ANNALES
POLONICI MATHEMATICI
107.3 (2013)

Projectively flat Finsler metrics with orthogonal invariance

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Abstract. We study Finsler metrics with orthogonal invariance. By determining an expression of these Finsler metrics we find a PDE equivalent to these metrics being locally projectively flat. After investigating this PDE we manufacture projectively flat Finsler metrics with orthogonal invariance in terms of error functions.

1. Introduction. It is Hilbert's Fourth Problem in the smooth case to study and characterize the projectively flat Finsler metrics on an open domain in $\mathbb{R}^n$. Beltrami's theorem tells us that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. However the situation is much more complicated for Finsler metrics. In fact, there are lots of projectively flat Finsler metrics which are not of constant flag curvature [15]. Conversely, there are infinitely many non-projectively flat Finsler metrics with constant flag curvature [9, 4, 19, 16, 3]. The flag curvature is the most important Riemannian quantity in Finsler geometry because it is an analogue of sectional curvature in Riemannian geometry [2].

Below are three important examples.

(a) Consider the following Randers metric defined near the origin in $\mathbb{R}^n$:

$$F := \frac{\sqrt{|y|^2 - (|xQ|^2|y|^2 - \langle y, xQ\rangle^2)}}{1 - |xQ|^2} - \frac{\langle y, xQ\rangle}{1 - |xQ|^2}$$

where $Q = (q^i_j)$ is an anti-symmetric matrix. When $Q \neq 0$, F is not projectively flat with zero flag curvature.

(b) Let $F = \sqrt{\sqrt{A} + B}$ be a generalized fourth root metric on $\mathbb{B}^n \subset \mathbb{R}^n$ defined by [11]

$$A := \frac{|y|^4 + (|x|^2|y|^2 - \langle x, y\rangle^2)^2}{4(1 + |x|^4)^2}, B := \frac{(1 + |x|^4)|x|^2|y|^2 + (1 - |x|^4)\langle x, y\rangle^2}{2(1 + |x|^4)^2}.$$

2010 *Mathematics Subject Classification*: Primary 53B40; Secondary 58E20.
Key words and phrases: Finsler metric, projectively flat, error function, orthogonal invariance.

DOI: 10.4064/ap107-3-3

Then F is projectively flat Finsler metric with scalar flag curvature

$$K = \frac{6\sqrt{A}\langle x,y\rangle^2}{F^4(1+|x|^4)^2} - \frac{2(\sqrt{A}-2B)}{F^2}.$$

Hence F is not of constant flag curvature.

(c) Let ε be an arbitrary number with $|\varepsilon| < 1$. Let

$$F_\varepsilon := \frac{1}{\Psi}\left\{\sqrt{\Psi\left[\frac{1}{2}(\sqrt{\Phi^2+(1-\varepsilon^2)|y|^4}+\Phi)\right]} + \sqrt{1-\varepsilon^2}\langle x,y\rangle\right\}$$

where

$$\Phi := \varepsilon|y|^2 + |x|^2|y|^2 - \langle x,y\rangle^2, \qquad \Psi := 1 + 2\varepsilon|x|^2 + |x|^4.$$

One can verify that F_ε is a projectively flat Finsler metric with constant flag curvature $K = 1$ [5, 14, 21]. Note that if $\varepsilon = 1$, then F_1 is the spherical metric on $\mathbb{R}^n$.

Thus locally projectively flat Finsler metrics form a rich class of Finsler metrics. On the other hand, all Finsler metrics we mentioned above satisfy

$$F(Ax, Ay) = F(x,y) \tag{1.1}$$

for all $A \in O(n)$. These three examples inspire us to study projectively flat Finsler metrics which satisfy (1.1). A Finsler metric F is said to be *orthogonally invariant* if F satisfies (1.1) for all $A \in O(n)$, equivalently, the orthogonal group $O(n)$ acts as isometries of F.

The aim of this paper is to study and characterize projectively flat orthogonally invariant Finsler metrics. First, we give a characterization of orthogonally invariant Finsler metrics (see Proposition 3.1). In particular, we show that all such metrics are general (α,β)-metrics.

Recall that *general* (α,β)*-metrics* are Finsler metrics of the form $F = \alpha\phi(\|\beta\|_\alpha, \beta/\alpha)$ where α is a Riemannian metric and β is a 1-form (for exact definition, see Section 2) [14, 21, 8]. In this paper, we obtain a second-order PDE for ϕ equivalent to the general (α,β)-metric $F = \alpha\phi(\|\beta\|_\alpha, \beta/\alpha)$ being locally projectively flat where α has constant sectional curvature and β is closed and conformal with respect to α. The sufficiency of our condition has been shown in [21]. In particular, we have the following

THEOREM 1.1. *Let $F = |y|\phi(|x|, \langle x,y\rangle/|y|)$ be an orthogonally invariant Finsler metric on $\mathbb{B}^n(r)$. Then $F = F(x,y)$ is projectively flat if and only if $\phi = \phi(b,s)$ satisfies*

$$s\phi_{bs} + b\phi_{ss} - \phi_b = 0. \tag{1.2}$$

In the special case of $\phi = \epsilon + b^\mu f(s/b)$, our criterion has been obtained in [8].

The *error function* (also called the *Gauss error function* or *probability integral*) is a special (non-elementary) function of sigmoid shape [1, 12].

It has numerous applications in probability, statistics and partial differential equations [1]. In Section 5, we find the general solution ϕ of (1.2) (see Proposition 5.1). Then we give a lot of new projectively flat Finsler metrics in terms of error functions (see Theorem 5.2). In particular, we have the following

THEOREM 1.2. *Let $\phi(b,s)$ be a function defined by*

$$\phi(b,s) = sg(b) + e^{\lambda b^2}[e^{-\lambda s^2} + \sqrt{\lambda\pi}\, s\, \mathrm{erf}(\sqrt{\lambda}\, s)]$$

where $\lambda > 0$, $\mathrm{erf}(\ ,\ )$ denotes the error function and g is any function. Then the orthogonally invariant Finsler metric

$$F = |y|\phi(|x|, \langle x, y\rangle/|y|)$$

on an open subset in $\mathbb{R}^n$ is projectively flat.

2. Preliminaries. A Finsler metric on a manifold is a family of Minkowski norms on the tangent spaces. By definition, a *Minkowski norm* on a vector space V is a nonnegative function $F : V \to [0,\infty)$ with the following properties:

(i) F is positively y-homogeneous of degree one, i.e., for any $y \in V$ and any $\lambda > 0$,

$$F(\lambda y) = \lambda F(y).$$

(ii) F is C^∞ on $V \setminus \{0\}$ and for any tangent vector $y \in V \setminus \{0\}$, the following bilinear symmetric form $\mathbf{g}_y : V \times V \to \mathbb{R}$ is positive definite:

$$\mathbf{g}_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}[F^2(y + su + tv)]_{s=t=0}.$$

Let M be a manifold. Let $TM = \bigcup_{x\in M} T_xM$ be the tangent bundle of M, where T_xM is the tangent space at $x \in M$. We set $TM_o := TM \setminus \{0\}$ where $\{0\}$ stands for $\{(x,0) \mid x \in M,\, 0 \in T_xM\}$. A *Finsler metric* on M is a function $F : TM \to [0,\infty)$ with the following properties:

(a) F is C^∞ on TM_o.
(b) At each point $x \in M$, the restriction $F_x := F|_{T_xM}$ is a Minkowski norm on T_xM.

For instance, let $\phi = \phi(y)$ be a Minkowski norm on $\mathbb{R}^N$. Define

$$\varPhi(x,y) := \phi(y), \qquad y \in T_x\mathbb{R}^N \cong \mathbb{R}^N.$$

Then $\varPhi = \varPhi(x,y)$ is a Finsler metric. We call $\varPhi$ the *Minkowski metric* on $\mathbb{R}^N$ [7, 18].

Riemannian metrics are a special case of Finsler metrics: they are Finsler metrics with the quadratic restriction [7].

A Finsler metric is said to be *locally projectively flat* if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. It is known that every locally projectively flat Finsler metric is of scalar curvature [6,7]. Similar results on projectively flat Finsler metrics have been discussed by Bryant, Shen, Li, Yu, Yıldırım, Chen and Mo in [5, 6, 18, 10, 13, 15, 17, 20].

Definition 2.1. Let F be a Finsler metric on $\mathbb{B}^n(r)$. F is said to be *orthogonally invariant* if it satisfies

$$F(Ax, Ay) = F(x, y)$$

for all $x \in \mathbb{B}^n(r)$, $y \in T_x\mathbb{B}^n(r)$ and $A \in O(n)$.

A Finsler metric on a manifold M is said to be of *general* (α, β) *type* if

$$F = \alpha\phi(b, \beta/\alpha)$$

where α is a Riemannian metric, β is a 1-form on M, $b = \|\beta\|_\alpha$ and $\phi(b, s)$ is a C^∞ function satisfying (see [21, 8, 14])

$$\phi(s) - s\phi_s(s) > 0, \quad \phi(s) - s\phi_s(s) + (b^2 - s^2)\phi_{ss}(s) > 0, \quad |s| \le b < b_o$$

when $n \ge 3$, and

$$\phi(s) - s\phi_s(s) + (b^2 - s^2)\phi_{ss}(s) > 0, \quad |s| \le b < b_o$$

when $n = 2$ [21]. The reader should note that the general (α, β)-metric as defined here differs from those of Yu–Zhu and Mo [21, 14], defined by

$$F = \alpha\phi(b^2, \beta/\alpha).$$

A 1-form is said to be a *conformal* (resp. *Killing*) *form* with respect to a Riemannian metric α if its dual vector field with respect to α is of conformal (resp. Killing) type.

3. Finsler metrics with orthogonal invariance. In this section, we determine an expression of orthogonally invariant Finsler metrics. Let $|\cdot|$ and $\langle\,,\,\rangle$ be the standard Euclidean norm and inner product on $\mathbb{R}^n$.

Proposition 3.1. *A Finsler metric F on $\mathbb{B}^n(r)$ is orthogonally invariant if and only if there is a function $\phi : [0, r) \times \mathbb{R} \to \mathbb{R}$ such that*

$$F(x, y) = |y|\phi(|x|, \langle x, y\rangle/|y|) \tag{3.1}$$

where $(x, y) \in T\mathbb{B}^n(r) \setminus \{0\}$. In particular, all orthogonally invariant Finsler metrics are general (α, β)-metrics.

Proof. Assume that $F(x, y) = |y|\phi(|x|, \langle x, y\rangle/|y|)$ for some $\phi : [0, r) \times \mathbb{R} \to \mathbb{R}$. It is easy to see

$$\langle Ax, Ay\rangle = \langle x, A^\top Ay\rangle = \langle x, y\rangle$$

for $x, y \in \mathbb{R}^n$ and $A \in O(n)$. In particular, $|Ax| = |x|$ for $x \in \mathbb{R}^n$. Hence

$$F(Ax, Ay) = |Ay|\phi\bigg(|Ax|, \frac{\langle Ax, Ay\rangle}{|Ay|}\bigg) = |y|\phi\bigg(|x|, \frac{\langle x, y\rangle}{|y|}\bigg) = F(x, y).$$

Conversely, suppose that F is orthogonally invariant. Denote by $e_1,\ldots,e_n$ the standard orthonormal basis of $\mathbb{R}^n$, where

$$e_j=(0,\ldots,0,\underset{j}{1},0,\ldots,0),\qquad j=1,\ldots,n. \tag{3.2}$$

Put

$$\epsilon_1=\frac{x}{|x|},\qquad \epsilon_2=\frac{y-\frac{\langle y,x\rangle}{|x|^2}x}{\left|y-\frac{\langle y,x\rangle}{|x|^2}x\right|}. \tag{3.3}$$

Then ϵ_1 and ϵ_2 are orthonormal vectors in $\mathbb{R}^n$. It follows that there exists an $A\in O(n)$ such that

$$A\epsilon_1=e_1,\qquad A\epsilon_2=e_2. \tag{3.4}$$

A simple calculation gives

$$\left|y-\frac{\langle y,x\rangle}{|x|^2}x\right|^2=|y|^2-\frac{\langle x,y\rangle^2}{|x|^2}. \tag{3.5}$$

By using the first formula of (3.3) and the first formula of (3.4) we obtain

$$Ax=A(|x|\epsilon_1)=|x|A\epsilon_1=|x|e_1. \tag{3.6}$$

Together with (3.5), the second formula of (3.3) and the second formula of (3.4) we get

$$\begin{aligned}Ay&=A\left(\left|y-\frac{\langle y,x\rangle}{|x|^2}x\right|\epsilon_2+\frac{\langle y,x\rangle}{|x|^2}x\right)\\&=A\left(\frac{\langle x,y\rangle}{|x|^2}x+\frac{\sqrt{|x|^2|y|^2-\langle x,y\rangle^2}}{|x|}\epsilon_2\right)\\&=\frac{\langle x,y\rangle}{|x|^2}Ax+\frac{\sqrt{|x|^2|y|^2-\langle x,y\rangle^2}}{|x|}A\epsilon_2=\frac{\langle x,y\rangle}{|x|}e_1+\frac{\sqrt{|x|^2|y|^2-\langle x,y\rangle^2}}{|x|}e_2.\end{aligned} \tag{3.7}$$

Applying the orthogonal invariance of F we obtain

$$\begin{aligned}F(x,y)=F(Ax,Ay)&=F\left(|x|e_1,\frac{\langle x,y\rangle}{|x|}e_1+\frac{\sqrt{|x|^2|y|^2-\langle x,y\rangle^2}}{|x|}e_2\right)\\&=F\left(|x|,0,\ldots,0;\ \frac{\langle x,y\rangle}{|x|},\frac{\sqrt{|x|^2|y|^2-\langle x,y\rangle^2}}{|x|},0,\ldots,0\right)\\&=\psi(|x|,\langle x,y\rangle,|y|)\end{aligned} \tag{3.8}$$

where $\psi:[0,r)\times\mathbb{R}^2\to\mathbb{R}$ and we have used (3.2), (3.6) and (3.7). Note that F is homogeneous of degree one with respect to y. Hence

$$\begin{aligned}\lambda\psi(|x|,\langle x,y\rangle,|y|)&=\lambda F(x,y)=F(x,\lambda y)\\&=\psi(|x|,\langle x,\lambda y\rangle,|\lambda y|)=\psi(|x|,\lambda\langle x,y\rangle,\lambda|y|)\end{aligned}$$

for $\lambda\in[0,\infty)$. In particular,

$$\frac{1}{|y|}\psi(|x|,\langle x,y\rangle,|y|)=\psi\left(|x|,\frac{\langle x,y\rangle}{|y|},1\right):=\phi\left(|x|,\frac{\langle x,y\rangle}{|y|}\right)$$

where $y \in T_x\mathbb{B}^n(r) \setminus \{0\}$ and $\phi : [0,r) \times \mathbb{R} \to \mathbb{R}$. Plugging this into (3.8) yields (3.1). ■

In [8] we have studied a class of orthogonally invariant Finsler metrics. In particular, we produced such metrics in terms of hypergeometric functions.

4. Reducible differential equation. In this section, for a class of Finsler metrics, we find a partial differential equation equivalent to the metric being locally projectively flat (see Theorem 4.2 below).

If $\mathcal{U} \subset M$ is a coordinate neighborhood, a function ξ defined on $T\mathcal{U}$ can be expressed as $\xi(x^1, \ldots, x^n; y^1, \ldots, y^n)$. We use the notation

$$\xi_0 = \frac{\partial \xi}{\partial x^i} y^i.$$

It is easy to show the following (cf. [8, 7]):

LEMMA 4.1. *A Finsler metric $F = F(x,y)$ on a manifold M is locally projectively flat if and only if it satisfies the system of equations*

$$(F_0)_{y^i} = 2F_{x^i}. \tag{4.1}$$

THEOREM 4.2. *Let $F = \alpha\phi(\|\beta\|_\alpha, \beta/\alpha)$ be a general (α,β)-metric on an n-dimensional manifold M where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ has constant sectional curvature and $\beta = b_i(x)y^i$. Suppose that β is conformal with respect to α and satisfies $d\beta = 0$. Then:*

(i) *If ϕ satisfies*

$$s\phi_{bs} + b\phi_{ss} - \phi_b = 0 \tag{4.2}$$

where $b := \|\beta\|_\alpha$ and $s = \beta/\alpha$ then F is locally projectively flat.

(ii) *If β is not a Killing form and F is locally projectively flat then ϕ satisfies* (4.2).

Proof. Let $\nabla\beta = b_{i|j}dx^i \otimes dx^j$ denote the covariant derivative of β with respect to α and $b = \sqrt{a^{ij}b_ib_j}$ the length of β where $(a^{ij}) = (a_{ij})^{-1}$. Since β is conformal with respect to α, there is a scalar function $\lambda = \lambda(x)$ such that

$$b_{i|j} + b_{j|i} = \tfrac{1}{2}\lambda(x)a_{ij}.$$

Noticing that β is closed, we have $b_{i|j} = b_{j|i}$. It follows that

$$b_{i|j} = \lambda(x)a_{ij} \tag{4.3}$$

where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$. Since α is locally projectively flat, we have

$$(\alpha_0)_{y^i} = 2\alpha_{x^i} \tag{4.4}$$

where we have used Lemma 4.1. Denote the geodesic coefficients of α by G^i. Then the local projective flatness of α also implies that [7]

$$G^i = Py^i \tag{4.5}$$

where P is the projective factor of α. Furthermore, P is given by

$$P = \frac{\alpha_0}{2\alpha}. \tag{4.6}$$

A simple calculation gives

$$\beta_{x^i} = (b_{j|i} + \Gamma^k_{ji} b_k) y^j \tag{4.7}$$

where Γ^k_{ji} are the Christoffel symbols of the Levi-Civita connection of α. The connection coefficients N^i_j satisfy [7, (2.6) and (2.18)]

$$N^i_j = \Gamma^i_{jk} y^k = \frac{\partial G^i}{\partial y^j}. \tag{4.8}$$

Using (4.3) we get

$$b_{j|i} y^j = \lambda(x) y_i \tag{4.9}$$

where $y_i := a_{ji} y^j$. Plugging (4.5) into (4.8) yields

$$N^i_j = P_{y^j} y^i + P\delta^i_j. \tag{4.10}$$

Together with (4.8) we have

$$\Gamma^k_{ji} b_k y^j = N^k_i b_k = (P_{y^i} y^k + P\delta^k_i) b_k = P_{y^i}\beta + P b_i. \tag{4.11}$$

Substituting (4.9) and (4.11) into (4.7) yields

$$\beta_{x^i} = \lambda(x) y_i + P_{y^i}\beta + P b_i. \tag{4.12}$$

Using (4.4) and (4.6), we obtain

$$\begin{aligned} P_{y^i} &= \frac{1}{2}\left(\frac{\alpha_0}{\alpha}\right)_{y^i} = \frac{1}{2}\,\frac{(\alpha_0)_{y^i}\alpha - \alpha_0 \alpha_{y^i}}{\alpha^2} \\ &= \frac{\alpha_{x^i}}{\alpha} - \frac{\alpha_0}{2\alpha}\,\frac{\alpha_{y^i}}{\alpha} = \frac{1}{\alpha}(\alpha_{x^i} - P\alpha_{y^i}). \end{aligned} \tag{4.13}$$

Plugging (4.13) into (4.12) yields

$$\beta_{x^i} = \lambda(x) y_i + s(\alpha_{x^i} - P\alpha_{y^i}) + P b_i. \tag{4.14}$$

It follows that

$$\begin{aligned} s_{x^i} &= \left(\frac{\beta}{\alpha}\right)_{x^i} = \frac{1}{\alpha}(\beta_{x^i} - s\alpha_{x^i}) \\ &= \frac{1}{\alpha}[\lambda(x) y_i + s(\alpha_{x^i} - P\alpha_{y^i}) + P b_i - s\alpha_{x^i}] \\ &= \frac{1}{\alpha}[\lambda(x) y_i + P(b_i - s\alpha_{y^i})]. \end{aligned} \tag{4.15}$$

By direct calculations one obtains

$$\alpha_{y^i} = \frac{y_i}{\alpha}, \qquad \beta_{y^i} = b_i(x). \tag{4.16}$$

Thus

$$s_{y^i} = \left(\frac{\beta}{\alpha}\right)_{y^i} = \frac{b_i - s\alpha_{y^i}}{\alpha}. \tag{4.17}$$

Combining this with (4.15) and the first equation of (4.16) we have

$$s_{x^i} = \lambda(x)\alpha_{y^i} + Ps_{y^i}. \tag{4.18}$$

By a direct calculation we have (see [13, Lemma 3.1])

$$2bb_{x^i} = (b^2)_{x^i} = 2\lambda(x)b_i$$

where we have used (4.3). It follows that

$$b_{x^i} = \frac{\lambda}{b}b_i = \frac{\lambda}{b}(\alpha s_{y^i} + s\alpha_{y^i}) \tag{4.19}$$

where we have made use of (4.17). Combining (4.19) with (4.18) we obtain

$$\begin{aligned} F_{x^i} &= [\alpha\phi(b,s)]_{x^i} = \phi\alpha_{x^i} + \alpha(\phi_b b_{x^i} + \phi_s s_{x^i}) \\ &= \phi\alpha_{x^i} + \alpha\left[\phi_b\frac{\lambda}{b}(\alpha s_{y^i} + s\alpha_{y^i}) + \phi_s(\lambda\alpha_{y^i} + Ps_{y^i})\right] \\ &= \phi\alpha_{x^i} + \alpha\left[\left(\phi_b\frac{\lambda}{b}\alpha + \phi_s P\right)s_{y^i} + \lambda\left(\phi_b\frac{s}{b} + \phi_s\right)\alpha_{y^i}\right]. \end{aligned} \tag{4.20}$$

Note that s and α are positively homogeneous of degree 0 and 1 respectively. Hence

$$s_{y^i}y^i = 0, \qquad \alpha_{y^i}y^i = \alpha. \tag{4.21}$$

Contracting (4.20) with y^i and using (4.21), we get

$$F_0 = \phi\alpha_0 + \lambda\alpha^2\left(\phi_b\frac{s}{b} + \phi_s\right). \tag{4.22}$$

It follows that

$$(F_0)_{y^i} = \phi_{y^i}\alpha_0 + \phi(\alpha_0)_{y^i} + \lambda(\alpha^2)_{y^i}\left(\phi_b\frac{s}{b} + \phi_s\right) + \lambda\alpha^2\left(\phi_b\frac{s}{b} + \phi_s\right)_{y^i}. \tag{4.23}$$

Since $b_{y^i} = 0$, one obtains

$$\phi_{y^i} = \phi_s s_{y^i}, \tag{4.24}$$

$$\left(\phi_b\frac{s}{b} + \phi_s\right)_{y^i} = \frac{1}{b}(s\phi_{bs} + b\phi_{ss} + \phi_b)s_{y^i}. \tag{4.25}$$

Plugging (4.4), (4.24) and (4.25) into (4.23) yields

$$\begin{aligned} (F_0)_{y^i} = {} & 2\phi\alpha_{x^i} + 2\lambda\alpha\left(\phi_b\frac{s}{b} + \phi_s\right)\alpha_{y^i} \\ & + \left[\phi_s\alpha_0 + \frac{\lambda\alpha^2}{b}(s\phi_{bs} + b\phi_{ss} + \phi_b)\right]s_{y^i}. \end{aligned} \tag{4.26}$$

By (4.20), (4.26) and Lemma 4.1, $F = F(x, y)$ is locally projectively flat if and only if

$$\left[\phi_s\alpha_0 + \frac{\lambda\alpha^2}{b}(s\phi_{bs} + b\phi_{ss} + \phi_b)\right]s_{y^i} = 2\alpha\left(\phi_b\frac{\lambda}{b}\alpha + \phi_s P\right)s_{y^i}. \tag{4.27}$$

By (4.8), (4.27) holds if and only if

$$\frac{\lambda}{b}(s\phi_{bs} + b\phi_{ss} - \phi_b)s_{y^i} = 0. \tag{4.28}$$

Contracting (4.28) with b^i and using (4.16) and (4.17) we have

$$\frac{\lambda}{\alpha b}(s\phi_{bs} + b\phi_{ss} - \phi_b)(b^2 - s^2) = 0.$$

Taking $|s| < b$ we obtain

$$\lambda(s\phi_{bs} + b\phi_{ss} - \phi_b) = 0.$$

Thus we have proved Theorem 4.2. ■

It is worth mentioning the recent result by Yu and Zhu that for any general (α, β)-metric $F = \alpha\phi(\|\beta\|_\alpha, \beta/\alpha)$ where α is locally projectively flat and β is conformal with respect to α and satisfies $d\beta = 0$, the metric $F = F(x, y)$ is locally projectively flat if $\phi = \phi(b, s)$ satisfies (4.2) [21].

Proof of Theorem 1.1. Let us take a look at a special case: when $\alpha = |y|$, $\beta = \langle x, y\rangle$,

$$\|\beta\|_\alpha = |x|.$$

Then α is projectively flat and

$$d\beta = d\Big(\sum_i x^i dx^i\Big) = 0.$$

Furthermore, β is a non-Killing conformal form with respect to α. Now Theorem 1.1 is an immediate consequence of Theorem 4.2. ■

Taking $\phi(b, s) = \epsilon + b^\mu f(s/b)$ in Theorem 1.1 we have the following (see [8, Theorem 3.2]):

Corollary 4.3. *Let $F(x, y) := |y|\big\{\epsilon + |x|^\mu f\big(\frac{\langle x,y\rangle}{|x||y|}\big)\big\}$ be a general (α, β)-metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$. Then $F = F(x, y)$ is projectively flat if and only if*

$$(\lambda^2 - 1)f'' - \mu\lambda f' + \mu f = 0$$

where $\lambda = \langle x, y\rangle/(|x|\,|y|)$.

5. Projectively flat Finsler metrics in terms of error functions. In this section we are going to find the general solution ϕ of (4.2). Then we give a lot of new projectively flat general (α, β)-metrics in terms of error functions.

Proposition 5.1. *For $s > 0$, the general solution ϕ of (4.2) is given by*

$$\phi(b,s) = sg(b) - s\int_{s_0}^{s} t^{-2} f(b^2 - t^2)\,dt \tag{5.1}$$

where $s_0 \in (0, s]$.

Proof. Note that $s > 0$. We see that (4.2) is equivalent to

$$sz_b + bz_s = 0 \tag{5.2}$$

where

$$z := \phi - s\phi_s. \tag{5.3}$$

The characteristic equation of the quasi-linear PDE (5.2) is

$$\frac{db}{s} = \frac{ds}{b} = \frac{dz}{0}. \tag{5.4}$$

It follows that

$$b^2 - s^2 = c_1, \qquad z = c_2$$

are independent integrals of (5.4). Hence the solution of (5.2) is

$$z = f(b^2 - s^2) \tag{5.5}$$

where f is any continuously differentiable function. Hence

$$\phi - s\phi_s = f(b^2 - s^2). \tag{5.6}$$

It follows that every solution of (4.2) satisfies (5.6). Conversely, suppose that (5.6) holds. Then we obtain (5.2) and (5.3). Thus ϕ satisfies (4.2). We conclude that (5.6) and (4.2) are equivalent.

Now we consider $s \in [s_0, \infty)$ where $s_0 > 0$. Put

$$\phi = s\psi. \tag{5.7}$$

It follows that $\phi_s = \psi + s\psi_s$. Together with (5.6) this yields

$$f(b^2 - s^2) = s\psi - s(\psi + s\psi_s) = -s^2\psi_s.$$

Thus

$$\psi = g(b) - \int_{s_0}^{s} t^{-2} f(b^2 - t^2)\,dt.$$

Plugging this into (5.7) yields (5.1). ∎

Remark. Similarly, we can obtain the general solution of (4.2) for $s < 0$.

The *error function* is a (non-elementary) special function of sigmoid shape which occurs in probability, statistics and partial differential equations [1, 12]. It is defined by $\operatorname{erf}(x) := (2/\sqrt{\pi})\int_0^x e^{-t^2}\,dt$.

Now we manufacture projectively flat general (α,β)-metrics in terms of the error function.

Taking $f(u) = e^{\lambda u}$ in (5.1) where $\lambda \in \mathbb{R}^+$ we have

$$\begin{aligned}\int t^{-2} f(b^2 - t^2)\, dt &= \int t^{-2} e^{\lambda(b^2-t^2)}\, dt = e^{\lambda b^2} \int t^{-2} e^{-\lambda t^2}\, dt \\ &= e^{\lambda b^2}\Big(-\int e^{-\lambda t^2}\, dt^{-1}\Big) = e^{\lambda b^2}\Big(\int t^{-1} de^{-\lambda t^2} - t^{-1} e^{-\lambda t^2}\Big) \\ &= -e^{\lambda b^2}\Big(t^{-1} e^{-\lambda t^2} + 2\lambda \int e^{-\lambda t^2}\, dt\Big).\end{aligned}$$

Combining this with (5.1) we have

$$\begin{aligned}\phi(b,s) &= sg(b) + se^{\lambda b^2}\Big(t^{-1} e^{-\lambda t^2}\big|_{s_0}^{s} + 2\lambda \int_{s_0}^{s} e^{-\lambda t^2}\, dt\Big) \\ &= sg_1(b) + e^{\lambda b^2}\Big(e^{-\lambda s^2} + 2\lambda s \int_{s_0}^{s} e^{-\lambda t^2}\, dt\Big).\end{aligned} \tag{5.8}$$

On the other hand,

$$\int_0^r e^{-\lambda t^2}\, dt = \frac{1}{\sqrt{\lambda}} \int_0^{\sqrt{\lambda} r} e^{-\lambda x^2} dx = \frac{\sqrt{\pi}}{2\sqrt{\lambda}} \mathrm{erf}(\sqrt{\lambda}\, r).$$

Substituting this into (5.8) yields

$$\begin{aligned}\phi(b,s) &= sg_1(b) + e^{\lambda b^2}\{e^{-\lambda s^2} + \sqrt{\lambda\pi}[\mathrm{erf}(\sqrt{\lambda}\, s) - \mathrm{erf}(\sqrt{\lambda}\, s_0)]s\} \\ &= sg_2(b) + e^{\lambda b^2}[e^{-\lambda s^2} + \sqrt{\lambda\pi}\, s\, \mathrm{erf}(\sqrt{\lambda}\, s)].\end{aligned} \tag{5.9}$$

Together with Theorem 4.2 and Proposition 5.1 we obtain (see [14, 21])

THEOREM 5.2. *Define*

$$\alpha := \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2},$$

$$\beta := \frac{\langle a, y\rangle}{\sqrt{1 + \mu|x|^2}} + \frac{c - \mu\langle a, x\rangle}{(\sqrt{1 + \mu|x|})^3}\langle x, y\rangle$$

where c, μ are constants and $a \in \mathbb{R}^n$ is a constant vector. Let $\phi(b,s)$ be a function defined in (5.9)*. Then the general (α,β)-metric $F = \alpha\phi(\|\beta\|_\alpha, \beta/\alpha)$ on an open subset $\mathcal{U} \subseteq \mathbb{R}^n$ is projectively flat.*

Acknowledgments. The authors are grateful to the referee for the useful comments.

This work was supported by the National Natural Science Foundation of China 11071005 and Research Fund for the Doctoral Program of Higher Education of China 20110001110069.

References

[1] H. Alzer, *Error function inequalities*, Adv. Comput. Math. 33 (2010), 349–379.

[2] D. Bao and S. S. Chern, *On a notable connection in Finsler geometry*, Houston J. Math. 19 (1993), 135–180.
[3] D. Bao and C. Robles, *On Randers spaces of constant flag curvature*, Rep. Math. Phys. 51 (2003), 9–42.
[4] D. Bao and Z. Shen, *Finsler metrics of constant positive curvature on the Lie group* S^3, J. London Math. Soc. (2) 66 (2002), 453–467.
[5] R. Bryant, *Some remarks on Finsler manifolds with constant flag curvature*, Houston J. Math. 28 (2002), 221–262.
[6] X. Chen, X. Mo and Z. Shen, *On the flag curvature of Finsler metrics of scalar curvature*, J. London Math. Soc. (2) 68 (2003), 762–780.
[7] S. S. Chern and Z. Shen, *Riemann–Finsler geometry*, Nankai Tracts in Math. 6, World Sci., Hackensack, NJ, 2005.
[8] L. Huang and X. Mo, *A new class of projectively flat Finsler metrics in terms of hypergeometric functions*, Publ. Math. Debrecen 81 (2012), 421–434.
[9] A. Katok, *Ergodic perturbations of degenerate integrable Hamiltonian systems*, Izv. Akad. Nauk SSSR 37 (1973), 539–576 (in Russian).
[10] B. Li and Z. Shen, *On a class of projectively flat Finsler metrics with constant flag curvature*, Int. J. Math. 18 (2007), 749–760.
[11] B. Li and Z. Shen, *Projectively flat fourth root Finsler metrics*, Canad. Math. Bull. 55 (2012), 138–145.
[12] J. L. López and E. Pérez Sinusía, *The role of the error function in a singularly perturbed convection-diffusion problem in a rectangle with corner singularities*, Proc. Roy. Soc. Edinburgh Sect. A 137 (2007), 93–109.
[13] X. Mo, *On some projectively flat Finsler metrics in terms of hypergeometric functions*, Israel J. Math. 184 (2011), 59–78.
[14] X. Mo, *Finsler metrics with constant (or scalar) flag curvature*, Proc. Indian Acad. Sci. Math. Sci. 122 (2012), 411–427.
[15] X. Mo and C. Yu, *On some explicit constructions of Finsler metrics with scalar flag curvature*, Canad. J. Math. 62 (2010), 1325–1339.
[16] Z. Shen, *Two-dimensional Finsler metrics with constant flag curvature*, Manuscripta Math. 109 (2002), 349–366.
[17] Z. Shen, *Projectively flat Finsler metrics of constant flag curvature*, Trans. Amer. Math. Soc. 355 (2003), 1713–1728.
[18] Z. Shen, *Projectively flat Randers metrics with constant flag curvature*, Math. Ann. 325 (2003), 19–30.
[19] Z. Shen, *Finsler metrics with* $\mathbf{K} = 0$ *and* $\mathbf{S} = 0$, Canad. J. Math. 55 (2003), 112–132.
[20] Z. Shen and G. Ç. Yıldırım, *On a class of projectively flat metrics with constant flag curvature*, Canad. J. Math. 60 (2008), 443–456.
[21] C. Yu and H. Zhu, *On a new class of Finsler metrics*, Differential Geom. Appl. 29 (2011), 244–254.

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Received 13.11.2011
and in final form 27.3.2012 (2654)