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Convergence in capacity on smooth hypersurfaces of compact Kähler manifolds

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Abstract. We study restrictions of ω-plurisubharmonic functions to a smooth hypersurface S in a compact Kähler manifold X. The result obtained and the characterization of convergence in capacity due to S. Dinew and P. H. Hiep [to appear in Ann. Scuola Norm. Sup. Pisa Cl. Sci.] are used to study convergence in capacity on S.

1. Introduction. In [BT1, BT2], Bedford and Taylor laid the foundations of the theory of the complex Monge–Ampère operator which is nowadays a central part of pluripotential theory. Convergence in capacity C_n on domains in $\mathbb{C}^n$ was introduced in [BT2]. Initially Bedford and Taylor used this capacity to solve deep problems concerning small sets in pluripotential theory. It was soon realized, however, that capacities are very useful technical tools in solving Monge–Ampère equations with singular data. Especially, the discovery of Xing [Xi1] that the complex Monge–Ampère operator is continuous with respect to convergence in capacity attracted much interest. Recently, Kołodziej [Ko2] introduced the capacity C_X on a compact Kähler manifold X. In [GZ1], Guedj and Zeriahi proved that C_X is locally equivalent to C_n. In [DH], Dinew and Hiep gave characterizations of convergence in capacity C_X.

The main aim of the present note is to study restrictions of ω-plurisubharmonic functions to a smooth hypersurface S in a compact Kähler manifold X. The result obtained and the characterizations of convergence in capacity in [DH] are used to study convergence in capacity on the hypersurface S.

In Section 1, we introduce some definitions which can be found in [BK], [BT1]–[BT3], [Ce1], [Ce2], [De1], [De2], [GZ1], [GZ2], [HKH], [H1]–[H3], [Hö], [KH], [Kl], [Ko1]–[Ko3], [Si1], [Si2] and [Xi1], [Xi2].

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Our main results are the following theorems which generalize Theorem 2.4 in [H1]:

THEOREM A. *Let S be a smooth hypersurface in a compact Kähler manifold X and $u \in \mathcal{F}(X,\omega)$ be such that $\int_X |\varphi_S|\omega_u^n < \infty$ for some $\varphi_S \in \mathcal{D}(S,a)$ and $a > 0$. Then $u|_S \in \mathcal{E}_1(S,\omega_S)$.*

THEOREM B. *Let S be a smooth hypersurface in a compact Kähler manifold X and let $u_j, v_j \in \mathrm{PSH}^-(X,\omega) \cap L^\infty(X)$ and $u_0 \in \mathcal{F}(X,\omega)$ be such that $u_j - v_j \to 0$ in C_X as $j \to \infty$, $u_j, v_j \geq u_0$ for all $j \geq 1$, $\int_X |\varphi_S|\omega_u^n < \infty$ for some $\varphi_S \in \mathcal{D}(S,a)$, $a > 0$ and either*

(i) *for each $z \in S$ there exist a neighbourhood U of z and $\psi \in \mathrm{PSH}^-(U) \cap L^\infty(U)$, $\phi \in \mathcal{B}(U)$ such that $\omega_{u_j}^n + \omega_{v_j}^n \leq dd^c\psi \wedge (dd^c\phi)^{n-1}$ on U for all $j \geq 1$, or*

(ii) *there exist a neighbourhood U of S and an increasing function $F : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\int_1^\infty F(1/t)\,dt < \infty \quad \text{and} \quad \int_E [\omega_{u_j}^n + \omega_{v_j}^n] \leq F(C_X(E)),$$

for every Borel set $E \subset U$, and all $j \geq 1$.

Then $u_j|_S - v_j|_S \to 0$ in C_S as $j \to \infty$.

2. Preliminaries

2.1. Let Ω be a domain in $\mathbb{C}^n$. We denote by $\mathrm{PSH}(\Omega)$ the set of plurisubharmonic (psh) functions on Ω, and by $\mathrm{PSH}^-(\Omega)$ the subclass of negative functions.

2.2. Let X be a compact Kähler manifold with a fundamental form $\omega = \omega_X$ with $\int_X \omega^n = 1$. An upper semicontinuous function $\varphi : X \to [-\infty,\infty)$ is called *ω-plurisubharmonic* (ω-psh) if $\varphi \in L^1(X)$ and $\omega + dd^c\varphi \geq 0$. We denote by $\mathrm{PSH}(X,\omega)$ (resp. $\mathrm{PSH}^-(X,\omega)$) the set of ω-psh (resp. negative ω-psh) functions on X.

2.3. In [Ko2], Kołodziej introduced the capacity $C_{X,\omega}$ on X by

$$C_X(E) = C_{X,\omega}(E) = \sup\Big\{\int_E \omega_\varphi^n : \varphi \in \mathrm{PSH}(X,\omega),\ -1 \leq \varphi \leq 0\Big\}$$

where $\omega_\varphi^n = (\omega + dd^c\varphi)^n$ and $n = \dim X$. In [GZ1], Guedj and Zeriahi proved that C_X is a Choquet capacity on X and

$$C_X(E) = \int_X (-h_{E,\omega}^*)\omega_{h_{E,\omega}^*}^n$$

where $h^*_{E,\omega}$ denotes the upper semicontinuous regularization of $h_{E,\omega}$ given by

$$h_{E,\omega}(z) = \sup\{\varphi(z) : \varphi \in \mathrm{PSH}^-(X,\omega),\ \varphi|_E \le -1\}.$$

2.4. Let $u_j, v_j \in \mathrm{PSH}(X,\omega)$. We say that $u_j - v_j$ *converges to* 0 *in* C_X if

$$C_X(\{|u_j - v_j| > \delta\}) \to 0 \quad \text{as } j \to \infty$$

for all $\delta > 0$.

2.5. A family $\{\mu_i\}_{i\in I}$ of positive measures on X is said to be *uniformly absolutely continuous with respect to capacity* C_X if for every $\epsilon > 0$ there exists $\delta > 0$ such that for each Borel subset $E \subset X$ with $C_X(E) < \delta$ the inequality $\mu_i(E) < \epsilon$ holds for all $i \in I$. We then write $\mu_i \ll C_X$ uniformly for $i \in I$.

2.6. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. In [De2], Demailly introduced the class of psh functions which are bounded near the boundary:

$$\mathcal{B}(\Omega) = \{\varphi \in \mathrm{PSH}(\Omega) : \exists K \subset\subset \Omega \text{ such that } \varphi \in L^\infty(\Omega \setminus K)\}.$$

In [De2], he proved that $dd^c\varphi \wedge T$ is well-defined for each $\varphi \in \mathcal{B}(\Omega)$ and every non-negative closed current T on Ω.

The following classes of psh functions were introduced by Cegrell in [Ce1] and [Ce2]:

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \Big\{\varphi \in \mathrm{PSH}^-(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} \varphi(z) = 0,\ \int_\Omega (dd^c\varphi)^n < \infty\Big\},$$

$$\mathcal{F} = \mathcal{F}(\Omega) = \Big\{\varphi \in \mathrm{PSH}^-(\Omega) : \exists\, \mathcal{E}_0(\Omega) \ni \varphi_j \searrow \varphi,\ \sup_{j\ge1} \int_\Omega (dd^c\varphi_j)^n < \infty\},$$

$$\mathcal{E} = \mathcal{E}(\Omega) = \Big\{\varphi \in \mathrm{PSH}^-(\Omega) : \exists\, \mathcal{F}(\Omega) \ni \varphi_K\ = \varphi \text{ on } K,\ \forall K \subset\subset \Omega\Big\},$$

In [Ce2], he proved that the complex Monge–Ampère operator is well-defined on $\mathcal{E}(\Omega)$ and this is the largest possible domain of definition.

2.7. We define the class of ω-psh functions on which the complex Monge–Ampère operator is well-defined in every local coordinate:

$$\mathrm{DMA}_{\mathrm{loc}}(X,\omega) = \{\varphi \in \mathrm{PSH}^-(X,\omega) : \forall z \in X, \text{ there is a neighbourhood } U \text{ of } z \text{ and a potential } \theta \text{ of } \omega \text{ on } U \text{ such that } \varphi + \theta \in \mathcal{E}(U)\}.$$

2.8. In [GZ2], Guedj and Zeriahi introduced the classes of ω-psh functions

$$\mathcal{E}(X,\omega) = \Big\{\varphi \in \mathrm{PSH}^-(X,\omega) : \lim_{j\to\infty} \int_{\{\varphi>-j\}} \omega^n_{\max(\varphi,-j)} = 1\Big\},$$

$$\mathcal{E}_p(X,\omega) = \Big\{\varphi \in \mathrm{PSH}^-(X,\omega) : \exists\{\varphi_j\}_{j\geq 1} \subset \mathrm{PSH}\cap L^\infty(X,\omega) \text{ with } \varphi_j \searrow \varphi \text{ and } \sup_{j\geq 1}\int_X |\varphi_j|^p \omega_{\varphi_j}^n < \infty\Big\}.$$

They proved that the complex Monge–Ampère operator is well-defined on $\mathcal{E}(X,\omega)$ by

$$\omega_\varphi^n := \lim_{j\to\infty} 1_{\{\varphi>-j\}}\omega^n_{\max(\varphi,-j)}.$$

They showed that the complex Monge–Ampère operator is continuous under decreasing limits (see Theorem 1.9 in [GZ2] or Proposition 2.8 in [H3]). They also showed that

$$\mathcal{E}_p(X,\omega) = \Big\{\varphi \in \mathcal{E}(X,\omega) : \int_X |\varphi|^p\omega_\varphi^n < \infty\Big\}.$$

2.9. In [Xi2], Xing introduced the following class of ω-psh functions:

$$\mathcal{F}(X,\omega) = \{\varphi \in \mathrm{PSH}^-(X,\omega) : \varphi_j\omega_{\varphi_j}^{n-1}\wedge\omega \ll C_X \text{ uniformly for } j\geq 1\}$$

where $\varphi_j = \max(\varphi,-j)$, $j \geq 1$. He proved that the complex Monge–Ampère operator is well-defined on $\mathcal{F}(X,\omega)$ (see Proposition 1 and Theorem 5 in [Xi2]) by

$$\omega_\varphi^n := \omega\wedge\omega_\varphi^{n-1} + dd^c(\varphi\wedge\omega_\varphi^{n-1}).$$

He also proved that $\mathrm{DMA}_{\mathrm{loc}}(X,\omega) \subset \mathcal{F}(X,\omega)$ (see Theorem 2 in [Xi2]).

For $\mathcal{K}(X,\omega) \in \{\mathrm{DMA}_{\mathrm{loc}}(X,\omega), \mathcal{F}(X,\omega)\}$ we set

$$\mathcal{K}^a(X,\omega) = \{\varphi \in \mathcal{K}(X,\omega) : \omega_\varphi^n \ll C_X\}.$$

It is known that $\mathcal{E}_1(X,\omega) \subset \mathcal{F}^a(X,\omega)$.

2.10. Let S be a hypersurface in a compact Kähler manifold X. For each $a > 0$, we denote by $\mathcal{D}(S,a)$ the family of ω-psh functions $\varphi \in \mathrm{PSH}^-(X,\omega)\cap C(X\setminus S)$ such that for every $z \in S$, there exist a neighbourhood U of z and a holomorphic function f on U with $S\cap U = \{w \in U : f(w) = 0\}$, $\varphi - a\log|f| \in L^\infty(U\setminus S)$ and $f'(w) \neq 0$ for all $w \in \mathrm{Reg}(S)\cap U$. In Proposition 3.1 we show that if $a > 0$ is small enough then $\mathcal{D}(S,a) \neq \emptyset$.

2.11. Let S be a smooth hypersurface in a compact Kähler manifold X. For each $z \in S$ we can find a neighbourhood U of z and a strictly psh function θ on U such that $\omega = dd^c\theta$. Define $\omega_S = dd^c\theta|_S$ on $U\cap S$. Then ω_S is a fundamental form on S. Obviously if $u \in \mathrm{PSH}(X,\omega)$ is such that $u|_S \not\equiv -\infty$ then $u|_S \in \mathrm{PSH}(S,\omega_S)$.

Let X be a compact Kähler manifold and $u_1,\dots,u_{n-1} \in \mathrm{PSH}(X,\omega)\cap L^\infty(X)$, $\varphi \in \mathrm{PSH}(X,\omega)$. We can define the wedge product $\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge\omega_\varphi$ and it is continuous under decreasing limits (see Theorem 2.1 and Remark 2.2 in [BT3]). We prove that the integration by parts formula holds for these products.

2.12. Integration by parts formula

PROPOSITION 2.1. *Let X be a compact Kähler manifold and $u_1, \dots, u_{n-1}, v \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$, $\varphi \in \mathrm{PSH}(X,\omega)$. Then*

$$\int_X v\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge\omega_\varphi = \int_X v\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge\omega + \int_X \varphi\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge dd^c v.$$

Proof. First, we assume that $v \in \mathrm{PSH}(X,\omega) \cap C^\infty(X)$. Set $\varphi^j = \max(\varphi, -j)$. By integration by parts we have

$$\int_X v\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge\omega_{\varphi^j} = \int_X v\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge\omega + \int_X \varphi^j\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge dd^c v.$$

Letting $j \to \infty$, from $\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge\omega_{\varphi^j} \to \omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge\omega_\varphi$ weakly and from Lebesgue's convergence theorem we get

$$\int_X v\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge\omega_\varphi = \int_X v\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge\omega + \int_X \varphi\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge dd^c v.$$

In the general case, by Theorem 1 in [BK], we can choose $v_j \in \mathrm{PSH}(X,\omega) \cap C^\infty(X)$ such that $v_j \searrow v$ (see also [De1]). By the first case we have

$$\int_X v_j\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge\omega_\varphi = \int_X v_j\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge\omega + \int_X \varphi\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge dd^c v_j.$$

Letting $j \to \infty$, by Lebesgue's convergence theorem and by Corollary 5.2 in [Ce2] we get

$$\int_X v\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge\omega_\varphi = \int_X v\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge\omega + \int_X \varphi\omega_{u_1}\wedge\cdots\wedge\omega_{u_{n-1}}\wedge dd^c v. \ \blacksquare$$

3. Proof of main results. First, we state some results that will be used in the proofs of our main results. These are simple modifications of already known results, but we shall sketch some of the proofs for the sake of completeness:

PROPOSITION 3.1. *Let S be a hypersurface in a compact Kähler manifold X. Then $\mathcal{D}(S,a) \neq \emptyset$ for some $a > 0$.*

Proof. We can find finite open covers $W_i \subset\subset V_i \subset\subset U_i$, $1 \le i \le m$, of X and holomorphic functions f_i on U_i such that $X = \bigcup_{i=1}^m W_i$, $S \cap U_i = \{z \in U_i : f_i(z) = 0\}$, $f'(z) \neq 0$, for all $z \in \mathrm{Reg}(S) \cap U_i$ and $\|f_i\|_{L^\infty(U_i)} \le 1$. Set

$$M = \sup_{1\le i,j\le m} \sup\left\{\left|\log\frac{|f_i(z)|}{|f_j(z)|}\right| : z \in V_i \cap V_j\right\}.$$

We choose $h_i \in C_0^\infty(X)$ with $0 \le h_i \le 1$, $h_i|_{W_i} = 1$ and supp $h_i \subset\subset V_i$. We can find $\epsilon_0 > 0$ such that $\epsilon_0 dd^c h_i + \omega \ge 0$ for all $1 \le i \le m$. Set

$$\varphi_S(z) = \frac{\epsilon_0}{M}\sup_{1\le i\le m}\{\log|f_i(z)| + Mh_i(z) : z \in V_i\} - \epsilon_0.$$

We can check that

$$\varphi_S(z) = \frac{\epsilon_0}{M} \sup_{1\le i\le m} \{\log|f_i(z)| + Mh_i(z) : z \in \operatorname{supp} h_i\} - \epsilon_0.$$

Hence $\varphi_S \in \mathrm{PSH}^-(X,\omega) \cap C(X\setminus S)$. It is also easy to show that $\varphi_S \in \mathcal{D}(S,a)$ for $a = \epsilon_0/M > 0$. ∎

PROPOSITION 3.2. *Let S be a smooth hypersurface in a compact Kähler manifold X. Then $a\omega_u^{n-1}|_S \le \omega_u^{n-1} \wedge \omega_\varphi$ for all $u \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ and all $\varphi \in \mathcal{D}(S,a)$.*

Proof. Take $z \in S$. We can find a neighbourhood U of z, a holomorphic function f on U and a bounded psh function θ on U such that $S \cap U = \{w \in U : f(w) = 0\}$, $f'(w) \neq 0$ for all $w \in U$, $\varphi - a\log|f| \in L^\infty(U \setminus S)$ and $\omega = dd^c\theta$. Since $\varphi + \theta - a\log|f| \in \mathrm{PSH}(U \setminus S) \cap L^\infty(U \setminus S)$ we can extend it to a psh function h on U. By Corollary 4.2 in [BT3], we get

$$\begin{aligned}
\omega_u^{n-1} \wedge \omega_\varphi &= [dd^c(u+\theta)]^{n-1} \wedge dd^c(\varphi+\theta)\\
&= [dd^c(u+\theta)]^{n-1} \wedge dd^c(a\log|f| + h)\\
&\ge a[dd^c(u+\theta)]^{n-1} \wedge dd^c\log|f| = a[dd^c(u+\theta)]^{n-1}|_{S\cap U}\\
&= a\omega_u^{n-1}|_{S\cap U}. \ \blacksquare
\end{aligned}$$

REMARK. For each hypersurface S in a compact Kähler manifold X we set

$$m(S,X,\omega) = \sup\{a > 0 : \mathcal{D}(S,a) \neq \emptyset\}$$

By Propositions 3.1 and 3.2 we obtain

$$0 < m(S,X,\omega) \le \frac{\int_X \omega^n}{\int_S \omega_S^{n-1}}.$$

PROPOSITION 3.3. *Let $u_j^1,\dots,u_j^n,v_j^1,\dots,v_j^n,u_0 \in \mathcal{F}(X,\omega)$ be such that $u_j^k, v_j^k \ge u_0$ for all $1 \le k \le n$ and $j \ge 1$ and $u_j^k - v_j^k \to 0$ in C_X as $j \to \infty$, for all $1 \le k \le n$. Then*

$$\lim_{j\to\infty} \int_X f[\omega_{u_j^1} \wedge \cdots \wedge \omega_{u_j^n} - \omega_{v_j^1} \wedge \cdots \wedge \omega_{v_j^n}] = 0, \qquad \forall f \in C(X).$$

Proof. We only prove the equality in the case $f \in C^\infty(X)$. We choose $A > 0$ such that $A\omega + dd^c f \ge 0$ and $A\omega - dd^c f \ge 0$. By decomposing $\omega_{u_j^1} \wedge \cdots \wedge \omega_{u_j^n} - \omega_{v_j^1} \wedge \cdots \wedge \omega_{v_j^n}$ as a sum of n terms, we can assume that $u_j^k = v_j^k$ for all $2 \le k \le n$. To simplify the notation we set

$$T_j = \omega_{u_j^2} \wedge \cdots \wedge \omega_{u_j^n}$$

We have

$$\begin{aligned}\Big|\int\limits_X f[\omega_{u_j^1}\wedge\cdots\wedge\omega_{u_j^n}-\omega_{v_j^1}\wedge\cdots\wedge\omega_{v_j^n}]\Big| &= \Big|\int\limits_X f dd^c(u_j^1-v_j^1)\wedge T_j\Big| \\ &= \Big|\int\limits_X (u_j^1-v_j^1)dd^c f\wedge T_j\Big| \\ &\le 3A\int\limits_X |u_j^1-v_j^1|\omega\wedge T_j \\ &\le 3A\Big[\epsilon+\int\limits_{\{|u_j^1-v_j^1|>\epsilon\}} -u_0\omega\wedge T_j\Big],\end{aligned}$$

for all $\epsilon>0$. From $C_X(\{|u_j^1-v_j^1|>\epsilon\})\to 0$ as $j\to\infty$ and from Corollary 4 in [Xi2], we get

$$\overline{\lim_{j\to\infty}}\Big|\int\limits_X f[\omega_{u_j^1}\wedge\cdots\wedge\omega_{u_j^n}-\omega_{v_j^1}\wedge\cdots\wedge\omega_{v_j^n}]\Big|\le 3A\epsilon$$

for all $\epsilon>0$. Letting $\epsilon\to 0$ we obtain

$$\lim_{j\to\infty}\int\limits_X f[\omega_{u_j^1}\wedge\cdots\wedge\omega_{u_j^n}-\omega_{v_j^1}\wedge\cdots\wedge\omega_{v_j^n}]=0. \blacksquare$$

PROPOSITION 3.4. *Let Ω be a domain in $\mathbb{C}^n$ and $\psi\in\mathrm{PSH}^-(\Omega)\cap L^\infty(\Omega)$, $\phi\in\mathcal{B}(\Omega)$. Then $PSH(\Omega)\subset L^1_{\mathrm{loc}}(dd^c\psi\wedge(dd^c\phi)^{n-1})$.*

Proof. Without loss of generality, we can assume that $-1\le\psi\le 0$ on Ω and $-1\le\phi\le 0$ on $\Omega\setminus\Omega''$ for some $\Omega''\subset\subset\Omega'\subset\subset\Omega$. Set $\tilde\phi=\max(\phi,-1)$. Take $\varphi\in\mathrm{PSH}^-(\Omega)$ and $f\in C_0^\infty(\Omega)$ with $0\le f\le 1$, $f|_{\Omega'}=1$. We choose $A>0$ such that $Add^c|z|^2+dd^cf\ge 0$. Since $\operatorname{supp} df\subset\Omega\setminus\overline{\Omega''}$ and $\phi=\tilde\phi$ on $\Omega\setminus\overline{\Omega''}$, we have

$$\begin{aligned}\int\limits_\Omega -f\varphi dd^c\psi\wedge(dd^c\phi)^{n-1} &= \int\limits_\Omega -\psi dd^c(f\varphi)\wedge(dd^c\phi)^{n-1} \\ &= \int\limits_\Omega -\psi f dd^c\varphi\wedge(dd^c\phi)^{n-1}+2\int\limits_\Omega -\psi df\wedge d^c\varphi\wedge(dd^c\phi)^{n-1} \\ &\quad+\int\limits_\Omega -\psi\varphi dd^cf\wedge(dd^c\phi)^{n-1} \\ &\le \int\limits_\Omega f dd^c\varphi\wedge(dd^c\phi)^{n-1}+2\int\limits_\Omega -\psi df\wedge d^c\varphi\wedge(dd^c\tilde\phi)^{n-1} \\ &\quad+\int\limits_\Omega -\psi\varphi dd^cf\wedge(dd^c\tilde\phi)^{n-1} \\ &= \int\limits_\Omega \varphi dd^cf\wedge(dd^c\phi)^{n-1}+2\int\limits_\Omega -\psi df\wedge d^c\varphi\wedge(dd^c\tilde\phi)^{n-1} \\ &\quad+\int\limits_\Omega -\psi\varphi dd^cf\wedge(dd^c\tilde\phi)^{n-1}\end{aligned}$$

$$= \int_\Omega \varphi dd^c f \wedge (dd^c\tilde{\phi})^{n-1} + 2\int_\Omega -\psi df \wedge d^c\varphi \wedge (dd^c\tilde{\phi})^{n-1}$$
$$+ \int_\Omega -\psi\varphi dd^c f \wedge (dd^c\tilde{\phi})^{n-1}.$$

On the other hand, we have

$$\int_\Omega -f\varphi dd^c\psi \wedge (dd^c\tilde{\phi})^{n-1}$$
$$= \int_\Omega -\psi dd^c(f\varphi) \wedge (dd^c\tilde{\phi})^{n-1}$$
$$= \int_\Omega -\psi f dd^c\varphi \wedge (dd^c\tilde{\phi})^{n-1} + 2\int_\Omega -\psi df \wedge d^c\varphi \wedge (dd^c\tilde{\phi})^{n-1}$$
$$+ \int_\Omega -\psi\varphi dd^c f \wedge (dd^c\tilde{\phi})^{n-1}.$$

Therefore

$$\int_\Omega -f\varphi dd^c\psi \wedge (dd^c\phi)^{n-1}$$
$$\leq \int_\Omega \varphi dd^c f \wedge (dd^c\tilde{\phi})^{n-1} + \int_\Omega \psi f dd^c\varphi \wedge (dd^c\tilde{\phi})^{n-1}$$
$$+ \int_\Omega -f\varphi dd^c\psi \wedge (dd^c\tilde{\phi})^{n-1}$$
$$\leq \int_\Omega \varphi dd^c f \wedge (dd^c\tilde{\phi})^{n-1} + \int_\Omega -f\varphi dd^c\psi \wedge (dd^c\tilde{\phi})^{n-1}$$
$$\leq A \int_{\operatorname{supp} f} -\varphi dd^c|z|^2 \wedge (dd^c\tilde{\phi})^{n-1} + \int_{\operatorname{supp} f} -\varphi dd^c\psi \wedge (dd^c\tilde{\phi})^{n-1}.$$

By Theorem 2.1 in [BT3] (also see Proposition 1.11 in [De2]) we get

$$\int_\Omega -f\varphi dd^c\psi \wedge (dd^c\phi)^{n-1}$$
$$\leq C_{K,\operatorname{supp} f}(A\sup\{|z|^2 : z \in K\} + 1)\int_K -\varphi dV_{2n} < \infty$$

where $\operatorname{supp} f \subset\subset K \subset\subset \Omega$, $C_{K,\operatorname{supp} f} > 0$ is a constant and dV_{2n} is the Lebesgue measure in $\mathbb{C}^n$. ∎

3.1. Proof of Theorem A. Set $u^j = \max(u, -j)$ and $\varphi^j = \max(\varphi, -j)$. By Proposition 3.1 we choose $\varphi \in \mathcal{D}(S, a)$ for some $a > 0$. By Propositions 2.1 and 3.2 we get

$$\begin{aligned}
a\int_S -u^j|_S\omega_{u^j}^{n-1}|_S &\le \int_X -u^j\omega_{u^j}^{n-1}\wedge\omega_\varphi\\
&= \int_X -u^j\omega_{u^j}^{n-1}\wedge\omega + \int_X -\varphi\omega_{u^j}^{n-1}\wedge dd^c u^j\\
&\le \int_X -u^j\omega_{u^j}^{n-1}\wedge\omega + \int_X -\varphi\omega_{u^j}^{n}.
\end{aligned}$$

On the other hand, by Lebesgue's convergence theorem we have

$$\begin{aligned}
\int_X -\varphi\omega_{u^j}^n &= \lim_{m\to\infty}\int_X -\varphi^m\omega_{u^j}^n\\
&= \lim_{m\to\infty}\Big[\int_X -\varphi^m\omega_u^n + \int_X -\varphi^m dd^c(u^j-u)\wedge\sum_{k=0}^{n-1}\omega_{u^j}^k\wedge\omega_u^{n-k-1}\Big]\\
&= \int_X -\varphi\omega_u^n + \lim_{m\to\infty}\int_X (u-u^j)dd^c\varphi^m\wedge\sum_{k=0}^{n-1}\omega_{u^j}^k\wedge\omega_u^{n-k-1}\\
&= \int_X -\varphi\omega_u^n + \lim_{m\to\infty}\int_X (u-u^j)\omega_{\varphi^m}\wedge\sum_{k=0}^{n-1}\omega_{u^j}^k\wedge\omega_u^{n-k-1}\\
&\quad + \int_X (u^j-u)\omega\wedge\sum_{k=0}^{n-1}\omega_{u^j}^k\wedge\omega_u^{n-k-1}\\
&\le \int_X -\varphi\omega_u^n + \int_X -u\omega\wedge\sum_{k=0}^{n-1}\omega_{u^j}^k\wedge\omega_u^{n-k-1}.
\end{aligned}$$

Therefore

$$a\int_S -u^j|_S\omega_{u^j}^{n-1}|_S \le \int_X -u^j\omega_{u^j}^{n-1}\wedge\omega + \int_X -u\omega\wedge\sum_{k=0}^{n-1}\omega_{u^j}^k\wedge\omega_u^{n-k-1} + \int_X -\varphi\omega_u^n.$$

To end the proof, we need to prove that the right hand side of the last inequality is finite. This is equivalent to

$$\sup_{j\ge 1}\Big[\int_X -u^j\omega_{u^j}^{n-1}\wedge\omega + \int_X -u\omega\wedge\sum_{k=0}^{n-1}\omega_{u^j}^k\wedge\omega_u^{n-k-1}\Big] < \infty.$$

Indeed, by Corollary 4 in [Xi2] we get $-u\omega\wedge\sum_{k=0}^{n-1}\omega_{u^j}^k\wedge\omega_u^{n-k-1} \ll C_X$ uniformly for all $j\ge 1$. So we can find $\delta>0$ such that

$$\int_E -u^j\omega_{u^j}^{n-1}\wedge\omega + \int_E -u\omega\wedge\sum_{k=0}^{n-1}\omega_{u^j}^k\wedge\omega_u^{n-k-1} < 1$$

for all Borel sets $E \subset X$ with $C_X(E) < \delta$ and for all $j \geq 1$. For each $z \in X$ we choose a neighbourhood U_z such that $C_X(U_z) < \delta$. From the compactness of X we find a finite cover $\{U_t\}_{t=1}^m$ of X with $C_X(U_t) < \delta$ for all $t = 1, \dots, m$. We have

$$\int\limits_X -u^j \omega_{u^j}^{n-1} \wedge \omega + \int\limits_X -u\omega \wedge \sum_{k=0}^{n-1} \omega_{u^j}^k \wedge \omega_u^{n-k-1} < m$$

for all $j \geq 1$. So we have

$$\sup_{j \geq 1} \int\limits_S -u^j|_S \omega_{u^j}^{n-1}|_S < \infty.$$

Hence $u|_S \in \mathcal{E}_1(S, \omega_S)$. ■

3.2. Proof of Theorem B. By Theorem A and by Proposition 2.10 in [DH] we have $u_j|_S, v_j|_S, u_0|_S \in \mathcal{E}_1(S, \omega_S)$ for all $j \geq 1$, and $\omega_{u_j}^{n-1}|_S + \omega_{v_j}^{n-1}|_S \ll C_S$ uniformly for $j \geq 1$. We prove that

$$\lim_{j \to \infty} \int\limits_{\{u_j|_S < v_j|_S - \delta\}} \omega_{u_j}^{n-1}|_S = 0 \tag{1}$$

for all $\delta > 0$. Indeed, by Proposition 3.1 we pick $\varphi \in \mathcal{D}(S, a)$ for some $a > 0$. Set

$$w_j = \max(u_j, v_j), \quad j \geq 1.$$

By the proof of Theorem A we get

$$\begin{aligned} \int\limits_{\{u_j|_S < v_j|_S - \delta\}} \omega_{u_j}^{n-1}|_S &\leq \frac{1}{\delta} \int\limits_S (w_j|_S - u_j|_S) \omega_{u_j}^{n-1}|_S \\ &\leq \frac{1}{a\delta} \Big[\int\limits_X (w_j - u_j) \omega_{u_j}^{n-1} \wedge \omega + \int\limits_X -\varphi \omega_{u_j}^{n-1} \wedge (\omega_{u_j} - \omega_{w_j}) \Big]. \end{aligned} \tag{2}$$

Since $w_j - u_j \to 0$ in C_X as $j \to \infty$ and $-u_0 \omega_{u_j}^{n-1} \wedge \omega \ll C_X$ uniformly for $j \geq 1$ (Corollary 4 in [Xi2]), we get

$$\lim_{j \to \infty} \int\limits_X (w_j - u_j) \omega_{u_j}^{n-1} \wedge \omega = 0. \tag{3}$$

Next we show that

$$\overline{\lim_{j \to \infty}} \int\limits_X -\varphi \omega_{u_j}^{n-1} \wedge (\omega_{u_j} - \omega_{w_j}) \leq 0. \tag{4}$$

Indeed, setting $\varphi^t = \max(\varphi, -t)$, we have

$$\begin{aligned}\int_X -\varphi\omega_{u_j}^{n-1} \wedge (\omega_{u_j} - \omega_{w_j}) \\ &= \int_X (\varphi^t - \varphi)\omega_{u_j}^{n-1} \wedge (\omega_{u_j} - \omega_{w_j}) + \int_X -\varphi^t\omega_{u_j}^{n-1} \wedge (\omega_{u_j} - \omega_{w_j}) \\ &\le \int_X (\varphi^t - \varphi)\omega_{u_j}^{n} + \int_X -\varphi^t\omega_{u_j}^{n-1} \wedge (\omega_{u_j} - \omega_{w_j})\end{aligned}$$

for all $t > 0$. Moreover, by Proposition 3.3 we get

$$\varlimsup_{j\to\infty} \int_X -\varphi\omega_{u_j}^{n-1} \wedge (\omega_{u_j} - \omega_{w_j}) \le \varlimsup_{j\to\infty} \int_X (\varphi^t - \varphi)\omega_{u_j}^{n}$$

for all $t > 0$. This implies that to prove (4) we only need to show that

$$\lim_{t\to\infty} \varlimsup_{j\to\infty} \int_X (\varphi^t - \varphi)\omega_{u_j}^{n} = 0.$$

Case of hypothesis (i): We can find finite open covers $V_i \subset\subset U_i$, $1 \le i \le m$, of X and $\psi_i \in \mathrm{PSH}^-(U_i) \cap L^\infty(U_i)$, $\phi_i \in \mathcal{B}(U_i)$ such that $X = \bigcup_{i=1}^m V_i$ and $\omega_{u_j}^n + \omega_{v_j}^n \le dd^c\psi_i \wedge (dd^c\phi_i)^{n-1}$ on U_i, for all $j \ge 1$ and $1 \le i \le m$. By Proposition 3.4 and Lebesgue's monotone convergence theorem we get

$$\begin{aligned}\lim_{t\to\infty} \varlimsup_{j\to\infty} \int_X (\varphi^t - \varphi)\omega_{u_j}^{n} &\le \sum_{i=1}^m \lim_{t\to\infty} \varlimsup_{j\to\infty} \int_{V_i} (\varphi^t - \varphi)\omega_{u_j}^{n} \\ &\le \sum_{i=1}^m \lim_{t\to\infty} \int_{V_i} (\varphi^t - \varphi) dd^c\psi_i \wedge (dd^c\phi_i)^{n-1} = 0.\end{aligned}$$

Case of hypothesis (ii): We have

$$\begin{aligned}\int_X (\varphi^t - \varphi)\omega_{u_j}^{n} &= \int_t^\infty \int_{\{\varphi < -r\}} \omega_{u_j}^n \, dr \\ &\le \int_t^\infty F(C_X(\{\varphi < -r\}))\, dr \le \int_t^{+\infty} F\left(\frac{\sup_X |\varphi| + c}{r}\right) dr,\end{aligned}$$

where c is a positive constant. Letting $j \to \infty$ and $t \to \infty$ we get

$$\lim_{t\to\infty} \varlimsup_{j\to\infty} \int_X (\varphi^t - \varphi)\omega_{u_j}^{n} = 0.$$

From (2)–(4) we obtain (1). Similarly we get

$$\lim_{j\to\infty} \int_{\{v_j|_S < u_j|_S - \delta\}} \omega_{v_j}^{n-1}|_S = 0,$$

for all $\delta > 0$. Using Theorem 3.2 in [DH] we obtain $u_j|_S - v_j|_S \to 0$ in C_S as $j \to \infty$. ∎

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