrho(t^{(r)}, \omega_h^{(r)}), \quad 0 \le r \le K-1,$$

and $\omega_h^{(0)} \le \bar{\eta}$. The above relations and (3.25) imply (3.19). This completes the proof of the theorem. ∎

4. Convergence of difference schemes. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $W \in M_{n\times n}$, $W = [w_{ij}]_{i,j=1}^n$ we put

$$\|x\| = \sum_{i=1}^n |x_i|, \quad \|W\| = \max\Big\{\sum_{j=1}^n |w_{ij}| : 1 \le i \le n\Big\}.$$

Write $\eta_\star = \max\{\tilde{\eta}, \bar{\eta}\}$ and $C = \omega(a, \eta_\star)$ where $\omega(\cdot, \eta_\star)$ is the maximal solution to (3.2) with $\eta = \eta_\star$. Set $\Omega[C] = \{(t,x,w,q,s) \in \Omega : \|w\|_B \le C\}$.

Assumption $H[F,\sigma]$. There is $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

1) σ is continuous and it is nondecreasing with respect to both variables,
2) $\sigma(t,0) = 0$ for $t \in [0,a]$ and the function $\tilde{\omega}(t) = 0$ for $t \in [0,a]$ is the maximal solution of the Cauchy problem

$$\omega'(t) = \sigma(t, \omega(t)), \quad \omega(0) = 0, \tag{4.1}$$

3) the estimate

$$|F(t,x,w,q,s) - F(t,x,\tilde{w},q,s)| \le \sigma(t, \|w - \tilde{w}\|_B) \tag{4.2}$$

is satisfied on $\Omega[C]$.

Remark 4.1. It is important that we have assumed condition (4.2) for $\|w\|_B, \|\tilde{w}\|_B \le C$. There are differential equations with deviated variables and differential integral equations such that Assumption $H[F,\sigma]$ holds and condition (4.2) is not satisfied on Ω. We will give suitable examples.

Suppose that $\tilde{F} : E \times \mathbb{R}^n \times M_{n\times n} \to \mathbb{R}$ and $\tilde{L} : E \to \mathbb{R}$, $\phi : E \to \mathbb{R}^{1+n}$, $\phi = (\phi_0, \phi_1, \dots, \phi_n)$, are given functions. We assume that $\tilde{F}$ and $\tilde{L}$, ϕ are continuous and $\phi_0(t,x) \le t$ and $\phi(t,x) \in E$ for $(t,x) \in E$. Then $\phi(t,x) - (t,x) \in B$ for $(t,x) \in E$.

Suppose that the function $G : \mathbb{R} \to \mathbb{R}$ satisfies the conditions:

(i) G is of class C^1 on $\mathbb{R}$ and there is $\tilde{C} \in \mathbb{R}_+$ such that $|G(p)| \le \tilde{C}$ for $p \in \mathbb{R}$,
(ii) the function $\tilde{G}(p) = pG'(p)$, $p \in \mathbb{R}$, is unbounded on $\mathbb{R}$.

Let $F : \Omega \to \mathbb{R}$ be defined by

$$F(t,x,w,q,s) = \tilde{L}(t,x) w(\phi(t,x) - (t,x)) G(w(\phi(t,x) - (t,x))) + \tilde{F}(t,x,q,s). \tag{4.3}$$

Then (1.1) reduces to the differential equation with deviated variables

$$\partial_t z(t,x) = \tilde{L}(t,x)z(\phi(t,x))G(z(\phi(t,x))) + \tilde{F}(t,x,\partial_x z(t,x),\partial_{xx} z(t,x)).$$

Suppose that $\tilde{\beta},\tilde{\gamma} > 0$ are such that

$$|\tilde{L}(t,x)| \le \tilde{\beta}, \quad |\tilde{F}(t,x,0_{[n]},0_{[n\times n]})| \le \tilde{\gamma} \quad \text{for } (t,x)\in E.$$

Then

$$|F(t,x,w,0_{[n]},0_{[n\times n]})| \le \tilde{\beta}\|w\|_B + \tilde{\gamma} \tag{4.4}$$

where $(t,x,w)\in E\times C(B,\mathbb{R})$ and the solution of (3.2) is given by

$$\tilde{\omega}(t,\eta) = \eta\exp[\tilde{\beta}t] + \frac{\exp[\tilde{\beta}t]-1}{\tilde{\beta}}\tilde{\gamma}.$$

It follows that the function F given by (4.3) satisfies the Lipschitz condition with respect to the function variable on $\Omega[C]$ where $C = \tilde{\omega}(a,\eta)$ and the global Lipschitz condition is not satisfied.

Now we construct an integral functional equation. For the above $G : \mathbb{R}\to\mathbb{R}$ and $\tilde{F} : E\times\mathbb{R}^n\times M_{n\times n}\to\mathbb{R}$ we define

$$F(t,x,w,q,s) = \tilde{L}(t,x)\int_{D[t,x]} w(\tau,y)\,dy\,d\tau\, G\Big(\int_{D[t,x]} w(\tau,y)\,dy\,d\tau\Big) + \tilde{F}(t,x,q,s). \tag{4.5}$$

Then (1.1) reduces to the integral differential equation

$$\partial_t z(t,x) = \tilde{L}(t,x)\int_{D[t,x]} z(\tau,y)\,dy\,d\tau\, G\Big(\int_{D[t,x]} z(\tau,y)\,dy\,d\tau\Big) + \tilde{F}(t,x,\partial_x z(t,x),\partial_{xx}z(t,x)).$$

It is clear that there are $\tilde{\beta},\tilde{\alpha} > 0$ such that the function F defined by (4.5) satisfies condition (4.4) where $(t,x,w)\in E\times C(B,\mathbb{R})$. Then there is $L\ge 0$ such that Assumption $H[F,\sigma]$ holds for $\sigma(t,p) = Lp$ and the global Lipschitz condition is not satisfied.

Note that the function

$$G(p) = C_0\sin(C_1 p) + C_2\cos(C_3 p),$$

where $C_0,C_1,C_2,C_3\in\mathbb{R}$, satisfies the above conditions (i), (ii).

LEMMA 4.2. *If $z : E_0\cup E\to\mathbb{R}$ is of class $\mathbf{C}^{1.2}$ then there is $\alpha_\star : \Delta\to\mathbb{R}_+$ such that*

$$\|\delta z^{(r,m)} - \partial_x z^{(r,m)}\| \le \alpha_\star(h), \quad \|\delta^{(2)} z^{(r,m)} - \partial_{xx} z^{(r,m)}\| \le \alpha_\star(h)$$

where $(t^{(r)},x^{(m)})\in\theta_h\times Q_h$, $h\in\Delta$, and $\lim_{h\to 0}\alpha_\star(h) = 0$.

We omit a simple proof of the lemma. We now give sufficient conditions for the convergence of difference schemes (2.1), (2.2) and (2.3), (2.2).

Theorem 4.3. *Suppose that Assumptions $H[T_h]$, $H[F,\varphi,\varphi_h]$, $H[F,\sigma]$ are satisfied and $\tilde z : E_0 \cup E \to \mathbb{R}$ is a solution to* (1.1), (1.2) *and $\tilde z$ is of class $\mathbf{C}^{1.2}$.*

I. *There is $\alpha : \Delta \to \mathbb{R}_+$ such that*

$$|(\tilde z_h - v_h)(t,x)| \le \alpha(h) \text{ on } E_h \quad \text{and} \quad \lim_{h\to 0} \alpha(h) = 0, \tag{4.6}$$

where $v_h : E_{0.h} \cup E_h \to \mathbb{R}$ is a solution to (2.3), (2.2) *and $\tilde z$ is the restriction of $\tilde z$ to $E_{0.h} \cup E_h$.*

II. *Assume that the steps of the mesh satisfy the condition* (3.18) *where $X_0(P)$ is given by* (3.10). *Then there is $\alpha : \Delta \to \mathbb{R}_+$ such that*

$$|(\tilde z_h - u_h)(t,x)| \le \alpha(h) \text{ on } E_h \quad \text{and} \quad \lim_{h\to 0} \alpha(h) = 0, \tag{4.7}$$

where $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ is a solution to (2.1), (2.2).

Proof. The proof will be divided into two parts.

I. The existence and uniqueness of the solution $v_h : E_{0.h} \cup E_h \to \mathbb{R}$ to (2.3), (2.2) follows from Theorem 3.6. Let $\Gamma_{\mathrm{im}.h} : E'_h \to \mathbb{R}$ be defined by

$$\delta_0 \tilde z_h^{(r,m)} = \mathbb{F}_{\mathrm{im}.h}[\tilde z_h]^{(r,m)} + \Gamma_{\mathrm{im}.h}^{(r,m)} \quad \text{on } E'_h.$$

It follows from Lemma 4.2 and from Assumption $H[T_h]$ that there is $\gamma : \Delta \to \mathbb{R}_+$ such that

$$|\Gamma_{\mathrm{im}.h}^{(r,m)}| \le \gamma(h) \text{ on } E'_h \quad \text{and} \quad \lim_{h\to 0} \gamma(h) = 0.$$

Write $\vartheta_h = \tilde z_h - v_h$. Then

$$\vartheta_h^{(r+1,m)} = \vartheta_h^{(r,m)} + A_h[\tilde z_h, v_h]^{(r,m)} + B_h[\tilde z_h, v_h]^{(r,m)} \tag{4.8}$$

where

$$\begin{aligned} A_h[\tilde z_h, v_h]^{(r,m)} &= h_0[\mathbb{F}_{\mathrm{im}.h}[\tilde z_h]^{(r,m)} \\ &\qquad - F(t^{(r)}, x^{(m)}, (T_h v_h)_{[r,m]}, \delta \tilde z_h^{(r+1,m)}, \delta^{(2)} \tilde z_h^{(r+1,m)})] + \Gamma_{\mathrm{im}.h}^{(r,m)}, \\ B_h[\tilde z_h, v_h]^{(r,m)} &= h_0[F(t^{(r)}, x^{(m)}, (T_h v_h)_{[r,m]}, \delta \tilde z_h^{(r+1,m)}, \delta^{(2)} \tilde z_h^{(r+1,m)}) \\ &\qquad - \mathbb{F}_{\mathrm{im}.h}[v_h]^{(r,m)}]. \end{aligned}$$

According to Assumption $H[F,\varphi,\varphi_h]$ we have

$$B_h[\tilde z_h, v_h]^{(r,m)} = h_0 \sum_{i=1}^n \partial_{q_i} F(P) \delta_i \vartheta_h^{(r+1,m)} + h_0 \sum_{i,j=1}^n \partial_{s_{ij}} F(P)\, \delta_{ij} \vartheta_h^{(r+1,m)}$$

where $P \in \Omega$ is an intermediate point. We conclude from Lemma 3.5 that

$$B_h[\tilde z_h, v_h]^{(r,m)} = \Theta_h^{(m)}[\vartheta(t^{(r+1)}, \cdot), P]. \tag{4.9}$$

Write $\varepsilon_h^{(r)} = \|\vartheta_h\|_{h.r}$, $0 \le r \le K$. It follows from Assumption $H[T_h]$ and Lemma 3.1 that

(4.10)

$$\|(T_h\tilde{z}_h)_{[r,m]}\|_B \le \|T_h\tilde{z}_h\|_{t^{(r)}} \le \|\tilde{z}_h\|_{h.r} \le \omega(t^{(r)},\tilde{\eta}), \quad (t^{(r)},x^{(m)}) \in \theta_h \times Q_h.$$

According to Assumption $H[T_h]$ and (3.17) we have

$$\|(T_hv_h)_{[r,m]}\|_B \le \|T_hv_h\|_{t^{(r)}} \le \|v_h\|_{h.r} \le \omega(t^{(r)},\bar{\eta}), \quad (t^{(r)},x^{(m)}) \in \theta_h \times Q_h.$$

Thus we see that

$$\|(T_h\tilde{z}_h)_{[r,m]}\|_B \le C, \quad \|(T_hv_h)_{[r,m]}\|_B \le C, \quad (t^{(r)},x^{(m)}) \in \theta_h \times Q_h.$$

It follows from the above estimates and Assumption $H[F,\sigma]$ that

$$|A_h[\tilde{z}_h,v_h]^{(r,m)}| \le h_0\,\sigma(t^{(r)},\varepsilon_h^{(r)}) + h_0\gamma(h). \tag{4.11}$$

We conclude from (3.13)–(3.16) and from (4.8)–(4.11) that the function ε_h satisfies the recurrent inequality

$$\varepsilon_h^{(r+1)} \le \varepsilon_h^{(r)} + h_0\sigma(t^{(r)},\varepsilon_h^{(r)}) + h_0\gamma(h), \quad 0 \le r \le K-1,$$

and $\varepsilon_h^{(0)} \le \alpha_0(h)$. Let us denote by $\omega_h(\,\cdot\,,\gamma,\alpha_0)$ the maximal solution of the Cauchy problem

$$\omega'(t) = \sigma(t,\omega(t)) + \gamma(h), \quad \omega(0) = \alpha_0(h). \tag{4.12}$$

Then

$$\lim_{h\to 0} \omega_h(t,\gamma,\alpha_0) = 0 \quad \text{uniformly on } [0,a]$$

and

$$\omega_h(t^{(r+1)},\gamma,\alpha_0) \ge \omega_h(t^{(r)},\gamma,\alpha_0) + h_0\sigma(t^{(r)},\omega_h(t^{(r)},\gamma,\alpha_0)) + h_0\gamma(h), \quad 0 \le r \le K-1.$$

This gives $\varepsilon_h(t^{(r)}) \le \omega_h(t^{(r)},\gamma,\alpha_0)$ for $0 \le r \le K$. Thus we see that assertion (4.6) is satisfied with $\alpha(h) = \omega_h(a,\gamma,\alpha_0)$.

II. We prove (4.7). Let $\varGamma_{\mathrm{ex}.h} : E'_h \to \mathbb{R}$ be defined by the relation

$$\delta_0\tilde{z}_h^{(r,m)} = \mathbb{F}_{\mathrm{ex}.h}[\tilde{z}_h]^{(r,m)} + \varGamma_{\mathrm{ex}.h}^{(r,m)} \quad \text{on } E'_h.$$

It follows that there is $\gamma : \varDelta \to \mathbb{R}_+$ such that

$$|\varGamma_{\mathrm{ex}.h}^{(r,m)}| \le \gamma(h) \quad \text{on } E'_h, \quad \lim_{h\to 0}\gamma(h) = 0.$$

Write $\tilde{\vartheta}_h = \tilde{z}_h - u_h$. Then we have

$$\tilde{\vartheta}_h^{(r+1,m)} = \tilde{\vartheta}_h^{(r,m)} + \tilde{A}_h[\tilde{z}_h,u_h]^{(r,m)} + \tilde{B}_h[\tilde{z}_h,u_h]^{(r,m)} \tag{4.13}$$

where

$$\tilde{A}_h[\tilde{z}_h,v_h]^{(r,m)} = h_0[\mathbb{F}_{\mathrm{ex}.h}[\tilde{z}_h]^{(r,m)} - F(t^{(r)},x^{(m)},(T_hu_h)_{[r,m]},\delta\tilde{z}_h^{(r,m)},\delta^{(2)}\tilde{z}_h^{(r,m)})] + \varGamma_{\mathrm{ex}.h}^{(r,m)},$$

$$\tilde{B}_h[\tilde{z}_h,u_h]^{(r,m)} = h_0[F(t^{(r)},x^{(m)},(T_hu_h)_{[r,m]},\delta\tilde{z}_h^{(r,m)},\delta^{(2)}\tilde{z}_h^{(r,m)}) - \mathbb{F}_{\mathrm{ex}.h}[v_h]^{(r,m)}].$$

It follows from Lemma 3.5 that

$$(4.14)\quad \tilde{\vartheta}_h^{(r,m)} + \tilde{B}_h[\tilde{z}_h, u_h]^{(r,m)} = \tilde{\vartheta}_h^{(r,m)} + \Theta_h^{(m)}[\tilde{\vartheta}(t^{(r)}, \cdot), P]$$
$$= (1 + X_0(P))\tilde{\vartheta}_h^{(r,m)} + \sum_{i=1}^{n}[X_+^{(i)}(P)\,\tilde{\vartheta}_h^{(r,m+\lambda_{i+}e_i)} + X_-^{(i)}(P)\tilde{\vartheta}_h^{(r,m-\lambda_{i-}e_i)}]$$
$$+ \sum_{(i,j)\in J_+}[Y_+^{(i,j)}(P)\tilde{\vartheta}_h^{(r,m+\lambda_{i+j+}(e_i+e_j))} + Z_+^{(i,j)}(P)\tilde{\vartheta}_h^{(r,m-\lambda_{i-j-}(e_i+e_j))}]$$
$$+ \sum_{i,j)\in J_-}[Y_-^{(i,j)}(P)\tilde{\vartheta}_h^{(r,m-\lambda_{i-j+}(e_i-e_j))} + Z_-^{(i,j)}(P)\tilde{\vartheta}_h^{(r,m+\lambda_{i+j-}(e_i-e_j))}]$$

where $P \in \Omega$ is an intermediate point. Write $\tilde{\varepsilon}_h^{(r)} = \|\tilde{\vartheta}_h\|_{h.r}$, $0 \le r \le K$. We conclude from Assumption $H[T_h]$ and (3.19) that

$$\|(T_h u_h)_{[r,m]}\|_B \le \|T_h u_h\|_{t^{(r)}} \le \|u_h\|_{h.r} \le \omega(t^{(r)}, \bar{\eta}), \qquad (t^{(r)}, x^{(m)}) \in \theta_h \times Q_h.$$

The above relations and (4.10) imply

$$\|(T_h \tilde{z}_h)_{[r,m]}\|_B \le C, \qquad \|(T_h u_h)_{[r,m]}\|_B \le C, \qquad (t^{(r)}, x^{(m)}) \in \theta_h \times Q_h.$$

It follows from the above estimates and Assumption $H[F, \sigma]$ that

$$(4.15)\qquad |\tilde{A}_h[\tilde{z}_h, u_h]^{(r,m)}| \le h_0\sigma(t^{(r)}, \tilde{\varepsilon}_h^{(r)}) + h_0\gamma(h).$$

We conclude from (3.13)–(3.16) and from (3.18), (4.13)–(4.15) that

$$|\tilde{\vartheta}_h^{(r,m)} + \tilde{B}_h[\tilde{z}_h, u_h]^{(r,m)}| \le \tilde{\varepsilon}_h^{(r)}.$$

Thus we see that the function $\tilde{\varepsilon}_h$ satisfies the recurrent inequality

$$\tilde{\varepsilon}_h^{(r+1)} \le \tilde{\varepsilon}_h^{(r)} + h_0\sigma(t^{(r)}, \tilde{\varepsilon}_h^{(r)}) + h_0\gamma(h), \qquad 0 \le r \le K-1,$$

and $\tilde{\varepsilon}_h^{(0)} \le \alpha_0(h)$. Then we obtain (4.7) with $\alpha(h) = \omega_h(a, \gamma, \alpha_0)$ where $\omega_h(\,\cdot\,, \gamma, \alpha_0)$ is the maximal solution to (4.12) with the above given γ. This completes the proof of the theorem. ■

REMARK 4.4. Relations between h_0 and h' are required in (3.18). Suppose that the steps $(h_1, \ldots, h_n)$ are given and we have constructed the mesh X_h on $\overline{Q}$, the coefficients

$$\lambda_{i-}, \lambda_{i+} \quad \text{for } i = 1, \ldots, n, \qquad \lambda_{i-j+}, \lambda_{i+j-} \quad \text{for } (i,j) \in J_-,$$
$$\lambda_{i-j-}, \lambda_{i+j+} \quad \text{for } (i,j) \in J_+$$

are given and the function

$$\tilde{X}_0(P) = \frac{1}{h_0} X_0(P), \qquad P = (t, x, w, q, s) \in \Omega,$$

is bounded. It follows that there is $\varepsilon_0 > 0$ such that condition (3.18) is satisfied for $0 < h_0 < \varepsilon_0$.

REMARK 4.5. Condition (3.18) shows that the conditions on the mesh for explicit difference schemes are more restrictive than the suitable assumptions for implicit methods.

REMARK 4.6. Note that assumption (3.18) for $(t^{(r)}, x^{(m)}) \in \theta_h \times \operatorname{Int} Q_h$, $(w, q, s) \in \mathbf{C}(B, \mathbb{R}) \times \mathbb{R}^n \times M_{n\times n}$ is equivalent to the following inequality:

$$1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2}\, \partial_{s_{ii}} F(P) + h_0 \sum_{(i,j)\in J} \frac{1}{h_i h_j} |\partial_{s_{ij}} F(P)| \ge 0. \tag{4.16}$$

The conditions (3.18) and (4.16) are complicated because we consider functional differential equations with all the derivatives $[\partial_{x_i x_j} z]_{i,j=1}^n$. Let us consider the equation

$$\partial_t z(t,x) = \sum_{i=1}^{n} \partial_{x_i x_i} z(t,x) + f(t, x, z_{(t,x)}, \partial_x z(t,x))$$

where $f : E \times \mathbf{C}(B, \mathbb{R}) \times \mathbb{R}^n$ is a given function. Then condition (3.18) has the form

$$1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2}\, \frac{1}{\lambda_{i-} \lambda_{i+}} \ge 0.$$

LEMMA 4.7. *Suppose that Assumptions $H[T_h]$, $H[F, \varphi, \varphi_h]$, $H[F, \sigma]$ are satisfied with $\sigma(t, p) = Lp$ on $[0, a] \times \mathbb{R}_+$ and $\tilde{z} : E_0 \cup E \to \mathbb{R}$ is a solution to* (1.1), (1.2) *and $\tilde{z}$ is of class $\mathbf{C}^{1.2}$. Then*

$$|(\tilde{z}_h - v_h)(t,x)| \le \tilde{\alpha}(h) \qquad \text{on } E_h$$

where $v_h : E_{0.h} \cup E_h \to \mathbb{R}$ is a solution to (2.3), (2.2) *and $\tilde{z}_h$ is the restriction of $\tilde{z}$ to $E_{0.h} \cup E_h$ and*

$$\tilde{\alpha}(h) = \begin{cases} \alpha_0(h) e^{La} + \dfrac{\gamma(h)}{L}(e^{La} - 1) & \text{if } L > 0, \\ \alpha_0(h) e^{La} + a\gamma(h) & \text{if } L = 0. \end{cases}$$

If we assume that the steps of the mesh satisfy condition (3.18) *then*

$$|(\tilde{z}_h - u_h)(t,x)| \le \tilde{\alpha}(h) \qquad \text{on } E_h$$

with the above given $\tilde{\alpha}(h)$ where $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ is a solution to (2.1), (2.2).

We obtain the above estimates by solving problem (4.12) with $\sigma(t, p) = Lp$.

Lemma 4.7 shows that we have obtained the same error estimates for implicit and for explicit difference schemes.

REMARK 4.8. The results presented in the paper can be extended to weakly coupled functional differential systems.

5. Numerical examples. Write

$$Q = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\},$$
$$E = [0, 0.25] \times \overline{Q}, \qquad E_0 = \{0\} \times \overline{Q}, \qquad \partial_0 E = [0, 0.25] \times \partial Q.$$

We consider initial-boundary value problems for functional differential equations with solutions defined on E. Let us denote by z an unknown function of the variables (t, x, y).

Implicit difference methods lead to nonlinear systems of algebraic equations. In our experiments we have obtained approximate solutions of suitable nonlinear systems by using the Newton method. We have calculated three Newton iterations.

EXAMPLE 5.1. Consider the differential integral equation

$$\begin{aligned}\partial_t z(t, x, y) &= 2\partial_{xx} z(t, x, y) + 2\partial_{yy} z(t, x, y)\\ &\quad + \sin[\partial_{xx} z(t, x, y) - \partial_{yy} z(t, x, y) - 4t^2(x^2 - y^2) z(t, x, y)] + \partial_{xy} z(t, x, y)\\ &\quad + t\int_0^x s z(t, s, y)ds - t\int_0^y s z(t, x, s)\, ds\\ &\quad - z(t, x, y) \sin z(t, x, y) + f(t, x, y) z(t, x, y) + g(t, x, y)\end{aligned}$$

with the initial-boundary condition

$$z(t, x, y) = 1 \qquad \text{for } (t, x, y) \in E_0 \cup \partial_0 E.$$

where

$$\begin{aligned} f(t, x, y) &= 1 - x^2 - y^2 + 8t - 4xyt^2 - 8t^2(x^2 + y^2)\\ &\qquad + \sin\exp\{t(1 - x^2 - y^2)\},\\ g(t, x, y) &= \tfrac{1}{2}\exp[t(1 - x^2)] - \tfrac{1}{2}\exp[t(1 - y^2)].\end{aligned}$$

The solution of the above problem is known: it is

$$\tilde{z}(t, x, y) = \exp\{t(1 - x^2 - y^2)\}.$$

The following tables show the maximal values of errors for several step sizes.

Table 1. Explicit difference method

$h_1 = h_2$	h_0	Maximal error	Time
$2^{-1}\, 10^{-1}$	$2^{-3}\, 10^{-4}$	$9.094634 \cdot 10^{-4}$	6 min
$4 \cdot 10^{-2}$	$2^{-4}\, 10^{-4}$	$7.431544 \cdot 10^{-4}$	17 min
2^{-5}	$2^{-5}\, 10^{-4}$	$5.910445 \cdot 10^{-4}$	60 min

Now we consider the implicit difference schemes with steps of the mesh given in Table 2.

Table 2. Explicit difference method, condition (3.18) violated

$h_1 = h_2$	h_0	Maximal error
$2^{-1}\,10^{-1}$	10^{-3}	$1.373362 \cdot 10^{11}$
$2^2\,10^{-2}$	10^{-3}	$5.287499 \cdot 10^{13}$
2^{-5}	10^{-3}	$3.140010 \cdot 10^{16}$

Table 3. Implicit difference method

$h_1 = h_2$	h_0	Maximal error	Time
$2^{-1}\,10^{-1}$	10^{-3}	$9.113438 \cdot 10^{-4}$	18 s
$2^2\,10^{-2}$	10^{-3}	$7.441051 \cdot 10^{-4}$	28 s
2^{-5}	10^{-3}	$5.913871 \cdot 10^{-4}$	50 s

EXAMPLE 5.2. Let us consider the differential equation with deviated variables

$$\begin{aligned}\partial_t z(t,x,y) = 2\partial_{xx} z(t,x,y) + 2\partial_{yy} z(t,x,y) - \partial_{xy} z(t,x,y)\\ - \cos[\partial_{xx} z(t,x,y) - \partial_{yy} z(t,x,y)]\\ + z(t, \tfrac{1}{2}(\sqrt{3}x + y), \tfrac{1}{2}(x - \sqrt{3}y)) \cos z(t, \tfrac{1}{2}(x + \sqrt{3}y), \tfrac{1}{2}(\sqrt{3}x - y))\\ + f(t,x,y) z(t,x,y) + g(t,x,y),\end{aligned}$$

with the initial-boundary condition

$$z(t,x,y) = 1 \qquad \text{for } (t,x,y) \in E_0 \cup \partial_0 E,$$

where

$$\begin{aligned}f(t,x,y) &= x^2 - y^2 - 1 - 8t - 8t^2(x^2+y^2) + 4xyt^2 - \cos\exp\{t(x^2+y^2-1)\},\\ g(t,x,y) &= \cos\{4t^2(x^2-y^2)\exp[t(x^2+y^2-1)]\}.\end{aligned}$$

The solution of the above problem is $\tilde{z}(t,x,y) = e^{t(x^2+y^2-1)}$.

Table 4. Explicit difference method

$h_1 = h_2$	h_0	Maximal error	Time
10^{-1}	10^{-4}	$1.227446 \cdot 10^{-4}$	1 min
$2^{-1}\,10^{-1}$	10^{-5}	$8.719288 \cdot 10^{-5}$	3 min
$2^{-2}\,10^{-1}$	$2^{-5}\,10^{-4}$	$4.751054 \cdot 10^{-5}$	20 min

Table 5. Explicit difference method, condition (3.18) violated

$h_1 = h_2$	h_0	Maximal error
10^{-1}	$2 \cdot 10^{-3}$	$1.312591 \cdot 10^{-4}$
$2^{-1} \cdot 10^{-1}$	10^{-4}	$1.952779 \cdot 10^{0}$
$2^{-2} \cdot 10^{-1}$	10^{-4}	$1.026864 \cdot 10^{9}$

Now we consider implicit difference schemes with steps of the mesh given in Table 5.

Table 6. Implicit difference method

$h_1 = h_2$	h_0	Maximal error	Time
10^{-1}	$2 \cdot 10^{-3}$	$1.325853 \cdot 10^{-4}$	2 s.
$2^{-1}\, 10^{-1}$	10^{-4}	$8.771281 \cdot 10^{-5}$	18 s.
$2^{-2}\, 10^{-1}$	10^{-4}	$4.757541 \cdot 10^{-5}$	76 s.

REMARK 5.3. Note that the right hand sides of the equations considered in this section satisfy the assumptions of Theorem 4.3. The local Lipschitz condition with respect to the function variable holds and the global Lipschitz condition is not satisfied.

Our considerations show that there are the following relations between explicit and implicit difference methods for (1.1), (1.2). Assumptions on the regularity of given functions are the same in the theorems on the convergence of explicit and implicit difference schemes. We need condition (3.18) on the mesh for explicit difference methods, while this condition is not needed in the case of implicit difference methods. Error estimates are the same for both methods. Tables 2, 3 and 5, 6 show that there are implicit difference methods which are convergent, while the corresponding explicit difference schemes are not.

Acknowledgements. The authors are greatly indebted to the referee for valuable comments and suggestions.

References

[1] H. Brunner, *The numerical treatment of ordinary and partial Volterra integro-differential equations*, in: Proc. First Int. Colloq. Numer. Anal. (Plovdiv, 1992), D. Bainov (ed.), VSP, Utrecht, 1993, 13–26.

[2] S. Brzychczy, *Existence of solutions for nonlinear systems of differential-functional equations of parabolic type in an arbitrary domain*, Ann. Polon. Math. 47 (1987), 309–317.

[3] R. Ciarski, *Numerical approximations of parabolic differential functional equations with initial boundary conditions of the Neumann type*, ibid. 84 (2004), 103–119.

[4] W. Czernous and Z. Kamont, *Implicit difference methods for parabolic functional differential equations*, Z. Angew. Math. Mech. 85 (2005), 326–338.

[5] —, —, *Comparison of explicit and implicit difference methods for quasilinear functional differential equations*, Appl. Math. (Warsaw) 38 (2011), 315–340.

[6] Z. Kamont, *Implicit difference schemes for evolution functional differential equations*, Z. Anal. Anwend. 30 (2011), 105–128.

[7] —, *Hyperbolic Fuctional Differential Inequalities*, Kluwer, Dordrecht, 1999.

[8] Z. Kamont, *Difference schemes for evolution functional differential equations*, Funct. Differential Equations 16 (2009), 315–351.

[9] Z. Kamont and K. Kropielnicka, *Implicit difference functional inequalities and applications*, J. Math. Inequal. 2 (2008), 407–427.

[10] Z. Kamont and H. Leszczyński, *Stability of difference equations generated by parabolic differential-functional problems*, Rend. Mat. Appl. (7) 16 (1996), 265–287.

[11] W. Kohl, *On a class of parabolic integro-differential equations*, Z. Anal. Anwend. 19 (2000), 159–201.

[12] K. Kropielnicka, *Difference methods for parabolic functional differential problems of the Neumann type*, Ann. Polon. Math. 92 (2007), 163–178.

[13] —, *Implicit difference methods for quasilinear parabolic functional differential problems of the Dirichlet type*, Appl. Math. (Warsaw) 35 (2008), 155–175.

[14] —, *Implicit difference methods for parabolic functional differential problems of the Neumann type*, Nonlinear Oscill. 11 (2008), 345–364.

[15] X. Lu, *Combined iterative methods for numerical solutions of parabolic problems with time delays*, Appl. Math. Comput. 89 (1998), 213–224.

[16] M. Malec et A. Schiaffino, *Méthode aux différences finies pour une équation non-linéaire différentielle fonctionnelle du type parabolique avec une condition initiale de Cauchy*, Boll. Un. Mat. Ital. B (7) (1987), 99–209.

[17] M. Netka, *Implicit difference inequalities corresponding to parabolic functional differential equations*, Differential Equations Appl. 3 (2011), 21–41.

[18] C. V. Pao, *Numerical methods for systems of nonlinear parabolic equations with time delays*, J. Math. Anal. Appl. 240 (1999), 249–270.

[19] —, *Finite difference reaction-diffusion systems with coupled boundary conditions and time delays*, J. Math. Anal. Appl. 272 (2002), 407–434.

[20] R. Redheffer and W. Walter, *Stability of the null solution of parabolic functional differential inequalities*, Trans. Amer. Math. Soc. 262 (1980), 285–302.

[21] R. Redlinger, *Existence theorems for semilinear parabolic systems with functionals*, Nonlinear Anal. 8 (1984), 667–682.

[22] L. Sapa, *A finite difference method for quasilinear and nonlinear differential functional equations with Dirichlet's condition*, Ann. Polon. Math. 93 (2008), 113–133.

[23] L. Simon, *Application of monotone type operators to parabolic and functional parabolic PDE's*, in: Evolutionary Equations, C. M. Dafermos and M. Pokorný (eds.), Handb. Differ. Equ., Elsevier, Amsterdam, 2008, 267–321.

[24] A. L. Skubachevskiĭ and R. V. Shamin, *Second order parabolic differential-difference equations*, Dokl. Akad. Nauk 379 (2001), 595–598 (in Russian).

[25] J. Szarski, *Uniqueness of solutions of a mixed problem for parabolic differential-functional equations*, Ann. Polon. Math. 28 (1973), 57–65.

[26] K. Topolski, *On the existence of classical solutions for differential-functional IBVP*, Abstr. Appl. Anal. 3 (1998), 363–375.

[27] —, *Parabolic differential-functional inequalities in viscosity sense*, Ann. Polon. Math. 68 (1998), 17–25.

[28] W. Voigt, *Finite-difference schemes for parabolic problems with first and mixed second order derivatives*, Z. Angew. Math. Mech. 68 (1988), 281–288.

[29] —, *On finite-difference methods for parabolic functional differential equations on unbounded domains*, in: Numerical Methods and Applications, Publ. House Bulgar. Acad. Sci., Sofia, 1989, 559–567.

[30] W. M. Wang and C. V. Pao, *Time delayed finite difference reaction-diffusion systems with nonquasimonotone functions*, Numer. Math. 103 (2006), 485–513.

[31] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer, Berlin, 1996.

Zdzisław Kamont, Karolina Kropielnicka
Institute of Mathematics
University of Gdańsk
Wit Stwosz Street 57
80-952 Gdańsk, Poland
E-mail: Zdzislaw.Kamont@mat.ug.edu.pl
Karolina.Kropielnicka@mat.ug.edu.pl

Received 10.1.2011
and in final form 15.4.2011 (2353)