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Regularity of solutions for a sixth order nonlinear parabolic equation in two space dimensions

by Changchun Liu (Changchun)

Abstract. We consider an initial-boundary problem for a sixth order nonlinear parabolic equation, which arises in oil-water-surfactant mixtures. Using Schauder type estimates and Campanato spaces, we prove the global existence of classical solutions for the problem in two space dimensions.

1. Introduction. In this paper, we investigate the sixth order nonlinear parabolic equation

$$\frac{\partial u}{\partial t}-\operatorname{div}\left[m(u)\left(k\nabla\Delta^2u+\nabla\left(-a(u)\Delta u-\frac{a'(u)}{2}|\nabla u|^2+h(u)\right)\right)\right]=0, \tag{1.1}$$

in a two-dimensional bounded domain $\Omega\subset\mathbb{R}^2$ with smooth boundary, where $k>0$, $a(u)=\gamma_1u^2+\gamma_2$, and $\gamma_1>0,\gamma_2>0$ are constants ([GG]). From physical considerations, we prefer to consider a typical case of the volumetric free energy $H(u)$, that is, $H'(u)=h(u)$, in the following form ([GG, PZ]):

$$H(u)=(u+1)^2(u^2+h_0)(u-1)^2. \tag{H1}$$

The equation (1.1) is supplemented by the boundary value conditions

$$u|_{\partial\Omega}=\Delta u|_{\partial\Omega}=\Delta^2u|_{\partial\Omega}=0,\qquad t>0, \tag{1.2}$$

and the initial value condition

$$u(x,0)=u_0(x). \tag{1.3}$$

The equation (1.1) is a sixth order parabolic equation which describes the dynamics of phase transitions in ternary oil-water-surfactant systems [GG, GK, GK2]. Here $u(x,t)$ is a scalar order parameter which is proportional to the local difference between the oil and water concentrations. The surfactant has the property that one part of it is hydrophilic and the other lipophilic is called the amphiphile. In the system, almost pure oil, almost pure water

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and microemulsion which consists of a homogeneous, isotropic mixture of oil and water can coexist in equilibrium.

During the past years, only a few works have been devoted to sixth-order parabolic equations in general [BF, EGK1, EGK2, FK, KEMW, LT]. Pawłow and Zajączkowski [PZ] proved that the initial boundary value problem for (1.1) with $m(u) = 1$ admits a unique global smooth solution which depends continuously on the initial datum. F. Bernis and A. Friedman [BF] have studied the initial boundary value problem for the thin film equation

$$\frac{\partial u}{\partial t} + (-1)^{m-1}\partial_x(f(u)\partial_x^{2m+1}u) = 0,$$

where $f(u) = |u|^n f_0(u)$, $f_0(u) > 0$, $n \geq 1$, and proved the existence of weak solutions preserving nonnegativity. J. W. Barrett, S. Langdon and R. Nuernberg [BLN] considered the above equation with $m = 2$. A finite element method was presented which was proved to be well posed and convergent. Numerical experiments illustrated the theory.

Recently, Evans, Galaktionov and King [EGK1, EGK2] considered the sixth-order thin film equation containing an unstable (backward parabolic) second-order term

$$\frac{\partial u}{\partial t} = \operatorname{div}[|u|^n \nabla \Delta^2 u] - \Delta(|u|^{p-1}u), \qquad n > 0,\ p > 1. \tag{1.4}$$

By a formal matched expansion technique, they showed that, for the first critical exponent $p = p_0 = n+1+4/N$ for $n \in (0, 5/4)$, where N is the space dimension, the free-boundary problem with zero-height, zero-contact-angle, zero-moment, and zero-flux conditions at the interface admits a countable set of continuous branches of radially symmetric self-similar blow-up solutions $u_k(x,t) = (T-t)^{-N/(nN+6)} f_k(y)$, $y = x/(T-t)^{1/(nN+6)}$, where $T > 0$ is the blow-up time. Some other results can be found in [JM, L, SP].

Our main purpose is to establish the global existence of classical solutions under much general assumptions. The main difficulties in treating the regularized problem are caused by the nonlinearity of the principal part and the lack of maximum principle. The key step is to get a priori estimates on the Hölder norm of Δu. The method used in [PZ] seems not applicable to the present situation. Our method is based on uniform Schauder type estimates for local in time solutions in the framework of Campanato spaces. For this purpose, we require some delicate local integral estimates rather than the global energy estimates used in the discussion of the Cahn–Hilliard equation with constant mobility.

Now, we state the main results of this paper.

Theorem 1.1. *Assume that*

$$m \in C^{1+\alpha}(\mathbb{R}), \qquad m(s) \geq M_1, \qquad |m'(s)|^2 \leq M_2 m(s), \tag{H2}$$

where M_1, M_2, α are positive constants, and $u_0|_{\partial\Omega} = \Delta u_0|_{\partial\Omega} = \Delta^2 u_0|_{\partial\Omega} = 0$. Then the problem (1.1)–(1.3) *admits a unique classical solution $u \in C^{6+\alpha,1+\alpha/6}(\overline{Q}_T)$ for any smooth initial data u_0, where $Q_T = \Omega \times (0,T)$.*

This paper is organized as follows. We first present in Section 2 a key step, yielding a priori estimates on the Hölder norm of solutions, and then give the proof of our main theorem in Section 3.

2. Hölder estimates. As an important step, in this section we give Hölder norm estimates on local in time solutions. From the classical approach, it is not difficult to conclude that the problem admits a unique classical solution local in time. So it is sufficient to find a priori estimates.

PROPOSITION 2.1. *Assume that* (H1), (H2) *hold, and u is a smooth solution of the problem* (1.1)–(1.3). *Then there exists a constant C, depending only on the known quantities, such that for any $(x_1,t_1),(x_2,t_2) \in Q_T$ and some $0 < \alpha < 1$,*

$$|u(x_1,t_1) - u(x_2,t_2)| \le C(|t_1-t_2|^{\alpha/6} + |x_1-x_2|^{\alpha}), \tag{2.1}$$

$$|\nabla u(x_1,t_1) - \nabla u(x_2,t_2)| \le C(|t_1-t_2|^{1/12} + |x_1-x_2|^{1/2}). \tag{2.2}$$

Proof. We set

$$F(t) = \int_\Omega \left[\frac{k}{2}(\Delta u)^2 + \frac{a(u)}{2}|\nabla u|^2 + H(u)\right] dx.$$

Integrating by parts and using the equation (1.1) itself and the boundary condition (1.2), we see that

$$\begin{aligned}\frac{dF(t)}{dt} &= \int_\Omega \left[k\Delta u \Delta u_t + a(u)\nabla u \nabla u_t + \frac{a'(u)}{2}|\nabla u|^2 u_t + h(u)u_t\right] dx \\ &= \int_\Omega \left[k\Delta^2 u - a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + h(u)\right]\frac{\partial u}{\partial t}\, dx \\ &= -\int_\Omega m(u)\left[k\nabla\Delta^2 u + \nabla(-a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + h(u))\right]^2 dx \le 0.\end{aligned}$$

On the other hand, we have

$$\int_\Omega |\nabla u(x,t)|^2\, dx \le \varepsilon \int_\Omega (\Delta u)^2\, dx + C(\varepsilon)\int_\Omega u^2\, dx.$$

By the Young inequality

$$u^2 \le \varepsilon u^6 + C_{1\varepsilon}, \qquad u^4 \le \varepsilon u^6 + C_{2\varepsilon}.$$

Combining the above inequalities and using $a(u) = \gamma_1 u^2 + \gamma_2$, $\gamma_1 > 0$, yields

$$\sup_{0<t<T} \int_\Omega u^2\, dx \le C, \tag{2.3}$$

(2.4) $$\sup_{0<t<T} \int_\Omega |\nabla u|^2 \, dx \le C,$$

(2.5) $$\sup_{0<t<T} \int_\Omega (\Delta u)^2 \, dx \le C.$$

By the Sobolev imbedding theorem,

(2.6) $$\sup_{Q_T} |u| \le C,$$

(2.7) $$\sup_{0<t<T} \int_\Omega |\nabla u|^q \, dx \le C, \qquad 2 \le q < \infty.$$

Multiplying both sides of the equation (1.1) by $\Delta^2 u$ and then integrating the resulting relation with respect to x over Ω, after integrating by parts, and using the boundary condition, we derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega (\Delta u)^2 \, dx &+ \int_\Omega k m(u) |\nabla \Delta^2 u|^2 \, dx \\ &= \int_\Omega m(u) a(u) \nabla \Delta u \nabla \Delta^2 u \, dx + 2 \int_\Omega m(u) a'(u) \nabla u \Delta u \nabla \Delta^2 u \, dx \\ &\quad + \frac{1}{2} \int_\Omega m(u) a''(u) |\nabla u|^3 \nabla \Delta^2 u \, dx - \int_\Omega m(u) h'(u) \nabla u \nabla \Delta^2 u \, dx.\end{aligned}$$

Using the Hölder inequality and (2.6), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega (\Delta u)^2 \, dx &+ \int_\Omega k m(u) |\nabla \Delta^2 u|^2 \, dx \\ &\le \frac{k}{2} \int_\Omega m(u) |\nabla \Delta^2 u|^2 \, dx + C \int_\Omega |\nabla \Delta u|^2 \, dx + C \int_\Omega |\nabla u|^4 \, dx \\ &\quad + C \int_\Omega |\Delta u|^4 \, dx + C \int_\Omega |\nabla u|^6 \, dx + C \int_\Omega |\nabla u|^2 \, dx.\end{aligned}$$

It follows by using the Gagliardo–Nirenberg inequalities (noticing that we consider only the two-dimensional case)

$$\Big(\int_\Omega |\nabla \Delta u|^2 \, dx \Big)^{1/2} \le C_1 \Big(\int_\Omega |\nabla \Delta^2 u|^2 \, dx \Big)^{1/6} \Big(\int_\Omega |\Delta u|^2 \, dx \Big)^{1/3},$$

$$\Big(\int_\Omega |\Delta u|^4 \, dx \Big)^{1/4} \le C_1 \Big(\int_\Omega |\nabla \Delta^2 u|^2 \, dx \Big)^{1/12} \Big(\int_\Omega |\Delta u|^2 \, dx \Big)^{5/12}.$$

By (2.5) and (2.7), we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\Delta u)^2 \, dx + \int_\Omega k m(u) |\nabla \Delta^2 u|^2 \, dx \le \frac{k}{2} \int_\Omega m(u) |\nabla \Delta^2 u|^2 \, dx + C.$$

Hence

$$\iint_{Q_T} m(u)|\nabla \Delta^2 u|^2 \, dx \, dt \le C. \tag{2.8}$$

(2.3) and (2.4) imply that

$$|u(x_1,t) - u(x_2,t)| \le C|x_1 - x_2|^\alpha, \quad 0 < \alpha < 1. \tag{2.9}$$

Integrating the equation (1.1) over $\Omega_y \times (t_1,t_2)$, where $0 < t_1 < t_2 < T$, $\Delta t = t_2 - t_1$, $\Omega_y = (y_1, y_1 + (\Delta t)^{1/12}) \times (y_2, y_2 + (\Delta t)^{1/12})$, we see that

$$\begin{aligned}
(2.10) \quad & \int_{\Omega_y} [u(z,t_2) - u(z,t_1)] \, dz \\
&= \int_{t_1}^{t_2} \int_{y_2}^{y_2+(\Delta t)^{1/12}} [F_1(y_1 + (\Delta t)^{1/12}, y, s) - F_1(y_1, y, s)] \, dy \, ds \\
&\quad + \int_{t_1}^{t_2} \int_{y_1}^{y_1+(\Delta t)^{1/12}} [F_2(y, y_2 + (\Delta t)^{1/12}, s) - F_2(y, y_2, s)] \, dy \, ds \\
&= \int_{t_1}^{t_2} (\Delta t)^{1/12} [F_1(y_1 + (\Delta t)^{1/12}, y_2 + \theta_1^* (\Delta t)^{1/12}, s) \\
&\quad - F_1(y_1, y_2 + \theta_1^*(\Delta t)^{1/12}, s) + F_2(y_1 + \theta_2^*(\Delta t)^{1/12}, y_2 + (\Delta t)^{1/12}, s) \\
&\quad - F_2(y_1 + \theta_2^*(\Delta t)^{1/12}, y_2, s)] \, ds,
\end{aligned}$$

where

$$m(u(x,s))\Big(k\nabla\Delta^2 u + \nabla\Big(-a(u)\Delta u - \frac{a'(u)}{2}|\nabla u|^2 + h(u)\Big)\Big)(x,s) = (F_1, F_2).$$

Set

$$\begin{aligned}
& N(s, y_1, y_2) \\
&= (\Delta t)^{1/12}[F_1(y_1 + (\Delta t)^{1/12}, y_2 + \theta_1^*(\Delta t)^{1/12}, s) - F_1(y_1, y_2 + \theta_1^*(\Delta t)^{1/12}, s) \\
&\quad + F_2(y_1 + \theta_2^*(\Delta t)^{1/12}, y_2 + (\Delta t)^{1/12}, s) - F_2(y_1 + \theta_2^*(\Delta t)^{1/12}, y_2, s)].
\end{aligned}$$

Then (2.10) is converted into

$$\begin{aligned}
(\Delta t)^{1/6} \int_{I=(0,1)\times(0,1)} & [u(y + \theta(\Delta t)^{1/12}, t_2) - u(y + \theta(\Delta t)^{1/12}, t_1)] \, d\theta \\
& = \int_{t_1}^{t_2} N(s, y_1, y_2) \, ds.
\end{aligned}$$

Integrating the above equality over Ω_x, we get

$$(\Delta t)^{1/3}(u(x^*,t_2)-u(x^*,t_1)) = \int_{t_1}^{t_2}\int_{\Omega_x} N(s,y)\,dy\,ds.$$

Here, we have used the mean value theorem, where $x^* = y^* + \theta^*(\Delta t)^{1/12}$. Hence by the Hölder inequality and (2.5), (2.6), (2.8), we get

$$|u(x^*,t_2)-u(x^*,t_1)| \le C(\Delta t)^{\alpha/6}, \qquad 0<\alpha<1.$$

Again multiplying both sides of (1.1) by $\Delta^3 u$ and integrating the resulting relation with respect to x over Ω, integrating by parts, and using the boundary condition, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\Delta u|^2\,dx + \int_\Omega km(u)(\Delta^3 u)^2\,dx + \int_\Omega km'(u)\nabla u\cdot\nabla\Delta^2 u\Delta^3 u\,dx\\
&\quad - \int_\Omega [2(m(u)a'(u))' + 3m(u)a''(u)]|\nabla u|^2\Delta u\Delta^3 u\,dx\\
&\quad - \int_\Omega m(u)a(u)\Delta^2 u\Delta^3 u\,dx - \int_\Omega [2m(u)a'(u) + (m(u)a(u))']\nabla u\nabla\Delta u\Delta^3 u\,dx\\
&\quad - 2\int_\Omega m(u)a'(u)(\Delta u)^2\Delta^3 u\,dx - \int_\Omega (m(u)a''(u))'|\nabla u|^4\Delta^3 u\,dx\\
&\quad + \int_\Omega (m(u)h'(u))'|\nabla u|^2\Delta^3 u\,dx + \int_\Omega m(u)h'(u)\Delta u\Delta^3 u\,dx = 0.
\end{aligned}$$

The Hölder inequality and the assumption (H2) yield

$$\begin{aligned}
&\Big|\int_\Omega m'(u)\nabla u\nabla\Delta^2 u\Delta^3 u\,dx\Big|\\
&\qquad \le \frac{1}{16}\int_\Omega m(u)(\Delta^3 u)^2\,dx + C\int_\Omega \frac{|m'(u)|^2}{m(u)}|\nabla u|^2|\nabla\Delta^2 u|^2\,dx\\
&\qquad \le \frac{1}{16}\int_\Omega m(u)(\Delta^2 u)^2\,dx + CM_2\int_\Omega |\nabla u|^2|\nabla\Delta^2 u|^2\,dx\\
&\qquad \le \frac{1}{16}\int_\Omega m(u)(\Delta^3 u)^2\,dx + CM_2\Big(\int_\Omega |\nabla u|^8\,dx\Big)^{1/4}\Big(\int_\Omega |\nabla\Delta^2 u|^{8/3}\,dx\Big)^{3/4}.
\end{aligned}$$

It follows by using the Gagliardo–Nirenberg inequality (noticing that we consider only the two-dimensional case)

$$\Big(\int_\Omega |\nabla\Delta^2 u|^{8/3}\,dx\Big)^{3/8} \le C_1\Big(\int_\Omega |\Delta^3 u|^2\,dx\Big)^{13/32}\Big(\int_\Omega |\Delta u|^2\,dx\Big)^{3/32},$$

and by (2.7),

$$\Big|\int_\Omega m'(u)\nabla u\nabla \Delta^2 u\Delta^3 u\,dx\Big| \le \frac{1}{8}\int_\Omega m(u)(\Delta^3 u)^2\,dx + C.$$

Again using (2.3) and the Hölder inequality, we have

$$\begin{aligned}\Big|\int_\Omega [2(m(u)a'(u))' + 3m(u)a''(u)]|\nabla u|^2\Delta u\Delta^3 u\,dx\Big| \\ \le C\int_\Omega |\nabla u|^4|\Delta u|^2\,dx + \frac{1}{16}\int_\Omega m(u)(\Delta^3 u)^2\,dx \\ \le C\int_\Omega |\nabla u|^8\,dx + C\int_\Omega |\Delta u|^4\,dx + \frac{1}{16}\int_\Omega m(u)(\Delta^3 u)^2\,dx.\end{aligned}$$

Using the Gagliardo–Nirenberg inequality, we have

$$\int_\Omega |\Delta u|^4\,dx \le C_1\Big(\int_\Omega |\Delta^3 u|^2\,dx\Big)^{1/4}\Big(\int_\Omega |\Delta u|^2\,dx\Big)^{7/4}.$$

By (2.5), we obtain

$$\begin{aligned}\Big|\int_\Omega [2(m(u)a'(u))' + 3m(u)a''(u)]|\nabla u|^2\Delta u\Delta^3 u\,dx\Big| \\ \le \frac{1}{8}\int_\Omega m(u)(\Delta^3 u)^2\,dx + C.\end{aligned}$$

Similarly, we get

$$\Big|-2\int_\Omega m(u)a'(u)(\Delta u)^2\Delta^3 u\,dx\Big| \le \frac{1}{16}\int_\Omega m(u)(\Delta^3 u)^2\,dx + C,$$

$$\begin{aligned}\Big|-\int_\Omega [2m(u)a'(u) + (m(u)a(u))']\nabla u\nabla \Delta u\Delta^3 u\,dx\Big| \\ \le \frac{1}{16}\int_\Omega m(u)(\Delta^3 u)^2\,dx + C,\end{aligned}$$

$$\Big|-\int_\Omega m(u)a(u)\Delta^2 u\Delta^3 u\,dx\Big| \le \frac{1}{16}\int_\Omega m(u)(\Delta^3 u)^2\,dx + C,$$

$$\Big|-\int_\Omega (m(u)a''(u))'|\nabla u|^4\Delta^3 u\,dx\Big| \le \frac{1}{16}\int_\Omega m(u)(\Delta^3 u)^2\,dx + C,$$

$$\Big|\int_\Omega (m(u)h'(u))'|\nabla u|^2\Delta^3 u\,dx\Big| \le \frac{1}{16}\int_\Omega m(u)(\Delta^3 u)^2\,dx + C,$$

$$\Big|\int_\Omega m(u)h'(u)\Delta u\Delta^3 u\,dx\Big| \le \frac{1}{16}\int_\Omega m(u)(\Delta^3 u)^2\,dx + C.$$

Summing up, we have

$$\frac{d}{dt}\int\limits_{\Omega}|\nabla\Delta u|^2\,dx + C_1\int\limits_{\Omega}(\Delta^3 u)^2\,dx \le C_2.$$

By Gronwall's inequality, we obtain

$$\int\limits_{\Omega}|\nabla\Delta u|^2\,dx \le C, \qquad 0<t<T, \tag{2.11}$$

$$\iint\limits_{Q_T}(\Delta^3 u)^2\,dx\,dt \le C. \tag{2.12}$$

Similar to the discussion above, we have

$$|\nabla u(x_1,t_1)-\nabla u(x_2,t_2)| \le C(|x_1-x_2|^{1/2}+|t_1-t_2|^{1/12}). \tag{2.13}$$ ■

3. Proof of the main result. This section is devoted to the proof of Theorem 1.1. The key step is the Hölder estimate for Δu. We divide the argument into the following propositions.

Proposition 3.1. *If u, ∇u are Hölder continuous in the interior of Q_T, then u is classical in the interior of Q_T.*

We consider the following linear problem:

$$\frac{\partial u}{\partial t} - \nabla\Delta(a(x,t)\nabla\Delta u) + \nabla\Delta(b(x,t)\nabla u) = \nabla\Delta\vec{F}, \tag{3.1}$$

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = \Delta^2 u|_{\partial\Omega} = 0, \tag{3.2}$$

$$u(x,0) = 0. \tag{3.3}$$

Here we do not specify the smoothness of the given functions $a(x,t)$, $b(x,t)$ and $\vec{F}$, but simply assume that they are sufficiently smooth. Our main purpose is to find a relation between the Hölder norm of the solution u and $a(x,t)$, $b(x,t)$, $\vec{F}$.

The crucial step is to establish estimates on the Hölder norm of u. Fix $(x_0,t_0)\in\Omega\times(0,T)$ and define

$$\varphi(\rho) = \iint\limits_{S_\rho}(|u-u_\rho|^2+\rho^6|\nabla\Delta u|^2)\,dx\,dt \qquad (\rho>0),$$

where

$$S_\rho = B_\rho(x_0)\times(t_0-\rho^6,t_0+\rho^6), \qquad u_\rho = \frac{1}{|S_\rho|}\iint\limits_{S_\rho}u\,dx\,dt$$

and $B_\rho(x_0)$ is the ball centred at x_0 of radius ρ.

Let u be the solution of the problem (3.1)–(3.3). We split u on S_R into $u=u_1+u_2$, where u_1 is the solution of the problem

$$\frac{\partial u_1}{\partial t} - a(x_0,t_0)\Delta^3 u_1 + b(x_0,t_0)\Delta^2 u_1 = 0, \qquad (x,t)\in S_R, \tag{3.4}$$

(3.5) $$u_1 = u, \quad \frac{\partial u_1}{\partial n} = \frac{\partial u}{\partial n}, \quad \Delta u_1 = \Delta u, \quad x \in \partial B_R(x_0),$$

(3.6) $$u_1 = u, \quad t = t_0 - R^6,\ x \in B_R(x_0),$$

and u_2 solves the problem

(3.7) $$\begin{aligned}\frac{\partial u_2}{\partial t} - a(x_0,t_0)\Delta^3 u_2 + b(x_0,t_0)\Delta^2 u_2 \\ = -\nabla\Delta[(a(x_0,t_0) - a(x,t))\nabla\Delta u] \\ + \nabla\Delta[(b(x_0,t_0) - b(x,t))\nabla u] + \nabla\Delta\vec{F}, \quad (x,t) \in S_R,\end{aligned}$$

(3.8) $$u_2 = 0, \quad \frac{\partial u_2}{\partial n} = 0, \quad \Delta u_2 = 0, \quad (x,t) \in \partial B_R(x_0),$$

(3.9) $$u_2 = 0, \quad t = t_0 - R^6,\ x \in B_R(x_0).$$

By classical linear theory, the above decomposition is uniquely determined by u.

We need several lemmas on u_1 and u_2.

LEMMA 3.1. *Assume that*

$$|a(x,t) - a(x_0,t_0)| \le a_\sigma(|t-t_0|^{\sigma/6} + |x-x_0|^\sigma), \quad (x,t) \in B_R(x_0) \times J_R(t_0),$$

$$|b(x,t) - b(x_0,t_0)| \le b_\sigma(|t-t_0|^{\sigma/6} + |x-x_0|^\sigma), \quad (x,t) \in B_R(x_0) \times J_R(t_0),$$

where $J_R(t_0) = (t_0 - R^6, t_0 + R^6)$. *Then*

$$\begin{aligned}\sup_{(t_0-R^6,t_0+R^6)} \int_{B_R(x_0)} u_2^2(x,t)\,dx + \iint_{S_R} |\nabla\Delta u_2|^2\,dx\,dt \\ \le CR^{2\sigma} \iint_{S_R} |\nabla\Delta u|^2\,dx\,dt + CR^{2\sigma}\iint_{S_R} |\nabla u|^2\,dx\,dt + C\sup_{S_R}|\vec{F}|^2 R^5.\end{aligned}$$

Proof. Multiply the equation (3.7) by u_2 and integrate the resulting relation over $(t_0 - R^6, t) \times B_R(x_0)$. Integrating by parts, we have

$$\begin{aligned}\frac{1}{2}\int_{B_R} u_2^2\,dx + a(x_0,t_0)\int_{t_0-R^6}^{t} ds \int_{B_R} |\nabla\Delta u_2|^2\,dx + b(x_0,t_0)\int_{t_0-R^6}^{t} ds \int_{B_R} (\Delta u_2)^2\,dx \\ = \int_{t_0-R^6}^{t} ds \int_{B_R} [a(x_0,t_0) - a(x,t)]\nabla\Delta u \cdot \nabla\Delta u_2\,dx \\ + \int_{t_0-R^6}^{t} ds \int_{B_R} [b(x_0,t_0) - b(x,t)]\nabla u \cdot \nabla\Delta u_2\,dx \\ + \int_{t_0-R^6}^{t} ds \int_{B_R} \vec{F}\nabla\Delta u_2\,dx.\end{aligned}$$

Noticing that

$$\Big|\int_{t_0-R^6}^{t} ds \int_{B_R} [a(x_0,t_0)-a(x,t)]\nabla\Delta u\nabla\Delta u_2\,dx\Big| \le \varepsilon \iint_{S_R} |\nabla\Delta u_2|^2\,ds\,dx + C_\varepsilon a_\sigma^2 R^{2\sigma}\iint_{S_R} |\nabla\Delta u|^2\,dx\,ds,$$

and

$$\Big|\int_{t_0-R^6}^{t} ds \int_{B_R} \vec{F}\nabla\Delta u_2\,dx\Big| \le \varepsilon\iint_{S_R}|\nabla\Delta u_2|^2\,dx\,ds + C_\varepsilon R^5 \sup|\vec{F}|^2,$$

we hence obtain the estimate and the proof is complete. ∎

LEMMA 3.2. *For any* $(x_1,t_1),(x_2,t_2)\in S_\rho$,

$$\frac{|u_1(t_1,x_1)-u_1(t_2,x_2)|^2}{|t_1-t_2|^{1/6}+|x_1-x_2|} \le C\sup_{(t_0-\rho^6,t_0+\rho^6)}\int_{B_\rho(x_0)} (|\nabla u_1(x,t)|^2+\rho^4(\Delta u_1)^2)\,dx + C\iint_{S_\rho}(\Delta^2 u_1)^2\,dx\,dt.$$

Proof. By the Sobolev imbedding theorem, for any $(x_1,t),(x_2,t)\in S_\rho$ we have

$$\frac{|u_1(x_1,t)-u_1(x_2,t)|^2}{|x_1-x_2|} \le C\sup_{(t_0-\rho^6,t_0+\rho^6)}\int_{B_\rho(x_0)}(|\nabla u_1(x,t)|^2+\rho^4(\Delta u_1)^2)\,dx. \tag{3.10}$$

Integrating the equation over $\Omega_y\times(t_1,t_2)$, where $0<t_1<t_2<T$, $\Delta t = t_2-t_1$, $\Omega_y=(y_1,y_1+(\Delta t)^{1/12})\times(y_2,y_2+(\Delta t)^{1/12})$, we see that

$$\begin{aligned}\int_{\Omega_y}&[u_1(z,t_2)-u_1(z,t_1)]\,dz\\ &= \int_{t_1}^{t_2}\int_{y_2}^{y_2+(\Delta t)^{1/12}} [G_1(y_1+(\Delta t)^{1/12},y,s)-G_1(y_1,y,s)]\,dy\,ds\\ &\quad+\int_{t_1}^{t_2}\int_{y_1}^{y_1+(\Delta t)^{1/12}} [G_2(y,y_2+(\Delta t)^{1/12},s)-G_2(y,y_2,s)]\,dy\,ds\\ &=\int_{t_1}^{t_2}(\Delta t)^{1/12}[G_1(y_1+(\Delta t)^{1/12},y_2+\theta_1^*(\Delta t)^{1/12},s)\\ &\quad-G_1(y_1,y_2+\theta_1^*(\Delta t)^{1/12},s)+G_2(y_1+\theta_2^*(\Delta t)^{1/12},y_2+(\Delta t)^{1/12},s)\\ &\quad-G_2(y_1+\theta_2^*(\Delta t)^{1/12},y_2,s)]\,ds,\end{aligned}$$

where

$$a(x_0,t_0)\nabla\Delta^2 u_1(x,s) - b(x_0,t_0)\nabla\Delta u_1(x,s) = (G_1,G_2).$$

Similar to the proof of Proposition 2.1, integrating the above equality over Ω_x, we get

$$|u_1(x^*,t_2) - u_1(x^*,t_1)| \le C|t_1-t_2|^{1/6}\Big[\iint_{S_\rho}(\Delta^2 u_1)^2\,dx\,dt + \iint_{S_\rho}(\Delta u_1)^2\,dx\,dt\Big],$$

where $x^* = y^* + \theta^*(\Delta t)^{1/12}$. This and (3.10) yield the desired conclusion. ■

Lemma 3.3 (Caccioppoli type inequality).

$$\sup_{(t_0-(R/2)^6,t_0+(R/2)^6)}\int_{B_{R/2}(x_0)} |u_1(x,t)-(u_1)_R|^2\,dx + \iint_{S_{R/2}} |\nabla\Delta u_1|^2\,dx\,dt \le \frac{C}{R^6}\iint_{S_R}|u_1(x,t)-(u_1)_R|^2\,dx\,dt,$$

$$\sup_{(t_0-(R/2)^6,t_0+(R/2)^6)}\int_{B_{R/2}(x_0)} |\nabla u_1|^2\,dx + \iint_{S_{R/2}} |\Delta^2 u_1|^2\,dx\,dt \le \frac{C}{R^6}\iint_{S_R}|\nabla u_1|^2\,dx\,dt \le \frac{C}{R^8}\iint_{S_{2R}}|u_1(x,t)-(u_1)_R|^2\,dx\,dt,$$

$$\sup_{(t_0-(R/2)^6,t_0+(R/2)^6)}\int_{B_{R/2}(x_0)} |\Delta u_1|^2\,dx + \iint_{S_{R/2}} |\nabla\Delta^2 u_1|^2\,dx\,dt \le \frac{C}{R^6}\iint_{S_R}|\Delta u_1|^2\,dx\,dt,$$

where

$$(u_1)_R = \frac{1}{|S_R|}\iint_{S_R} u_1\,dx\,dt.$$

Proof. For simplicity, we only prove the first inequality, since the other can be shown similarly. Choose a cut-off function $\chi(x)$ defined on $B_R(x_0)$ such that $\chi(x)=1$ in $B_{R/2}(x_0)$ and

$$|\nabla\chi| \le C/R, \qquad |D^2\chi| \le C/R^2,$$
$$|D^3\chi| \le C/R^3, \qquad |D^4\chi| \le C/R^4.$$

Let $g \in C_0^\infty(t_0,\infty)$ with $0 \le g(t) \le 1$, $0 \le g'(t) \le C/R^6$ and $g(t)=1$ for $t \ge t_0-(R/2)^6$. Multiplying (3.4) by $g(t)\chi^6[u_1(x,t)-(u_1)_R]$ and then integrating the resulting relation over $(t_0-R^6,t)\times B_R(x_0)$, we have

$$\int_{t_0-R^6}^{t} g(s)\,ds \int_{B_R(x_0)} \frac{\partial u_1}{\partial t}\chi^6[u_1(x,t)-(u_1)_R]\,dx$$
$$- a(x_0,t_0)\int_{t_0-R^6}^{t} g(s)\,ds \int_{B_R(x_0)} \Delta^3 u_1\chi^6[u_1(x,t)-(u_1)_R]\,dx$$
$$+ b(x_0,t_0)\int_{t_0-R^6}^{t} g(s)\,ds \int_{B_R(x_0)} \Delta^2 u_1\chi^6[u_1(x,t)-(u_1)_R]\,dx = 0.$$

It follows by integrating by parts that

$$\frac{1}{2}\int_{B_R(x_0)} g(s)\chi^6|u_1(x,t)-(u_1)_R|^2\,dx$$
$$+ a(x_0,t_0)\int_{t_0-R^6}^{t} g(s)\,ds \int_{B_R(x_0)} \nabla\Delta^2 u_1\nabla[\chi^6[u_1(x,t)-(u_1)_R]]\,dx$$
$$- b(x_0,t_0)\int_{t_0-R^6}^{t} g(s)\,ds \int_{B_R(x_0)} \nabla\Delta u_1\nabla[\chi^6[u_1(x,t)-(u_1)_R]]\,dx$$
$$= \frac{1}{2}\int_{t_0-R^6}^{t} g'\,ds \int_{B_R(x_0)} \chi^6|u_1(x,t)-(u_1)_R|^2\,dx.$$

Thus

$$\frac{1}{2}\int_{B_R(x_0)} g(s)\chi^6|u_1(x,t)-(u_1)_R|^2\,dx$$
$$+ a(x_0,t_0)\int_{t_0-R^6}^{t} g(s)\,ds \int_{B_R(x_0)} \chi^6|\nabla\Delta u_1|^2\,dx$$
$$+ b(x_0,t_0)\int_{t_0-R^6}^{t} g(s)\,ds \int_{B_R(x_0)} \chi^6(\Delta u_1)^2\,dx$$
$$+ a(x_0,t_0)\int_{t_0-R^6}^{t} g(s)\,ds \int_{B_R(x_0)} [18\chi^5\nabla\chi\Delta u_1\nabla\Delta u_1$$
$$+ (18\chi^5\Delta\chi + 90\chi^4|\nabla\chi|^2)\nabla u_1\nabla\Delta u_1$$
$$+ (6\chi^5\nabla\Delta\chi + 90\chi^4\nabla\chi\Delta\chi + 120\chi^3|\nabla\chi|^2\nabla\chi)(u_1(x,t)-(u_1)_R)\nabla\Delta u_1]\,dx$$
$$+ b(x_0,t_0)\int_{t_0-R^6}^{t} g(s)\,ds \int_{B_R(x_0)} [12\chi^5\nabla\chi\nabla u_1\Delta u_1$$
$$+ (30\chi^4|\nabla\chi|^2 + 6\chi^5\Delta\chi)(u_1(x,t)-(u_1)_R)\Delta u_1]\,dx$$
$$= \frac{1}{2}\int_{t_0-R^6}^{t} g'\,ds \int_{B_R(x_0)} \chi^6|u_1-(u_1)_R|^2\,dx.$$

By the Cauchy inequality, we have

$$\Big|18\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)a(x_0,t_0)\chi^5\nabla\chi\Delta u_1\nabla\Delta u_1\,dx\,ds\Big|$$
$$\le \frac{1}{4}a(x_0,t_0)\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)\chi^6|\nabla\Delta u_1|^2\,dx\,ds + C\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)\chi^4|\nabla\chi|^2(\Delta u_1)^2\,dx\,ds,$$

$$\Big|\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)a(x_0,t_0)(18\chi^5\Delta\chi+90\chi^4|\nabla\chi|^2)\nabla u_1\nabla\Delta u_1\,dx\,ds\Big|$$
$$\le \frac{1}{4}a(x_0,t_0)\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)\chi^6|\nabla\Delta u_1|^2\,dx\,ds + C\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)\chi^4|\Delta\chi|^2|\nabla u_1|^2\,dx\,ds + C\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)\chi^2|\nabla\chi|^4|\nabla u_1|^2\,dx\,ds$$

and

$$\Big|\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)a(x_0,t_0)(6\chi^5\nabla\Delta\chi+90\chi^4\nabla\chi\Delta\chi+120\chi^3|\nabla\chi|^2\nabla\chi)\cdot(u_1(x,t)-(u_1)_R)\nabla\Delta u_1\,dx\,ds\Big|$$
$$\le \frac{1}{4}a(x_0,t_0)\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)\chi^6|\nabla\Delta u_1|^2\,dx\,ds + \frac{C}{R^6}\int_{t_0-R^6}^{t}\int_{B_R(x_0)} (u_1(x,t)-(u_1)_R)^2\,dx\,ds.$$

Similarly, we obtain

$$\Big|12\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)b(x_0,t_0)\chi^5\nabla\chi\nabla u_1\Delta u_1\,dx\,ds\Big|$$
$$\le \frac{1}{4}b(x_0,t_0)\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)\chi^6(\Delta u_1)^2\,dx\,ds + C\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)\chi^4|\nabla\chi|^2|\nabla u_1|^2\,dx\,ds,$$

and

$$\Big|\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)b(x_0,t_0)(30\chi^4|\nabla\chi|^2+6\chi^5\Delta\chi)(u_1(x,t)-(u_1)_R)\Delta u_1\,dx\,ds\Big|$$

$$\le \frac{1}{4}b(x_0,t_0)\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)\chi^6(\Delta u_1)^2\,dx\,ds + \frac{C}{R^6}\int_{t_0-R^6}^{t}\int_{B_R(x_0)} (u_1(x,t)-(u_1)_R)^2\,dx\,ds.$$

Notice that

$$\begin{aligned}
&\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)\chi^4|\nabla\chi|^2|\nabla u_1|^2\,dx\,ds \\
&\quad = -\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)(u_1(x,t)-(u_1)_R)\nabla(\chi^4|\nabla\chi|^2\nabla u_1)\,dx\,ds \\
&\quad = -\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)(u_1(x,t)-(u_1)_R)\chi^4|\nabla\chi|^2\Delta u_1\,dx\,ds \\
&\qquad + \frac{1}{2}\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)(u_1(x,t)-(u_1)_R)^2\Delta(\chi^4|\nabla\chi|^2)\,dx\,ds \\
&\quad \le \frac{1}{4}b(x_0,t_0)\int_{t_0-R^6}^{t}\int_{B_R(x_0)} g(s)\chi^6(\Delta u_1)^2\,dx\,ds \\
&\qquad + \frac{C}{R^6}\int_{t_0-R^6}^{t}\int_{B_R(x_0)} (u_1(x,t)-(u_1)_R)^2\,dx\,ds.
\end{aligned}$$

Combining the above expressions yields

$$\begin{aligned}
&\int_{B_R(x_0)} g(s)\chi^6|u_1(x,t)-(u_1)_R|^2\,dx+\frac{1}{2}a(x_0,t_0)\int_{t_0-R^6}^{t} g(s)\,ds\int_{B_R(x_0)} \chi^6|\nabla\Delta u_1|^2\,dx \\
&\quad + b(x_0,t_0)\int_{t_0-R^6}^{t} g(s)\,ds\int_{B_R(x_0)} \chi^6(\Delta u_1)^2\,dx \\
&\le \int_{t_0-R^6}^{t} g'\,ds\int_{B_R(x_0)} \chi^6|u_1-(u_1)_R|^2\,dx + C\int_{t_0-R^6}^{t}\int_{B_R(x_0)} \chi^2|\nabla\chi|^4|\nabla u_1|^2\,dx\,ds
\end{aligned}$$

$$+ C \int_{t_0-R^6}^{t} \int_{B_R(x_0)} \chi^4 |\Delta \chi|^2 |\nabla u_1|^2 \, dx \, ds$$
$$+ C \int_{t_0-R^6}^{t} \int_{B_R(x_0)} \chi^4 |\nabla \chi|^2 (\Delta u_1)^2 \, dx \, ds$$
$$+ \frac{C}{R^6} \int_{t_0-R^6}^{t} \int_{B_R(x_0)} (u_1(x,t) - (u_1)_R)^2 \, dx \, ds$$
$$\equiv \int_{t_0-R^6}^{t} g' \, ds \int_{B_R(x_0)} \chi^6 |u_1 - (u_1)_R|^2 \, dx + C(I_1 + I_2 + I_3 + I_4).$$

As for I_1, we get

$$I_1 = -\int_{t_0-R^6}^{t} \int_{B_R(x_0)} u_1 \nabla(\chi^2 |\nabla \chi|^4 \nabla u_1) \, dx \, dt \tag{3.11}$$
$$= -\int_{t_0-R^6}^{t} \int_{B_R(x_0)} \chi^2 |\nabla \chi|^4 u_1 \Delta u_1 \, dx \, dt$$
$$- \int_{t_0-R^6}^{t} \int_{B_R(x_0)} \nabla(\chi^2 |\nabla \chi|^4) u_1 \nabla u_1 \, dx \, dt$$
$$\le \varepsilon_1 I_3 + C \int_{t_0-R^6}^{t} \int_{B_R(x_0)} |\nabla \chi|^6 u_1^2 \, dx \, dt$$
$$+ \frac{1}{2} \int_{t_0-R^6}^{t} \int_{B_R(x_0)} D^2(\chi^2 |\nabla \chi|^4) u_1^2 \, dx \, dt$$
$$\le \varepsilon_1 I_3 + C I_4.$$

As for I_2, we have

$$I_2 = -\int_{t_0-R^6}^{t} \int_{B_R(x_0)} u_1 \nabla(\chi^4 |\Delta \chi|^2 \nabla u_1) \, dx \, dt$$
$$= \int_{t_0-R^6}^{t} \int_{B_R(x_0)} \nabla \chi \nabla(\chi^4 \Delta \chi u_1 \Delta u_1) \, dx \, dt$$
$$+ \frac{1}{2} \int_{t_0-R^6}^{t} \int_{B_R(x_0)} \Delta(\chi^4 |\Delta \chi|^2) u_1^2 \, dx \, dt$$

$$\leq \int_{t_0-R^6}^{t} \int_{B_R(x_0)} \nabla\chi\nabla(\chi^4 \Delta\chi) u_1 \Delta u_1 \, dx\, dt$$

$$+ \int_{t_0-R^6}^{t} \int_{B_R(x_0)} \chi^4 \nabla\chi \Delta\chi (\nabla u_1 \Delta u_1 + u_1 \nabla\Delta u_1) \, dx\, dt + CI_4$$

$$= \varepsilon_2 I_3 + CI_4 + \varepsilon_3 \int_{t_0-R^6}^{t} \int_{B_R(x_0)} \chi^6 |\nabla\Delta u_1|^2 \, dx\, dt$$

$$- \frac{1}{2} \int_{t_0-R^6}^{t} \int_{B_R(x_0)} \nabla(\chi^4 \nabla\chi \Delta\chi) |\nabla u_1|^2 \, dx\, dt$$

$$= \varepsilon_2 I_3 + CI_4 + \varepsilon_3 \int_{t_0-R^6}^{t} \int_{B_R(x_0)} \chi^6 |\nabla\Delta u_1|^2 \, dx\, dt - \frac{1}{2} I_2$$

$$- \frac{1}{2} \int_{t_0-R^6}^{t} \int_{B_R(x_0)} (\chi^4 \nabla\chi \nabla\Delta\chi + 4\chi^3 |\nabla\chi|^2 \Delta\chi) |\nabla u_1|^2 \, dx\, dt,$$

that is,

$$I_2 \leq \varepsilon_2 I_3 + CI_4 + \varepsilon_3 \int_{t_0-R^6}^{t} \int_{B_R(x_0)} \chi^6 |\nabla\Delta u_1|^2 \, dx\, dt$$

$$- \frac{1}{3} \int_{t_0-R^6}^{t} \int_{B_R(x_0)} (\chi^4 \nabla\chi \nabla\Delta\chi + 4\chi^3 |\nabla\chi|^2 \Delta\chi) |\nabla u_1|^2 \, dx\, dt.$$

On the other hand,

$$- \int_{t_0-R^6}^{t} \int_{B_R(x_0)} (\chi^4 \nabla\chi \nabla\Delta\chi + 4\chi^3 |\nabla\chi|^2 \Delta\chi) |\nabla u_1|^2 \, dx\, dt$$

$$= \int_{t_0-R^6}^{t} \int_{B_R(x_0)} (\chi^4 \nabla\chi \nabla\Delta\chi + 4\chi^3 |\nabla\chi|^2 \Delta\chi) u_1 \Delta u_1 \, dx\, dt$$

$$+ \int_{t_0-R^6}^{t} \int_{B_R(x_0)} (\nabla(\chi^4 \nabla\chi \nabla\Delta\chi) + 4\nabla(\chi^3 |\nabla\chi|^2 \Delta\chi)) u_1 \nabla u_1 \, dx\, dt$$

$$\leq \varepsilon I_3 + CI_4.$$

Combining the above two yields

$$I_2 \leq \varepsilon_4 I_3 + CI_4 + \varepsilon_3 \int_{t_0-R^6}^{t} \int_{B_R(x_0)} \chi^6 |\nabla\Delta u_1|^2 \, dx\, dt. \tag{3.12}$$

Notice that

$$I_3 = -\int_{t_0-R^6}^{t}\int_{B_R(x_0)} \chi^4|\nabla\chi|^2\nabla u_1\nabla\Delta u_1\,dx\,dt$$
$$-\int_{t_0-R^6}^{t}\int_{B_R(x_0)} (4\chi^3|\nabla\chi|^2\nabla\chi + 2\chi^4\nabla\chi\Delta\chi)\nabla u_1\Delta u_1\,dx\,dt$$
$$\le \varepsilon_5\int_{t_0-R^6}^{t}\int_{B_R(x_0)} \chi^6|\nabla\Delta u_1|^2\,dx\,dt + C(\varepsilon_5)I_1 + \frac{1}{4}I_3 + CI_1 + \frac{1}{4}I_3 + CI_2,$$

that is,

$$I_3 \le 2C(\varepsilon_5)I_1 + CI_2 + 2\varepsilon_5\int_{t_0-R^6}^{t}\int_{B_R(x_0)} \chi^6|\nabla\Delta u_1|^2\,dx\,dt. \tag{3.13}$$

Finally, from (3.11)–(3.13), choosing $\varepsilon_1, \varepsilon_3, \varepsilon_4$ small enough, we see that

$$I_i \le \varepsilon\int_{t_0-R^6}^{t}\int_{B_R(x_0)} \chi^6|\nabla\Delta u_1|^2\,dx\,dt + CI_4, \qquad i = 1,2,3.$$

Hence we immediately obtain the desired first inequality of the lemma. ■

LEMMA 3.4. *Assume that*

$$|a(x,t) - a(x_0,t_0)| \le a_\sigma(|t-t_0|^{\sigma/6} + |x-x_0|^\sigma),$$
$$t \in (t_0 - R^6, t_0 + R^6), \qquad x \in B_R(x_0).$$

Then for any $\rho \in (0,R)$,

$$\frac{1}{\rho^8}\iint_{S_\rho}(|u_1 - (u_1)_\rho|^2 + \rho^6|\nabla\Delta u_1|^2)\,dx\,dt$$
$$\le \frac{C}{R^8}\iint_{S_R}(|u_1 - (u_1)_R|^2 + R^6|\nabla\Delta u_1|^2)\,dx\,dt.$$

Proof. One only needs to check the inequality for $\rho \le R/2$. From Lemmas 3.2 and 3.3, we have

$$\frac{1}{\rho^8}\iint_{S_\rho}|u_1 - (u_1)_\rho|^2\,dx\,dt \le C\sup_{(t_0-(R/2)^6,t_0+(R/2)^6)}\int_{B_{R/2}(x_0)}(|\nabla u_1(x,t)|^2$$
$$+ R^4(\Delta u_1)^2)\,dx + C\iint_{S_{R/2}}|\Delta^2 u_1|^2\,dx\,dt$$
$$\le \frac{C}{R^8}\iint_{S_R}(|u_1 - (u_1)_R|^2 + R^6|\nabla\Delta u_1|^2)\,dx\,dt.$$

On the other hand,

$$\begin{aligned}
&\iint\limits_{S_\rho}\rho^6|\nabla\Delta u_1|^2\,dx\,dt\\
&\quad\le C_1\iint\limits_{S_\rho}\rho^8(\Delta^2u_1)^2\,dx\,dt+C_2\iint\limits_{S_\rho}\rho^2|\nabla u_1|^2\,dx\,dt\\
&\quad\le C_1\rho^8\iint\limits_{S_{R/2}}(\Delta^2u_1)^2\,dx\,dt+C_2\rho^8\sup_{(t_0-(R/2)^6,t_0+(R/2)^6)}\int\limits_{B_{R/2}(x_0)}|\nabla u_1|^2\,dx\\
&\quad\le C\Big(\frac{\rho}{R}\Big)^8\iint\limits_{S_{R/2}}R^2|\nabla u_1|^2\,dx\,dt\\
&\quad\le C\Big(\frac{\rho}{R}\Big)^8\Big[\iint\limits_{S_R}R^6|\nabla\Delta u_1|^2\,dx\,dt+\iint\limits_{S_R}(u_1-(u_1)_R)^2\,dx\,dt\Big].
\end{aligned}$$

The conclusion of the lemma follows at once. ∎

LEMMA 3.5. *For $\lambda\in(5,6)$,*

$$\varphi(\rho)\le C_\lambda\Big(\varphi(R_0)+\sup_{S_{R_0}}|\vec F|^2\Big)\rho^\lambda,\qquad \rho\le R_0=\min(\operatorname{dist}(x_0,\partial\Omega),t_0^{1/6}),$$

where C_λ depends on λ, R_0 and the known quantities.

Proof. By Lemma 3.4,

$$\begin{aligned}
\varphi(\rho)&=\iint\limits_{S_\rho}(|u-(u)_\rho|^2+\rho^6|\nabla\Delta u|^2)\,dx\,dt\\
&=\iint\limits_{S_\rho}(|u_1-(u_1)_\rho|^2+\rho^6|\nabla\Delta u_1|^2)\,dx\,dt\\
&\quad+\iint\limits_{S_\rho}(|u_2-(u_2)_\rho|^2+\rho^6|\nabla\Delta u_2|^2)\,dx\,dt\\
&\le C\Big(\frac{\rho}{R}\Big)^8\iint\limits_{S_R}(|u-(u)_R|^2+R^6|\nabla\Delta u|^2)\,dx\,dt\\
&\quad+C\iint\limits_{S_R}(|u_2|^2+R^6|\nabla\Delta u_2|^2)\,dx\,dt\\
&\le C[(\rho/R)^8+R^{2\sigma}]\varphi(R)+C\sup_{S_{R_0}}|\vec F|^2R^{13}.
\end{aligned}$$

The conclusion follows immediately from [GS]. ∎

Similar to the discussion involving Campanato spaces in [GS], we first deduce from Lemma 3.5 the following:

THEOREM 3.6. *Let $\vec{F}$ be an appropriately smooth function and u be a smooth solution of the problem* (3.1)–(3.3)*. Then for any $\alpha \in (0,1/2)$, there exists a coefficient K, depending only on α, a_σ, b_σ, $\iint_{Q_T} u^2\,dx\,dt$ and $\iint_{Q_T} |\nabla\Delta u|^2\,dx\,dt$, such that*

$$|u(x_1,t_1)-u(x_2,t_2)| \le K(1+\sup|\vec{F}|)(|x_1-x_2|^{\alpha}+|t_1-t_2|^{\alpha/6}). \tag{3.14}$$

Proof of Proposition 3.1. Let $w = \Delta u - \Delta u_0$. Then w satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nabla\Delta(a(x,t)\nabla\Delta u) + \nabla\Delta(b(x,t)\nabla u) = \nabla\Delta\vec{F}, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = \Delta^2 u|_{\partial\Omega} = 0, \\ u(x,0) = 0, \end{cases}$$

where $a(x,t) = km(u)$, $b(x,t) = m(u)a(u)$ and $\vec{F} = m(u)\big(k\nabla\Delta^2 u_0 - a(u)\nabla\Delta u_0 - a'(u)\nabla u w - a'(u)\nabla u\Delta u_0 - a'(u)\nabla u w - a'(u)\nabla u\Delta u_0 + h'(u)\nabla u - \frac{a''(u)}{2}|\nabla u|^2\nabla u\big)$. Hence, using (2.5)–(2.8) and Theorem 3.6, we conclude that

$$|\Delta u(x_1,t_1) - \Delta u(x_2,t_2)| \le C(|x_1-x_2|^{\alpha/2} + |t_1-t_2|^{\alpha/12}). \tag{3.15}$$

The conclusion follows immediately from the classical theory, since we can transform the equation (1.1) into the form

$$\frac{\partial u}{\partial t} + a_1(x,t)\Delta^3 u + \vec{b}_1(x,t)\nabla\Delta^2 u + a_2(x,t)\Delta^2 u + \vec{b}_2(x,t)\nabla\Delta u \\ + a_3(x,t)\Delta u + \vec{b}_3(x,t)\nabla u = 0,$$

where the Hölder norms of

$$\begin{aligned} &a_1(x,t) = -km(u(x,t)), && \vec{b}_1(x,t) = -km'(u(x,t))\nabla u(x,t),\\ &a_2(x,t) = m(u(x,t))a(u(x,t)), && \vec{b}_2(x,t) = [m'(u)a(u) + 3m(u)a'(u)]\nabla u,\\ &a_3(x,t) = 2m(u)a'(u)\Delta u + \big(2m'(u)a'(u) + \tfrac{7}{2}m(u)a''(u)\big)|\nabla u|^2 - h'(u), \\ &\vec{b}_3(x,t) = \big(\tfrac{1}{2}m'(u)a''(u) + \tfrac{1}{2}m''(u)a'''(u)\big)|\nabla u|^2\nabla u - h''(u)\nabla u \end{aligned}$$

have been estimated in the above discussion. ∎

PROPOSITION 3.2. *If u, ∇u are Hölder continuous in $\overline{Q}_T$, then u is classical in $\overline{Q}_T$.*

Proof. Fix $(x_0,t_0) \in \partial\Omega\times(0,T)$ and assume that in some neighbourhood of x_0, $\partial\Omega$ is explicitly expressed by a function $y = \varphi(x)$. We split u as u_1+u_2 in $\Omega_R(x_0)\times(t_0-R^6, t_0+R^6)$ with $\Omega_R(x_0) = B_R(x_0)\cap\Omega$, where

$$\begin{aligned} &\frac{\partial u_1}{\partial t} - a(x_0,t_0)\Delta^3 u_1 + b(x_0,t_0)\Delta^2 u_1 = 0, \quad && (x,t)\in S_R,\\ &u_1 = u, \quad \Delta u_1 = \Delta u, \quad \Delta^2 u_1 = \Delta^2 u, && x\in\partial B_R(x_0),\\ &u_1 = u, \quad t = t_0 - R^6, && x\in B_R(x_0), \end{aligned}$$

and u_2 solves the problem

$$\frac{\partial u_2}{\partial t} - a(x_0,t_0)\Delta^3 u_2 + b(x_0,t_0)\Delta^2 u_2 = -\nabla\Delta[(a(x_0,t_0)-a(x,t))\nabla\Delta u] + \nabla\Delta[(b(x_0,t_0)-b(x,t))\nabla u] + \nabla\Delta\vec{F}, \quad (x,t)\in S_R,$$

$$u_2 = 0,\ \Delta u_2 = 0,\ \Delta^2 u_2 = 0, \quad (x,t)\in\partial B_R(x_0)\times(t_0-R^6,t_0+R^6),$$
$$u_2 = 0, \quad t = t_0 - R^6,\ x\in B_R(x_0).$$

Define the normal and tangential derivatives as

$$\partial_n = \varphi'(x)\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, \qquad \partial_\tau = \frac{\partial}{\partial x_1} + \varphi'(x)\frac{\partial}{\partial x_2}.$$

Now, we modify the function $\varphi(\rho)$ as

$$\varphi(\rho) = \iint_{S_\rho} (|\partial_n u|^2 + |\partial_\tau u - (\partial_\tau u)_\rho|^2 + \rho^6|\nabla\Delta u|^2)\,dx\,dt.$$

Similar to the proof of Proposition 3.1, we conclude that

$$\frac{|u_1(x_1,t_1)-u_1(x_2,t_2)|^2}{|t_1-t_2|^{1/6}+|x_1-x_2|} \le C \sup_{(t_0-\rho^6,t_0+\rho^6)} \int_{B_\rho(x_0)} (|\partial_n u_1|^2 + |\partial_\tau u_1 - (\partial_\tau u_1)_\rho|^2 + \rho^4(\Delta u_1)^2)\,dx + C\iint_{S_\rho} (\Delta^2 u_1)^2\,dx\,dt,$$

and

$$\sup_{(t_0-(R/2)^6,t_0+(R/2)^6)} \int_{\Omega_{R/2}(x_0)} (|\partial_n u_1|^2 + |\partial_\tau u_1 - (\partial_\tau u_1)_{1/2}|^2)\,dx + \iint_{S_{R/2}} |\Delta^2 u_1|^2\,dx\,dt \le \frac{C}{R^6}\iint_{S_R} (|\partial_n u_1|^2 + |\partial_\tau u_1 - (\partial_\tau u_1)_{1/2}|^2)\,dx\,dt.$$

The remaining part of the proof is similar to that of Proposition 3.1, and we omit the details. ∎

Proof of Theorem 1.1. Combining Proposition 3.1 with Proposition 3.2 completes the proof. ∎

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Changchun Liu
Department of Mathematics
Jilin University
Changchun 130012, China
E-mail: liucc@jlu.edu.cn

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