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The global existence of mild solutions for semilinear fractional Cauchy problems in the α-norm

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Abstract. We study the local and global existence of mild solutions to a class of semilinear fractional Cauchy problems in the α-norm assuming that the operator in the linear part is the generator of a compact analytic C_0-semigroup. A suitable notion of mild solution for this class of problems is also introduced. The results obtained are a generalization and continuation of some recent results on this issue.

1. Introduction. Let $A : D(A) \to X$ be the infinitesimal generator of a compact analytic semigroup $\{T(t)\}_{t\geq 0}$ of uniformly bounded linear operators on a Banach space $(X, \|\cdot\|)$ and suppose $0 \in \rho(A)$. Denote by X_α with $0 < \alpha < 1$ the Banach space $D(A^\alpha)$ endowed with the graph norm $\|u\|_\alpha = \|A^\alpha u\|$ for $u \in X_\alpha$. We consider the Cauchy problem for a semilinear fractional integro-differential equation

$$(1.1)\qquad \begin{cases} {}^cD_t^\beta u(t) = Au(t) + F(t, u(t)) + \int_0^t K(t-s)H(s, u(s))\, ds, & t > 0, \\ u(0) = u_0 \end{cases}$$

in X_α, where ${}^cD_t^\beta$, $0 < \beta < 1$, is the Caputo fractional derivative of order β, $K \geq 0$ is an integrable function defined on $[0, \infty)$, and $F, H : [0, T]\times X_\alpha \to X$ are given operators to be specified later.

As indicated in [Hi, KST, MR, Po] and references therein, differential equations involving fractional derivatives in time, compared with those of integer order in time, are more realistic to describe many phenomena in nature and they can provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. The advantages of fractional derivatives become apparent in modeling mechanical and electrical properties of real materials, as well as in the description

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of rheological properties of rocks, and in many other fields (see for details [M, Po]). This class of equations has been the object of extensive study in recent years. In particular, by using fractional powers of operators and some fixed point theorems, the existence of mild solutions has been studied in [ZJ2] for fractional evolution equations with nonlocal initial conditions, and in [ZJ1] for fractional neutral evolution equations with nonlocal initial conditions and time delays. The existence of mild solutions for fractional differential equations with nonlocal initial conditions in the α-norm has been investigated in [DMN] using the contraction mapping principle and the Schauder fixed point theorem. Other investigations regarding this class of equations, their applications and various generalizations are reported in [ALN, AZH, CD, CL, HOB, LCL] and the references therein.

However, to the best of our knowledge, the global existence of mild solutions to the Cauchy problems for fractional differential equations is still an untreated topic in the literature. Moreover, let us point out that in the treatment of global existence of mild solutions for fractional Cauchy problems, one of the difficult points is to give a reasonable concept of mild solution. Motivated by these, in the present paper we will study the local and global existence of mild solutions for the fractional Cauchy problem (1.1). To this end, firstly, a more appropriate definition of mild solution for the fractional Cauchy problem (1.1) will be introduced. We shall then use fractional powers of operators and the Schauder fixed point theorem to obtain local existence, and a singular version of the Gronwall inequality to obtain global existence of mild solutions for (1.1). The results obtained in this paper may be considered as a generalization and continuation of some recent results on this issue.

Remark 1.1. Let us note that our results can be easily extended to the case when $X_\alpha = X$.

The paper is organized as follows. In Section 2, some required notation, definitions and lemmas are given. In Section 3, we present our main results.

2. Preliminaries and notation. Throughout this paper, we let $0 < \beta < 1$, and let $A : D(A) \to X$ be the infinitesimal generator of a compact analytic semigroup $\{T(t)\}_{t\geq 0}$ of uniformly bounded linear operators on X with $0 \in \rho(A)$, which allows us to define the fractional power A^α, for $0 < \alpha < 1$, as a closed linear operator on its domain $D(A^\alpha)$ with inverse $A^{-\alpha}$.

Let X_α denote the Banach space $D(A^\alpha)$ endowed with the graph norm $\|u\|_\alpha = \|A^\alpha u\|$ for $u \in X_\alpha$, and let $C([0,T]; X_\alpha)$ for $0 < T < \infty$ be the Banach space of all continuous functions from $[0,T]$ into X_α with the uniform norm $|u|_\alpha = \sup\{\|u(t)\|_\alpha : t \in [0,T]\}$. $\mathcal{L}(X)$ stands for the Banach space of all bounded linear operators on X. Let M be a constant such that

$$M = \sup\{\|T(t)\|_{\mathcal{L}(X)} : t \in [0, \infty)\}.$$

The following are the basic properties of A^α.

THEOREM 2.1 ([P, pp. 69–75]).

(a) *$T(t) : X \to X_\alpha$ for each $t > 0$, and $A^\alpha T(t)x = T(t)A^\alpha x$ for each $x \in X_\alpha$ and $t \geq 0$.*
(b) *$A^\alpha T(t)$ is bounded on X for every $t > 0$ and there exist $M_\alpha > 0$ and $\delta > 0$ such that $\|A^\alpha T(t)\|_{\mathcal{L}(X)} \leq (M_\alpha/t^\alpha)e^{-\delta t}$.*
(c) *$A^{-\alpha}$ is a bounded linear operator in X with $D(A^\alpha) = \mathrm{Im}(A^{-\alpha})$.*
(d) *If $0 < \alpha_1 \leq \alpha_2$, then $X_{\alpha_2} \hookrightarrow X_{\alpha_1}$.*

LEMMA 2.2 ([LC]). *The restriction $T_\alpha(t)$ of $T(t)$ to X_α is exactly the part of $T(t)$ in X_α and is an immediately compact semigroup in X_α, and hence it is immediately norm-continuous.*

REMARK 2.3. Recall that the semigroup $T_\alpha(t)$ is called *immediately compact* if $T_\alpha(t)$ is a compact operator for all $t > 0$.

In the following we recall some definitions of fractional calculus (see e.g. [KST, LV] for more details).

DEFINITION 2.4. The *Riemann–Liouville fractional integral* of order $\alpha > 0$ of a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

DEFINITION 2.5. The *Caputo fractional derivative* of order α ($m - 1 < \alpha < m$, $m \in \mathbb{N}$) of a function f is defined as

$${}^cD^\alpha f(t) = I^{m-\alpha} D_t^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} D_s^m f(s)\, ds,$$

where $D_t^m := d^m/dt^m$. If $0 < \alpha < 1$, then

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha}\, ds.$$

Define two families $\{\mathcal{S}_\beta(t)\}_{t\geq 0}$ and $\{\mathcal{P}_\beta(t)\}_{t\geq 0}$ of linear operators by

$$\mathcal{S}_\beta(t)x = \int_0^\infty \Psi_\beta(s) T(t^\beta s)x\, ds, \qquad \mathcal{P}_\beta(t)x = \int_0^\infty \beta s \Psi_\beta(s) T(t^\beta s)x\, ds$$

for $x \in X$, $t \geq 0$, where

$$\Psi_\beta(s) = \frac{1}{\pi\beta}\sum_{n=1}^{\infty}(-s)^{n-1}\frac{\Gamma(1+\beta n)}{n!}\sin(n\pi\beta), \qquad s \in (0,\infty),$$

is the function of Wright type defined on $(0,\infty)$ which satisfies

$$\begin{aligned} &\Psi_\beta(s) \geq 0, \qquad s \in (0,\infty), \qquad \int_0^\infty \Psi_\beta(s)\,ds = 1, \\ &\int_0^\infty s^\zeta \Psi_\beta(s)\,ds = \frac{\Gamma(1+\zeta)}{\Gamma(1+\beta\zeta)}, \qquad \zeta \in (-1,\infty). \end{aligned} \tag{2.1}$$

The following lemma follows from the results in [ZJ1].

LEMMA 2.6. *The following properties are valid.*

(1) *For every $t \geq 0$, $\mathcal{S}_\beta(t)$ and $\mathcal{P}_\beta(t)$ are bounded linear operators on X: for all $x \in X$ and $0 \leq t < \infty$,*

$$\|\mathcal{S}_\beta(t)x\| \leq M\|x\|, \qquad \|\mathcal{P}_\beta(t)x\| \leq \frac{\beta M}{\Gamma(1+\beta)}\|x\|.$$

(2) *For every $x \in X$, $t \mapsto \mathcal{S}_\beta(t)x$ and $t \mapsto \mathcal{P}_\beta(t)x$ are continuous functions from $[0,\infty)$ into X.*
(3) *$\mathcal{S}_\beta(t)$ and $\mathcal{P}_\beta(t)$ are compact operators on X for all $t > 0$.*
(4) *For all $x \in X$,*

$$\|A^\alpha \mathcal{P}_\beta(t)x\| \leq C_\alpha t^{-\alpha\beta}\|x\|, \quad \textit{where} \quad C_\alpha = \frac{M_\alpha \beta \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}.$$

In this paper we introduce the following definition of a mild solution of the Cauchy problem (1.1).

DEFINITION 2.7. Let $u_0 \in X_\alpha$ and $0 < T < \infty$. A function $u \in C([0,T];X_\alpha)$ is said to be a *mild solution* of the fractional Cauchy problem (1.1) on $[0,T]$ if $u(0) = u_0$ and whenever we split $[0,T] = \bigcup_{i\in\mathbb{N}\cup\{0\}}[\tau_i,\tau_{i+1}]$ with $\tau_0 = 0$, then

$$u(t) = \mathcal{S}_\beta(t-\tau_i)u(\tau_i) + \int_{\tau_i}^{t}(t-s)^{\beta-1}\mathcal{P}_\beta(t-s)\Big(F(s,u(s)) + \int_{\tau_i}^{s} K(s-\tau)H(\tau,u(\tau))\,d\tau\Big)\,ds$$

for $t \in [\tau_i,\tau_{i+1}]$, $i \in \mathbb{N}\cup\{0\}$.

REMARK 2.8. It is important to note that the operators $\mathcal{S}_\beta(t)$ and $\mathcal{P}_\beta(t)$ do not have the semigroup property.

REMARK 2.9. It is easy to see that the definition of a mild solution given here is more general than those used in previous research (see, e.g., [ZJ1, Lemma 3.1 and Definition 3.1]).

3. Main results. We first prove the following result.

LEMMA 3.1. *The functions* $t \mapsto A^\alpha \mathcal{P}_\beta(t)$ *and* $t \mapsto A^\alpha \mathcal{S}_\beta(t)$ *are continuous on* $(0, \infty)$ *in the uniform operator topology.*

Proof. Let $\epsilon > 0$. For every $r > 0$, from (2.1) we may choose $\delta_1, \delta_2 > 0$ such that

$$\frac{M_\alpha}{r^{\alpha\beta}} \int_0^{\delta_1} \Psi_\beta(s) s^{-\alpha}\, ds \le \frac{\epsilon}{6}, \quad \frac{M_\alpha}{r^{\alpha\beta}} \int_{\delta_2}^{\infty} \Psi_\beta(s) s^{-\alpha}\, ds \le \frac{\epsilon}{6}. \tag{3.1}$$

Since $t \mapsto A^\alpha T(t)$ is continuous on $(0, \infty)$ in the uniform operator topology (see [HRH, Lemma 2.1]), we deduce that there exists a constant $\delta > 0$ such that

$$\int_{\delta_1}^{\delta_2} \Psi_\beta(s) \| A^\alpha T(t_1^\beta s) - A^\alpha T(t_2^\beta s)\|_{\mathcal{L}(X)}\, ds \le \frac{\epsilon}{3} \tag{3.2}$$

for $t_1, t_2 \ge r$ and $|t_1 - t_2| < \delta$.

On the other hand, for any $x \in X$, we write

$$\begin{aligned} \mathcal{S}_\beta(t_1)x - \mathcal{S}_\beta(t_2)x &= \int_0^{\delta_1} \Psi_\beta(s)(T(t_1^\beta s)x - T(t_2^\beta s)x)\, ds \\ &\quad + \int_{\delta_1}^{\delta_2} \Psi_\beta(s)(T(t_1^\beta s)x - T(t_2^\beta s)x)\, ds \\ &\quad + \int_{\delta_2}^{\infty} \Psi_\beta(s)(T(t_1^\beta s)x - T(t_2^\beta s)x)\, ds. \end{aligned}$$

Therefore, using (3.1), (3.2) and Lemma 2.6 we get

$$\begin{aligned} &\| A^\alpha \mathcal{S}_\beta(t_1)x - A^\alpha \mathcal{S}_\beta(t_2)x\| \\ &\quad \le \int_0^{\delta_1} \Psi_\beta(s)(\|A^\alpha T(t_1^\beta s)\|_{\mathcal{L}(X)} + \|A^\alpha T(t_2^\beta s)\|_{\mathcal{L}(X)})\|x\|\, ds \\ &\qquad + \int_{\delta_1}^{\delta_2} \Psi_\beta(s)\|A^\alpha T(t_1^\beta s) - A^\alpha T(t_2^\beta s)\|_{\mathcal{L}(X)}\|x\|\, ds \\ &\qquad + \int_{\delta_2}^{\infty} \Psi_\beta(s)(\|A^\alpha T(t_1^\beta s)\|_{\mathcal{L}(X)} + \|A^\alpha T(t_2^\beta s)\|_{\mathcal{L}(X)})\|x\|\, ds \end{aligned}$$

$$\le \frac{2M_\alpha}{r^{\alpha\beta}} \int_0^{\delta_1} \Psi_\beta(s) s^{-\alpha} \|x\|\, ds$$

$$+ \int_{\delta_1}^{\delta_2} \Psi_\beta(s) \|T(t_1^\beta s) - T(t_2^\beta s)\|_{\mathcal{L}(X)} \|x\|\, ds + \frac{2M_\alpha}{r^{\alpha\beta}} \int_{\delta_2}^{\infty} \Psi_\beta(s) s^{-\alpha} \|x\|\, ds \le \epsilon \|x\|,$$

that is,

$$\|A^\alpha \mathcal{S}_\beta(t_1) - A^\alpha \mathcal{S}_\beta(t_2)\|_{\mathcal{L}(X)} \le \epsilon \quad \text{for } t_1, t_2 \ge r \text{ and } |t_1 - t_2| < \delta,$$

which together with the arbitrariness of $r > 0$ implies that $A^\alpha \mathcal{S}_\beta(t)$ is continuous for $t > 0$ in the uniform operator topology. A similar argument gives the continuity of $A^\alpha \mathcal{P}_\beta(t)$. ∎

Now, we are in a position to present our first result.

THEOREM 3.2. *Assume that $F, H : [0,\infty) \times X_\alpha \to X$ are continuous. Then for $u_0 \in X_\alpha$, the fractional Cauchy problem* (1.1) *has a mild solution $u \in C([0, t_{\max}); X_\alpha)$ defined on a maximal interval of existence* $[0, t_{\max})$.

Proof. Fix $\delta > 0$ and $u_0 \in X_\alpha$. From Lemma 2.6(2) it follows that there exists an $\eta_0 > 0$ such that

$$\|\mathcal{S}_\beta(t)u_0 - u_0\|_\alpha = \|\mathcal{S}_\beta(t) A^{1/2} u_0 - A^{1/2} u_0\| \le \delta/2 \text{ for all } 0 \le t \le \eta_0. \tag{3.3}$$

Given $\eta_1 > 0$, assume that $M', M'' > 0$ are two constants such that

$$M' = \sup\{\|F(t,x)\| : 0 \le t \le \eta_1,\ \|x - u_0\| < \delta\},$$
$$M'' = \sup\{\|H(t,x)\| : 0 \le t \le \eta_1,\ \|x - u_0\| < \delta\}.$$

We set $\widetilde{k} = \int_0^{\eta_0} K(t)\, dt$ and denote by E the Banach space $C([0,\eta]; X_\alpha)$ endowed with the norm

$$\|u\|_E = \sup\{\|u(t)\|_\alpha : t \in [0,\eta]\},$$

where

$$\eta = \min\left\{\eta_0, \eta_1, \left(\frac{\delta(1-\alpha)\beta}{2C_\alpha(M' + \widetilde{k}M'')}\right)^{1/(1-\alpha)\beta}\right\}.$$

Let $\Omega(u_0)$ be the closed and bounded subset of E defined by

$$\Omega(u_0) = \{u \in E : u(0) = u_0, \|u - u_0\|_\alpha \le \delta\}.$$

On $\Omega(u_0)$ we define a map Φ by

$$(\Phi u)(t) = \mathcal{S}_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1} \mathcal{P}_\beta(t-s) \Big(F(s, u(s)) + \int_0^s K(s-\tau) H(\tau, u(\tau))\, d\tau \Big)\, ds$$

$$=: \mathcal{S}_\beta(t)u_0 + (\Phi_1 u)(t), \qquad t \in [0,\eta].$$

It is evident that $\Phi u \in E$ and $(\Phi u)(0) = u_0$ for all $u \in E$. Also, for $t \in [0, \eta]$ and $u \in \Omega(u_0)$, by (3.3) and Lemma 2.6(4) we have

$$\begin{aligned}\|(\Phi u)(t) - u_0\|_\alpha &\le \|\mathcal{S}_\beta(t)u_0 - u_0\|_\alpha + \Big\| \int_0^t (t-s)^{\beta-1}\mathcal{P}_\beta(t-s)\Big(F(s,u(s)) \\ &\qquad + \int_0^s K(s-\tau)H(\tau,u(\tau))\,d\tau\Big)\,ds\Big\|_\alpha \\ &\le \frac{\delta}{2} + \int_0^t (t-s)^{\beta-1}\|A^\alpha \mathcal{P}_\beta(t-s)\|_{\mathcal{L}(X)} \\ &\qquad \times \Big\|F(s,u(s)) + \int_0^s K(s-\tau)H(\tau,u(\tau))\,d\tau\Big\|\,ds \\ &\le \frac{\delta}{2} + C_\alpha \int_0^t (t-s)^{\beta-1-\alpha\beta}(M' + \widetilde{k}M'')\,ds \\ &\le \frac{\delta}{2} + \frac{C_\alpha(M' + \widetilde{k}M'')t^{(1-\alpha)\beta}}{(1-\alpha)\beta} \le \delta,\end{aligned}$$

which implies that Φ maps $\Omega(u_0)$ into itself.

Next, we prove that Φ is continuous on $\Omega(u_0)$. Fix $\varepsilon > 0$ and $u_1 \in \Omega(u_0)$. It follows from the continuity of F, H that there exists a $\mu = \mu(u_1)$ such that

$$\begin{aligned}\|F(t,u_1(t)) - F(t,u_2(t))\| &\le \frac{\beta(1-\alpha)\varepsilon}{2C_\alpha \eta^{\beta(1-\alpha)}}, \\ \|H(t,u_1(t)) - H(t,u_2(t))\| &\le \frac{\beta(1-\alpha)\varepsilon}{2\widetilde{k}C_\alpha \eta^{\beta(1-\alpha)}}\end{aligned}$$

for any $u_2 \in \Omega(u_0)$ satisfying $\|u_1 - u_2\|_\alpha \le \mu$ and hence

$$\begin{aligned}&\|(\Phi u_1)(t) - \Phi u_2)(t)\|_\alpha \\ &\quad \le \int_0^t (t-s)^{\beta-1}\|A^\alpha \mathcal{P}_\beta(t-s)\|_{\mathcal{L}(X)}\|F(s,u_1(s)) - F(s,u_2(s))\|\,ds \\ &\qquad + \int_0^t (t-s)^{\beta-1}\|A^\alpha \mathcal{P}_\beta(t-s)\|_{\mathcal{L}(X)} \\ &\qquad\qquad \times \int_0^s K(s-\tau)\|H(\tau,u_1(\tau)) - H(\tau,u_2(\tau))\|\,d\tau\,ds \\ &\quad \le \frac{\beta(1-\alpha)\varepsilon}{C_\alpha \eta^{\beta(1-\alpha)}} \int_0^t (t-s)^{\beta(1-\alpha)-1}\,ds \le \varepsilon\end{aligned}$$

for $t \in [0, \eta]$. That is, Φ is continuous on $\Omega(u_0)$.

In what follows we show that Φ is compact on $\Omega(u_0)$. It is sufficient to show that Φ_1 is compact. Fix $t \in (0,\eta]$ and let $\epsilon, \epsilon_1 > 0$ be small enough. For $u \in \Omega(u_0)$, define the map $\Phi^{\epsilon,\epsilon_1}$ by

$$\begin{aligned}(\Phi^{\epsilon,\epsilon_1}u)(t) = \int_0^{t-\epsilon}\int_{\epsilon_1}^{\infty} \beta\tau\Psi_\beta(\tau)T((t-s)^\beta\tau)\Big(F(s,u(s)) \\ + \int_0^s K(s-\tau')H(\tau',u(\tau'))\,d\tau'\Big)\,d\tau\,ds \\ = T(\epsilon^\beta\epsilon_1)\int_0^{t-\epsilon}\int_{\epsilon_1}^{\infty} \beta\tau\Psi_\beta(\tau)T((t-s)^\beta\tau-\epsilon^\beta\epsilon_1)\Big(F(s,u(s)) \\ + \int_0^s K(s-\tau')H(\tau,u(\tau'))\,d\tau'\Big)\,d\tau\,ds.\end{aligned}$$

From Lemma 2.2 we see that for each $t \in (0,\eta]$, the set $\{(\Phi^{\epsilon,\epsilon_1}u)(t) : u \in \Omega(u_0)\}$ is relatively compact in X_α. Then, as

$$\begin{aligned}&\|(\Phi_1 u)(t) - (\Phi^{\epsilon,\epsilon_1}u)(t)\|_\alpha \\ &\le \Big\|\int_0^t\int_0^{\epsilon_1} \beta\tau(t-s)^{\beta-1}\Psi_\beta(\tau)T((t-s)^\beta\tau)\Big(F(s,u(s)) \\ &\qquad + \int_0^s K(s-\tau')H(\tau',u(\tau'))\,d\tau'\Big)\,d\tau\,ds\Big\|_\alpha \\ &\quad + \Big\|\int_{t-\epsilon}^t\int_{\epsilon_1}^{\infty} \beta\tau(t-s)^{\beta-1}\Psi_\beta(\tau)T((t-s)^\beta\tau)\Big(F(s,u(s)) \\ &\qquad + \int_0^s K(s-\tau')H(\tau',u(\tau'))\,d\tau'\Big)\,d\tau\,ds\Big\|_\alpha \\ &\le \int_0^t\int_0^{\epsilon_1} \beta\tau(t-s)^{\beta-1}\Psi_\beta(\tau)\|A^\alpha T((t-s)^\beta\tau)\|_{\mathcal{L}(X)} \\ &\qquad \times \Big\|F(s,u(s)) + \int_0^s K(s-\tau')H(\tau',u(\tau'))\,d\tau'\Big\|\,d\tau\,ds \\ &\quad + \int_{t-\epsilon}^t\int_{\epsilon_1}^{\infty} \beta\tau(t-s)^{\beta-1}\Psi_\beta(\tau)\|A^\alpha T((t-s)^\beta\tau)\|_{\mathcal{L}(X)} \\ &\qquad \times \Big\|F(s,u(s)) + \int_0^s K(s-\tau')H(\tau',u(\tau'))\,d\tau'\Big\|\,d\tau\,ds \\ &\le \beta M_\alpha(M' + \widetilde{k}M'')\Big(\int_0^t (t-s)^{\beta(1-\alpha)-1}\,ds\int_0^{\epsilon_1}\tau^{1-\alpha}\Psi_\beta(\tau)\,d\tau \\ &\qquad + \int_{t-\epsilon}^t (t-s)^{\beta(1-\alpha)-1}\,ds\int_{\epsilon_1}^{\infty}\tau^{1-\alpha}\Psi_\beta(\tau)\,d\tau\Big)\end{aligned}$$

$$\leq \beta M_\alpha(M'+\widetilde{k}M'')\left(\frac{t^{\beta(1-\alpha)}}{\beta(1-\alpha)}\int_0^{\epsilon_1}\tau^{1-\alpha}\Psi_\beta(\tau)\,d\tau+\frac{\Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\frac{\epsilon^{\beta(1-\alpha)}}{\beta(1-\alpha)}\right)$$
$$\to 0 \quad \text{as } \epsilon,\epsilon_1\to 0^+$$

in view of (2.1), we conclude, using the total boundedness, that for each $t\in(0,\eta]$, the set $\{(\Phi_1 u)(t): u\in\Omega(u_0)\}$ is relatively compact in X_α.

On the other hand, for $0<t_1<t_2\leq\eta$ and $\epsilon'>0$ small enough, we have

$$\|(\Phi_1 u)(t_1)-(\Phi_1 u)(t_2)\|_\alpha \leq A_1+A_2+A_3+A_4,$$

where

$$A_1=\int_{t_1}^{t_2}(t_2-s)^{\beta-1}\|A^\alpha\mathcal{P}_\beta(t_2-s)\|_{\mathcal{L}(X)} \times\left\|F(s,u(s))+\int_0^s K(s-\tau)H(\tau,u(\tau))\,d\tau\right\|ds,$$

$$A_2=\int_0^{t_1-\epsilon'}(t_1-s)^{\beta-1}\|A^\alpha\mathcal{P}_\beta(t_2-s)-A^\alpha\mathcal{P}_\beta(t_1-s)\|_{\mathcal{L}(X)} \times\left\|F(s,u(s))+\int_0^s K(s-\tau)H(\tau,u(\tau))\,d\tau\right\|ds,$$

$$A_3=\int_{t_1-\epsilon'}^{t_1}(t_1-s)^{\beta-1}\|A^\alpha\mathcal{P}_\beta(t_2-s)-A^\alpha\mathcal{P}_\beta(t_1-s)\|_{\mathcal{L}(X)} \times\left\|F(s,u(s))+\int_0^s K(s-\tau)H(\tau,u(\tau))\,d\tau\right\|ds,$$

$$A_4=\int_0^{t_1}|(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}|\cdot\|A^\alpha\mathcal{P}_\beta(t_2-s)\|_{\mathcal{L}(X)} \times\left\|F(s,u(s))+\int_0^s K(s-\tau)H(\tau,u(\tau))\,d\tau\right\|ds.$$

It follows from Lemmas 2.6 and 3.1 that

$$A_1\leq C_\alpha(M'+\widetilde{k}M'')\int_{t_1}^{t_2}(t_2-s)^{\beta-1-\alpha\beta}\,ds\leq\frac{C_\alpha(M'+\widetilde{k}M'')(t_2-t_1)^{(1-\alpha)\beta}}{(1-\alpha)\beta},$$

$$A_2\leq(M'+\widetilde{k}M'')\sup_{s\in[0,t_1-\epsilon']}\|A^\alpha\mathcal{P}_\alpha(t_2-s)-A^\alpha\mathcal{P}_\alpha(t_1-s)\|_{\mathcal{L}(X)} \times\int_0^{t_1-\epsilon'}(t_1-s)^{\beta-1}\,ds$$
$$\leq(M'+\widetilde{k}M'')\left(\frac{t_1^\beta}{\beta}-\frac{\epsilon'^\beta}{\beta}\right)\sup_{s\in[0,t_1-\epsilon']}\|A^\alpha\mathcal{P}_\alpha(t_2-s)-A^\alpha\mathcal{P}_\alpha(t_1-s)\|_{\mathcal{L}(X)},$$

$$A_3 \le C_\alpha(M' + \widetilde{k}M'') \int_{t_1-\epsilon'}^{t_1} (t_1 - s)^{\beta-1}((t_2 - s)^{-\alpha\beta} + (t_1 - s)^{-\alpha\beta})\, ds$$

$$\le 2C_\alpha(M' + \widetilde{k}M'') \int_{t_1-\epsilon'}^{t_1} (t_1 - s)^{\beta-1-\alpha\beta}\, ds \le \frac{2C_\alpha(M' + \widetilde{k}M'')\epsilon'^{(1-\alpha)\beta}}{(1-\alpha)\beta},$$

$$A_4 \le C_\alpha(M' + \widetilde{k}M'') \int_0^{t_1} ((t_1 - s)^{\beta-1} - (t_2 - s)^{\beta-1})(t_2 - s)^{-\alpha\beta}\, ds$$

$$\le C_\alpha(M' + \widetilde{k}M'') \int_0^{t_1} ((t_1 - s)^{(1-\alpha)\beta-1} - (t_2 - s)^{(1-\alpha)\beta-1})\, ds$$

$$\le \frac{C_\alpha(M' + \widetilde{k}M'')}{(1-\alpha)\beta}({t_1}^{(1-\alpha)\beta} - {t_2}^{(1-\alpha)\beta} + (t_2 - t_1)^{(1-\alpha)\beta}),$$

from which it is easy to see that A_i $(i = 1, 2, 3, 4)$ tends to zero independently of $u \in \Omega(u_0)$ as $t_2 - t_1 \to 0$ and $\epsilon' \to 0$. Hence,

$$\|(\Phi_1 u)(t_1) - (\Phi_1 u)(t_2)\|_\alpha \to 0 \quad \text{as } t_2 - t_1 \to 0,$$

and the limit is independent of $u \in \Omega(u_0)$.

For the case when $0 = t_1 < t_2 \le \eta$, since

$$\Big\| \int_0^{t_2} (t_2 - s)^{\beta-1}\mathcal{P}_\beta(t_2 - s)\Big(F(s, u(s)) + \int_0^s K(s-\tau)H(\tau, u(\tau))\, d\tau\Big)\, ds\Big\|_\alpha$$

$$\le \int_0^{t_2} (t_2-s)^{\beta-1}\|A^\alpha \mathcal{P}_\beta(t_2-s)\|_{\mathcal{L}(X)} \Big\| F(s, u(s)) + \int_0^s K(s-\tau)H(\tau, u(\tau))\, d\tau \Big\|\, ds$$

$$\le C_\alpha(M' + \widetilde{k}M'') \int_0^{t_2} (t_2 - s)^{\beta-1-\alpha\beta}\, ds \le \frac{C_\alpha(M' + \widetilde{k}M''){t_2}^{(1-\alpha)\beta}}{(1-\alpha)\beta},$$

$\|(\Phi_1 u)(t_2)\|_\alpha$ can be made small when t_2 is small independently of $u \in \Omega(u_0)$. Consequently, the set $\{(\Phi_1 u)(t) : t \in [0, \eta],\ u \in \Omega(u_0)\}$ is equicontinuous. Now the Arzelà–Ascoli theorem shows that Φ_1 is compact on $\Omega(u_0)$.

By the above arguments, Φ is continuous and compact on $\Omega(u_0)$. Thus, Schauder's fixed point theorem implies that Φ has a fixed point $u_1 \in \Omega(u_0)$, which means u_1 is a mild solution of the fractional Cauchy problem (1.1) on $[0, \eta]$.

An analogous argument can be used to show that there exists a positive constant η' such that the integral equation

$$u(t) = \mathcal{S}_\beta(t-\eta)u_1(\eta) + \int_\eta^t (t-s)^{\beta-1}\mathcal{P}_\beta(t-s)\Big(F(s, u(s)) + \int_\eta^s K(s-\tau)H(\tau, u(\tau))\, d\tau\Big)\, ds, \qquad t \ge \eta,$$

has a solution u_2 defined on $[\eta, \eta']$. We set

$$u(t) = \begin{cases} u_1(t), & t \in [0, \eta], \\ u_2(t), & t \in [\eta, \eta']. \end{cases}$$

Clearly, $u \in C([0, \eta']; X_\alpha)$ is a mild solution of the Cauchy problem (1.1) on $[0, \eta']$. The above procedure may be repeated to construct a mild solution defined on a maximal interval of existence denoted by $[0, t_{\max})$, where

$$t_{\max} = \sup\{\eta'' : \text{there exists a mild solution } u \in C([0, \eta'']; X_\alpha) \text{ of } (1.1)\}. \blacksquare$$

In the following, we consider the global existence of mild solutions to (1.1).

THEOREM 3.3. *Assume that* F, $H : [0, \infty) \times X_\alpha \to X$ *are continuous and there exist functions* $\alpha_1, \alpha_2, \beta_1, \beta_2 : [0, \infty) \to [0, \infty)$ *with* α_1, α_2 *bounded and measurable, and* $\beta_1, \widetilde{\beta}_2 \in L^p[0, \infty)$ *with* $p > 1/\beta(1 - \alpha)$ *and* $\widetilde{\beta}_2(t) = \int_0^t K(t - \tau)\beta_2(\tau)\, d\tau$ *such that for all* $0 \le t < \infty$ *and* $u \in X_\alpha$,

$$\|F(t, u)\| \le \alpha_1 \|u\|_\alpha + \beta_1, \qquad \|H(t, u)\| \le \alpha_2 \|u\|_\alpha + \beta_2. \tag{3.4}$$

Then mild solutions of the fractional Cauchy problem (1.1) *exist globally.*

Proof. As F, $H : [0, \infty) \times X_\alpha \to X$ are continuous, it follows from Theorem 3.2 that for every $u_0 \in X_\alpha$, the fractional Cauchy problem (1.1) has a mild solution u which is defined on a maximal interval of existence $[0, t_{\max})$. Let $[t_0, t_{\max})$ be a subset of $[0, t_{\max})$ such that u satisfies

$$u(t) = \mathcal{S}_\beta(t - t_0)u(t_0) + \int_{t_0}^{t} (t - s)^{\beta - 1}\mathcal{P}_\beta(t - s)\Big(F(s, u(s)) + \int_{t_0}^{s} K(s - \tau)H(\tau, u(\tau))\, d\tau\Big)\, ds$$

for $t \in [t_0, t_{\max})$. Set

$$\Psi(t) = M\|u(t_0)\|_\alpha + C_\alpha \int_{t_0}^{t} (t - s)^{\beta - 1 - \alpha\beta}(\beta_1(s) + \widetilde{\beta}_2(s))\, ds.$$

Since $\beta_1, \widetilde{\beta}_2 \in L^p[0, \infty)$ with $p > 1/\beta(1 - \alpha)$, one has $\beta(1 - \alpha)q - q + 1 > 0$ with $q = p/(p - 1)$ and

$$\int_{t_0}^{t} (t - s)^{\beta - 1 - \alpha\beta}(\beta_1(s) + \widetilde{\beta}_2(s))\, ds$$
$$\le \left(\frac{1}{\beta(1 - \alpha)q - q + 1}\right)^{1/q} (t - t_0)^{\beta(1-\alpha)-1+1/q}(\|\beta_1\|_{L^p[t_0,\infty)} + \|\widetilde{\beta}_2\|_{L^p[t_0,\infty)}),$$

which implies that Ψ is a continuous function defined on $[0, \infty)$. Moreover, for all $t \in [t_0, t_{\max})$,

$$\begin{aligned}
\|u(t)\|_\alpha &\le \|\mathcal{S}_\beta(t)u(t_0)\|_\alpha + \int_{t_0}^{t} (t-s)^{\beta-1} \Big\| \mathcal{P}_\beta(t-s)\Big(F(s,u(s)) \\
&\qquad\qquad + \int_{t_0}^{s} K(s-\tau)H(\tau,u(\tau))\,d\tau\Big)\Big\|_\alpha\,ds \\
&\le M\|u(t_0)\|_\alpha + \int_{t_0}^{t} (t-s)^{\beta-1}\|A^\alpha\mathcal{P}_\beta(t-s)\|_{\mathcal{L}(X)} \\
&\qquad\qquad \times \Big\|F(s,u(s)) + \int_{t_0}^{s} K(s-\tau)H(\tau,u(\tau))\,d\tau\Big\|\,ds \\
&\le \Psi(t) + C_\alpha \int_{t_0}^{t} (t-s)^{\beta-1-\alpha\beta}\Big(\alpha_1(s)\|u(s)\|_\alpha \\
&\qquad\qquad + \int_{t_0}^{s} K(s-\tau)\alpha_2(\tau)\|u(\tau)\|_\alpha\,d\tau\Big)\,ds \\
&\le \Psi(t) + C_\alpha \int_{t_0}^{t} (t-s)^{\beta-1-\alpha\beta} \\
&\qquad\qquad \times \Big(\alpha_1(s) + \int_{t_0}^{s} K(s-\tau)\alpha_2(\tau)\,d\tau\Big)\max_{0\le r\le s}\|u(r)\|_\alpha\,ds
\end{aligned}$$

in view of Lemma 2.6(4) and (3.4), from which we see that

$$\max_{0\le r\le t}\|u(r)\|_\alpha \le \Psi(t) + C_\alpha L\int_{t_0}^{t}(t-s)^{\beta-1-\alpha\beta}\max_{0\le r\le s}\|u(r)\|_\alpha\,ds,$$

where

$$L = \sup_{t\ge t_0}\alpha_1(t) + \sup_{t\ge 0}\alpha_1(t)\cdot\int_{t_0}^{\infty}K(t)\,dt,$$

which together with Lemma 7.11 in [H] implies that $\|u(t)\|_\alpha$ is bounded by a continuous function which depends on Ψ and is defined on $[0,\infty)$. Hence, the mild solution u of (1.1) is global, i.e., $t_{\max} = \infty$. ∎

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