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A note on composition operators on spaces of real analytic functions

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Abstract. We characterize composition operators on spaces of real analytic functions which are open onto their images. We give an example of a semiproper map φ such that the associated composition operator is not open onto its image.

1. Introduction. Let M and N be real analytic manifolds and let $\varphi : M \to N$ be a real analytic map. We define the *composition operator* $C_\varphi(f) := f \circ \varphi$, where $f : N \to \mathbb{C}$. There is an extensive literature on $C_\varphi : C^\infty(N) \to C^\infty(M)$ starting from papers of Whitney [21] and Glaeser [12] and culminating in a characterization when the range of C_φ is closed in $C^\infty(M)$ (equivalently, when C_φ is open onto its image): see [2, Th. 1.13] and [3, Cor. 1.4, 1.5].

This paper is devoted to the real analytic case, i.e., $C_\varphi : \mathscr{A}(N) \to \mathscr{A}(M)$, where $\mathscr{A}(N)$ and $\mathscr{A}(M)$ denote the spaces of real analytic functions on N and M, respectively (equipped with their natural topologies described below, comp. [17]). This case is very different from the smooth case, in particular, having closed image is not the same as being open onto the image [7, Ex. 3.8].

In [7] we characterized when $C_\varphi : \mathscr{A}(N) \to \mathscr{A}(M)$ is a topological embedding, obtaining an analogue of the result of Glaeser [12] for the C^∞-case. Three conditions are important here:

1. φ is *semiproper* (i.e., for every compact set K in N there is a compact set L in M such that $\varphi(L) = \varphi(M) \cap K$);
2. $\varphi(M)$ has the "global extension property";
3. $\varphi(M)$ has the "semiglobal extension property".

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In [8] we showed that if C_φ has closed image then condition 2 holds, and in [9] the converse was shown under assumption 1. Moreover, if $\varphi(M)$ is analytic we showed that C_φ is open onto its image if and only if conditions 1 and 3 hold.

In the present paper we show that the assumption that $\varphi(M)$ is analytic in the last result is superfluous, so we fully characterize when $C_\varphi : \mathscr{A}(N) \to \mathscr{A}(M)$ is open onto its image without any additional assumption, in that way finishing one direction of research from [7]–[9]. Moreover, we show that if M is compact then C_φ is open onto its image if and only if it has a closed image, and that this is equivalent to the conditions that φ is semiproper and $\varphi(M)$ has the global (or, equivalently, semiglobal) extension property.

In [7, Ex. 3.8] we gave an example of a composition operator C_φ which was non-open onto its image because the underlying map φ failed to be semiproper. We show here that if the image of φ is $\mathbb{C}$-analytic of non-pure dimension, then the image of φ does not have the semiglobal extension property, therefore C_φ is non-open even though φ can be semiproper.

2. Preliminaries. In this section we will fix the notation and recall some basic facts needed in this paper. Some material presented here is taken from [8] and it is included for the sake of convenience.

Let M, N be real analytic manifolds and let $\varphi : M \to N$ be a real analytic map. Without loss of generality we may assume that any real analytic manifold is a submanifold of $\mathbb{R}^d$. Thus the following definition makes sense: if $S \subseteq N \subset \mathbb{R}^d \subset \mathbb{C}^d$ is an arbitrary set, then we define $\mathscr{A}(S)$ to be the space of real analytic functions on S, i.e., functions $f : S \to \mathbb{C}$ such that for every $x \in S$ the function f extends to a holomorphic function on some neighbourhood of x in $\mathbb{C}^d$.

Let us recall that the topology on the space of germs of holomorphic functions over a compact set $K \subseteq M$ (denoted by $H(K)$) is defined as the inductive limit topology $\operatorname{ind}_{n\in\mathbb{N}} H^\infty(U_n)$, where (U_n) is a basis of neighbourhoods of K in the complexification of M, and H^∞ denotes the Banach space of bounded holomorphic functions. If U is an open subset in a complex analytic manifold then $H(U)$ denotes the space of all holomorphic functions on U equipped with the compact-open topology. The natural topology on $\mathscr{A}(M)$ is defined to be the unique topology such that the restriction maps $H(U) \to \mathscr{A}(M)$ and $\mathscr{A}(M) \to H(K)$ are all continuous whenever U is an arbitrary neighbourhood of M in its complexification and K is an arbitrary compact subset of M (comp. [17, 6]). The space $\mathscr{A}(M)$ with that topology is separable, complete, nuclear, webbed, ultrabornological, and the Closed Graph Theorem holds for operators $T : \mathscr{A}(M) \to \mathscr{A}(N)$ etc.; see [6], [10] and [11]. Surprisingly, $\mathscr{A}(M)$ has no Schauder basis [10]. For basic facts from the general theory of locally convex spaces we refer to [18]. For general information on the spaces of real analytic functions see [6].

A subset S of a real analytic manifold N has the *global extension property* if every analytic function f on S extends to an analytic function on N. Similarly, we say that a subset S of a real analytic manifold N has the *semiglobal extension property* if for every relatively compact set $\Omega \Subset N$ there is an open set Δ, with $\Omega \subseteq \Delta \Subset N$, such that for every $f \in \mathscr{A}(S \cap \Delta)$ there is $g \in \mathscr{A}(\Omega)$ such that

$$f|_{\Omega \cap S} = g|_{\Omega \cap S}.$$

For the definition of a *Nash set* see [1, p. 544]. The only fact we will need about them is that all semianalytic sets are Nash but the converse is false (see [1, Prop. 2.3]). Summarizing, we have the following implications for any set S (none of them can be reversed):

$$\text{global ext. property} \Rightarrow \mathbb{C}\text{-analytic} \Rightarrow \text{analytic} \Rightarrow \text{semianalytic} \Rightarrow \text{Nash}.$$

For the definition of subanalytic and semianalytic sets see [16]. For more information on (real or complex) analytic sets see [13], [19], [4] or [5]. For complex analysis of several variables we refer to [14] and [15].

An analytic subset S of a real analytic manifold is called *non-pure-dimensional* at a point $a \in S$ if

- the germ S_a is irreducible and $\dim S_a = \dim S$;
- for every neighbourhood U of a there is a point $c \in U \cap S$ such that $\dim S_c < \dim S$.

An analytic subset S of a real analytic manifold M is called $\mathbb{C}$-*analytic* if S is the common zero set of a finite number of analytic functions defined on M.

3. Main results. We start with the following lemma.

LEMMA 3.1. *Every set with the semiglobal extension property is automatically analytic.*

Proof. Let N be a real analytic manifold; as usual we assume without loss of generality that $N \subset \mathbb{R}^d$. Let $S \subset N$ be a subset with the semiglobal extension property, i.e.,

$$\forall \Omega \Subset N\ \exists \Delta \Subset N,\ \Omega \subset \Delta\ \forall f \in \mathscr{A}(S \cap \Delta)\ \exists g \in \mathscr{A}(\Omega): \quad f|_{\Omega \cap S} = g|_{\Omega \cap S}.$$

By [22, p. 154], the intersection of any family of $\mathbb{C}$-analytic sets is $\mathbb{C}$-analytic. Assume that S is not analytic. Thus there is an open set $\Omega \Subset N$ such that $\Omega \cap S$ is not $\mathbb{C}$-analytic in Ω. Hence the intersection R of all $\mathbb{C}$-analytic subsets of Ω containing $\Omega \cap S$ is strictly bigger than $S \cap \Omega$. Let $x \in R \setminus S$. Clearly the function g defined by the formula

$$g(z) := \frac{1}{|x - z|^2}$$

is analytic on $N \setminus \{x\}$, thus on S. By the semiglobal extension property, it extends to an analytic function $h : \Omega \to \mathbb{C}$. Clearly, $h - g$ vanishes on S and it is defined on an open neighbourhood U of $S \cap \Omega$ in Ω. The zero set Z of $h - g$ is a $\mathbb{C}$-analytic subset of U. By [20, Lemma 1], the set Z is also $\mathbb{C}$-analytic in Ω and it contains S but not x. This contradicts the definition of R. ∎

The aim of this note is to prove the following improvement of [9, Thm. 2.3] (where we used the additional assumption that $\varphi(M)$ is Nash).

THEOREM 3.2. *Let $\varphi : M \to N$ be a real analytic map between real analytic manifolds. The composition operator $C_\varphi : \mathscr{A}(N) \to \mathscr{A}(M)$ is open onto its image if and only if φ is semiproper and $\varphi(M)$ has the semiglobal extension property.*

Proof. Necessity: By [9, proof of the necessity part of Thm. 2.3], if C_φ is open then $\varphi(M)$ has the semiglobal extension property (Nash property mentioned in the statement is not used in that part of the proof). By [9, Lemma 2.4], φ is also semiproper.

Sufficiency: By Lemma 3.1 above, $\varphi(M)$ is analytic thus Nash. Sufficiency follows from [9, Thm. 2.3]. ∎

LEMMA 3.3. *A compact set with the semiglobal extension property in a real analytic manifold is $\mathbb{C}$-analytic.*

Proof. We proceed as in the proof of Lemma 3.1 assuming that $S \subset \Omega$, Ω open, and deducing that S is $\mathbb{C}$-analytic in Ω. By [20, Lemma 1], S is $\mathbb{C}$-analytic in N. ∎

PROPOSITION 3.4. *Any compact set $S \subset N$, N a real analytic manifold, has the global extension property if and only if it has the semiglobal extension property.*

Proof. Necessity: Let $S \subset \Omega$ and let $\Omega \Subset \Delta \Subset N$ be open subsets. Let $f \in \mathscr{A}(S \cap \Delta) = \mathscr{A}(S)$. Then by the global extension property f extends to N.

Sufficiency: Let $f \in \mathscr{A}(S) = \mathscr{A}(S \cap \Delta)$. Then, by the semiglobal extension property, f extends to Ω, a neighbourhood of S. By Lemma 3.3, S is $\mathbb{C}$-analytic, and, by [20, Thm. 1], it has the so-called *weak extension property* (i.e., every function which extends to some neighbourhood extends to the whole space). So f extends to the whole N. ∎

COROLLARY 3.5. *Let $\varphi : M \to N$ be a real analytic map between real analytic manifolds, with M compact. The following conditions are equivalent:*

1. *The map $C_\varphi : \mathscr{A}(N) \to \mathscr{A}(M)$ has closed range.*
2. *The map $C_\varphi : \mathscr{A}(N) \to \mathscr{A}(M)$ is open onto its image.*
3. *φ is semiproper and $\varphi(M)$ has the global extension property.*

Proof. 1⇒2. The map φ is always semiproper. If C_φ has closed image, by [9, Thm. 2.2], it has the global extension property, thus by Proposition 3.4, it has the semiglobal extension property. By Theorem 3.2, C_φ is open onto its image

2⇒1. If C_φ is open onto its image then, by Theorem 3.2 and Proposition 3.4, $\varphi(M)$ has the global extension property and $\varphi(M)$ is analytic. By [9, Thm. 2.2], C_φ has closed image.

1&2⇔3. Use [9, Cor. 2.5] and Proposition 3.4. ∎

PROPOSITION 3.6. *If a* $\mathbb{C}$*-analytic subset* X *of an open set* Ω *in* $\mathbb{R}^d$ *with* $\dim X \geq 2$ *is non-pure-dimensional at a point* $a \in X$*, then for every neighbourhood* Δ *of* a *in* Ω *there is an analytic function on* X *which does not extend to* Δ.

Proof. One gets a proof of this by re-reading the proof of [20, Thm. 2]. Only minor changes, indicated below, are necessary. We follow the notation from [20] where possible. All undefined objects are as in [20].

We first assume that X is $\mathbb{C}$-irreducible.

From the definition of being non-pure-dimensional, we can assume that Δ is such that $K = \Delta \cap X$ is irreducible analytic, so K is $\mathbb{C}$-irreducible. We take $c \in \Delta \cap T$ with $X_c = T_c$.

We define analytic functions $q, \widetilde{q}, g, \widetilde{g}, \lambda$ exactly as in [20]. We will show that λ has no extension to Δ.

Assume to the contrary that $\mu : \Delta \to \mathbb{R}$ is real analytic and $\mu|_{\Delta\cap X} = \lambda|_{\Delta\cap X}$. The function μ extends to a holomorphic function on some neighbourhood of Δ in $\mathbb{C}^n$. Let $\widetilde{\Delta}$ be an open subset of $\widetilde{\Omega}$ such that $\widetilde{\Delta} \cap \mathbb{R}^n = \Delta$, and $\widetilde{\mu} \in H(\widetilde{\Delta})$ be such that $\widetilde{\mu}|_\Delta = \mu$.

Now let $\widetilde{K}$ be the smallest complex analytic subset of $\widetilde{\Delta}$ containing K. Certainly $\widetilde{K} \subseteq \widetilde{\Delta} \cap \widetilde{X}$. But $\dim \widetilde{K} = \dim \widetilde{\Delta} \cap \widetilde{X}$, so from [5, I.5.3 Cor. 2], $\widetilde{K}$ is an irreducible component of $\widetilde{\Delta} \cap \widetilde{X}$ containing $\Delta \cap X$.

Let

$$\widetilde{P} = \{z \in \widetilde{K} : \widetilde{q}(z) + \widetilde{g}(z) = 0\}, \qquad \widetilde{S} = \{z \in \widetilde{K} : \widetilde{q}(z) = 0\}.$$

We have $c \in \widetilde{P} \cap \widetilde{S}$ and $\widetilde{P} \subsetneq \widetilde{K}$ (because $a \notin \widetilde{P}$). Therefore $(\operatorname{reg} \widetilde{K}) \setminus \widetilde{P}$ is connected by [5, I.5.3 Prop., I.2.2 Prop. 3].

Let $\widetilde{\lambda} : \widetilde{K} \setminus \widetilde{P} \to \mathbb{C}$ be given by the formula

$$\widetilde{\lambda}(z) = \frac{\widetilde{q}(z)}{\widetilde{q}(z) + \widetilde{g}(z)}.$$

There is a point $x \in K$ which is regular for both K and $\widetilde{K}$, so by [20, Remark 1], $\widetilde{K}_x$ is the complexification of K_x. As $a \notin \widetilde{P}$, $\widetilde{\lambda}$ is defined in a neighbourhood of a, so we can assume that it is defined at x. Therefore

$(\widetilde{\lambda} - \widetilde{\mu})|_{K_x} = 0$. As $\widetilde{K}_x$ is the complexification of K_x, we get

$$(\widetilde{\lambda} - \widetilde{\mu})|_{\widetilde{K}_x} = 0.$$

From connectedness of $(\operatorname{reg} \widetilde{K}) \setminus \widetilde{P}$ and continuity we get $\widetilde{\lambda} = \widetilde{\mu}$ on $\widetilde{K} \setminus \widetilde{P}$.

The germ $\widetilde{g}_c$ does not vanish on the germs $\widetilde{S}^l_c$, so $\widetilde{S} \cap \widetilde{P}$ is a non-empty proper subset of $\widetilde{S}$. Thus $\widetilde{S} \setminus \widetilde{P}$ is an open subset of $\widetilde{S}$ with $c \in \overline{\widetilde{S} \setminus \widetilde{P}}$. But on $\widetilde{S} \setminus \widetilde{P}$ we have $\widetilde{\lambda} = \widetilde{\mu}$ and $\widetilde{\lambda} \equiv 0$ from its definition. By continuity $\widetilde{\mu}(c) = 0$, a contradiction because $\mu(c) = \lambda(c) = 1$.

If X is not $\mathbb{C}$-irreducible, then a lies in exactly one $\mathbb{C}$-irreducible component. Indeed, assume that $a \in A$ and $a \in B$, where $A, B \subseteq X$ are $\mathbb{C}$-analytic and $\mathbb{C}$-irreducible components of X. Let $\widetilde{\Omega}$ be a complex open neighbourhood of Ω and $\widetilde{A}, \widetilde{B} \subseteq \widetilde{\Omega}$ be complex analytic subsets with $A = \widetilde{A} \cap \mathbb{R}^n$, $B = \widetilde{B} \cap \mathbb{R}^n$. By [20, Remark 1(1)] we can assume that $\widetilde{A}, \widetilde{B}$ are irreducible. Since X_a is irreducible, we must have $A_a = B_a = X_a$. It follows that $\widetilde{A}_a = \widetilde{B}_a$. Now from [5, I.5.3 Cor. 2] it follows that $\widetilde{A} = \widetilde{B}$, therefore $A = B$.

Let X' be the $\mathbb{C}$-irreducible component containing a. As we can assume that the point c lies only in X', the function λ is defined on the whole of X and does not extend to Δ. ∎

Corollary 3.7. *If a $\mathbb{C}$-analytic subset S of a real analytic manifold N is non-pure-dimensional at a point $a \in S$, then for every neighbourhood Ω of a in N there is an analytic function on S which does not extend to Ω.*

Proof. By [13, VI.1.3], N can be embedded into $\mathbb{R}^n$ for n large enough. The manifold N is coherent ([13, p. 17]), so by [4, Sect. 7, (2)], analytic functions defining S can be extended to $\mathbb{R}^n$. Since N is also $\mathbb{C}$-analytic ([19, p. 104]), it follows that S is $\mathbb{C}$-analytic as a subset of $\mathbb{R}^n$.

Assume that every analytic function on S can be extended to an analytic function on Ω. Let $\widetilde{\Omega}$ be a neighbourhood of a in $\mathbb{R}^n$ such that $\widetilde{\Omega} \cap N \subseteq \Omega$. Clearly, once again by [4, Sect. 7, (2)], every analytic function on Ω extends to $\widetilde{\Omega}$, thus every analytic function on S extends to $\widetilde{\Omega}$, which contradicts Proposition 3.6. ∎

Corollary 3.8. *If a $\mathbb{C}$-analytic subset S of a real analytic manifold N with* $\dim S \geq 2$ *is non-pure-dimensional at a certain point $a \in S$, then S does not have the semiglobal extension property.*

Example 3.9. Consider the mapping ϕ from $\mathbb{R}^2 \times \{0, 1\}$ into $\mathbb{R}^3$ given by

$$(u, v, 0) \mapsto (u, uv, v^2), \qquad (u, v, 1) \mapsto (0, 0, u).$$

Then ϕ is semiproper, and the image of ϕ is the Whitney umbrella given in $\mathbb{R}^3$ by the equation $zx^2 = y^2$. Since the Whitney umbrella is irreducible and of non-pure dimension, it follows that the composition operator $C_\phi : \mathscr{A}(\mathbb{R}^3) \to \mathscr{A}(\mathbb{R}^2)^2$ is not open onto its image.

Let us observe that every coherent set has both the global and the semiglobal extension property. On the other hand, by [20, Cor. 3] for normal sets the global extension property implies coherence. Thus if the set $\varphi(M)$ is normal and the operator C_φ has closed range, then C_φ is open onto its image if and only if φ is semiproper.

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