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On the connectedness of boundary and complement for domains

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Abstract. This article gives a short and elementary proof of the fact that the connectedness of the boundary of an open domain in $\mathbb{R}^n$ is equivalent to the connectedness of its complement.

It is known that an open domain in $\mathbb{R}^2$ is connected if and only if its complement is connected. An analogous result for $n > 2$ is regarded as known for some mathematicians. However, to the best of the authors' knowledge, there is no handy citable reference that would establish this fact. This short note aims to provide such a reference while handling the problem in a fast and efficient way.

The case of $n = 2$ is proved in [N] and appears as an exercise in [B]. It was also solved independently by K. Rudol in 1981 and his proof was later simplified by K. Ciesielski, but this solution has not been published. Our approach is not based on these existing results and differs from them also in the case of $n = 2$.

An easy consequence of the theorem is that a bounded domain of holomorphy in $\mathbb{C}^n$, where $n \geq 2$, has a connected boundary [J].

Let $\overline{A}$, $\operatorname{Int} A$ and ∂A denote respectively the closure, the interior, and the boundary of the set A.

Theorem. *Let $U \subset \mathbb{R}^n$ be an open domain. Let $F = \mathbb{R}^n \setminus U$. Then ∂U is connected if and only if F is connected.*

Proof. The implication $\Rightarrow$ follows easily by contradiction from the normality of $\mathbb{R}^n$.

For the proof of $\Leftarrow$ assume that F is connected, but ∂U is not. Split ∂U into two closed disjoint nonempty sets C and D. Pick open sets A and B such that:

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- A is a neighbourhood of C and B is a neighbourhood of D,
- $\overline{A} \cap \overline{B} = \emptyset$,
- the intersection of C and each component of A is nonempty and the intersection of D and each component of B is nonempty.

Note that $U \cup A \cup B$ is open and connected because U is connected and the intersection of U and every component of A or B is nonempty. This implies that $U \cup A \cup B$ is also pathwise connected.

The set $F \cup A \cup B$ is connected for the same reason as above. Note that it is open, as it can be written as $A \cup B \cup \operatorname{Int} F \cup \partial F$, where the first three sets are open, and the fourth is a subset of $A \cup B$. Therefore it is also pathwise connected.

Denote by d_n the euclidean metric in $\mathbb{R}^n$. Let S^1 be the unit circle in $\mathbb{R}^2$ and let $a = (0, 1)$, $b = (0, -1)$. Define a map $f : \mathbb{R}^n \to S^1$ in the following way:

- $f(\overline{A}) = a$,
- $f(\overline{B}) = b$,
- for a point $x \in U \setminus (\overline{A} \cup \overline{B})$ define $f(x)$ as the point on the left half of S^1 for which

$$(*) \qquad \frac{d_n(x, A)}{d_n(x, B)} = \frac{d_2(f(x), a)}{d_2(f(x), b)},$$

- for a point $x \in F \setminus (\overline{A} \cup \overline{B})$ define $f(x)$ as the point on the right half of S^1 for which $(*)$ holds.

It is a simple consequence of the definition of A and B that f is well defined and continuous.

Pick $p_a \in A$ and $p_b \in B$. Let γ_1 be a path that joins p_a to p_b in $U \cup A \cup B$. Let γ_2 be a path that joins p_b to p_a in $F \cup A \cup B$. Let H be a homotopy in $\mathbb{R}^n$ between the loop $\gamma_1 * \gamma_2$ and a constant loop. Then $f \circ H$ is a homotopy in S^1 between $f(\gamma_1 * \gamma_2)$ and some constant loop—a contradiction, since $f(\gamma_1 * \gamma_2)$ is not null homotopic in S^1. ∎

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