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Some results on curvature and topology of Finsler manifolds

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Abstract. We investigate the curvature and topology of Finsler manifolds, mainly the growth of the fundamental group. By choosing a new counting function for the fundamental group that does not rely on the generators, we are able to discuss the topic in a more general case, namely, we do not demand that the manifold is compact or the fundamental group is finitely generated. Among other things, we prove that the fundamental group of a forward complete and noncompact Finsler n-manifold (M, F) with nonnegative Ricci curvature and finite uniformity constant has polynomial growth of order $\leq n-1$, and the first Betti number satisfies $b_1(M) \leq n-1$. We also obtain some sufficient conditions to ensure that the fundamental group is finite or is trivial. Most of the results are new even for Riemannian manifolds.

1. Introduction. Finsler geometry is just Riemannian geometry without the quadratic restriction. Instead of a Euclidean norm on each tangent space, one endows every tangent space of a differentiable manifold with a Minkowski norm. In recent years, global Finsler geometry have been developed tremendously, including geodesic theory [BCS], the sphere theorem [R], volume comparison theorems [S1, S2, W1, WX] and the global theory of submanifolds [S3, W2], etc.

In global Finsler geometry it is important to reveal the relationship between the topology and geometry invariants for Finsler manifolds. In this paper we shall investigate the curvature and topology of Finsler manifolds, mainly the growth of the fundamental group. The growth of the fundamental group for Riemannian manifolds was first discussed by Milnor [M]. By using volume comparison theorems he was able to prove that any finitely generated subgroup of the fundamental group of an n-dimensional complete Riemannian manifold with nonnegative Ricci curvature has polynomial growth of order $\leq n$, while the fundamental group of a compact Riemannian manifold of negative sectional curvature has exponential growth. Milnor's results

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have been generalized to Finsler manifolds by Shen in the case of nonnegative Ricci curvature [S1, S2], and by Xin and the author [WX] and Shen and Zhao [SZ] in the case of negative flag curvature. However, the fundamental group is assumed to be finitely generated and an additional condition on S-curvature is needed in these results. The additional condition on S-curvature has recently been removed by the author by using the maximal or minimal volume form [W1].

We shall consider this topic further and obtain some results on the growth of the fundamental group of Finsler manifolds. By choosing a new counting function that does not rely on the generators, we are able to discuss the topic in a more general case, namely, we do not demand that the manifold is compact or the fundamental group is finitely generated. For example, we prove that the fundamental group of a forward complete and noncompact Finsler n-manifold with nonnegative Ricci curvature and finite uniformity constant has polynomial growth of order $\leq n-1$, and the first Betti number satisfies $b_1(M) \leq n-1$. We also obtain some sufficient conditions for the fundamental group to be finite or trivial. Most of the results are new even for Riemannian manifolds.

2. Finsler geometry. In this section, we give a brief description of basic quantities and fundamental formulas in Finsler geometry; for more details one is referred to [BCS, CS]. Let (M, F) be a Finsler n-manifold with Finsler metric $F : TM \to [0, \infty)$. Let $(x, y) = (x^i, y^i)$ be the local coordinates on TM. The *fundamental tensor* g_{ij} is defined by

$$g_{ij}(x,y) := \frac{1}{2}\,\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}.$$

Let $\Gamma^i_{jk}(x,y)$ be the Chern connection coefficients. Then the *first Chern curvature tensor* $R_j{}^i{}_{kl}$ can be expressed as

$$R_j{}^i{}_{kl} = \frac{\delta \Gamma^i_{jl}}{\delta x^k} - \frac{\delta \Gamma^i_{jk}}{\delta x^l} + \Gamma^i_{ks}\Gamma^s_{jl} - \Gamma^s_{jk}\Gamma^i_{ls},$$

where $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - y^k \Gamma^j_{ik} \frac{\partial}{\partial y^j}$. Let $R_{ijkl} := g_{js} R_i{}^s{}_{kl}$, and write $\mathbf{g}_y = g_{ij}(x,y)dx^i \otimes dx^j$, $\mathbf{R}_y = R_{ijkl}(x,y)dx^i \otimes dx^j \otimes dx^k \otimes dx^l$. For a tangent plane $P \subset T_xM$, let

$$\mathbf{K}(P,y) = \mathbf{K}(y;u) := \frac{\mathbf{R}_y(y,u,u,y)}{\mathbf{g}_y(y,y)\mathbf{g}_y(u,u) - [\mathbf{g}_y(y,u)]^2},$$

where $y, u \in P$ are tangent vectors such that $P = \operatorname{span}\{y,u\}$. We call $\mathbf{K}(P,y)$ the *flag curvature of P with flag pole y*. Let

$$\mathbf{Ric}(y) = \sum_i \mathbf{K}(y; e_i),$$

where $e_1, \dots, e_n$ is a $\mathbf{g}_y$-orthogonal basis for the corresponding tangent space. We call $\mathbf{Ric}(y)$ the *Ricci curvature of* y.

A *volume form* $d\mu$ on the Finsler manifold (M,F) is nothing but a global nondegenerate n-form on M. For a more technical definition of volume form one is referred to [W2, W3]. In local coordinates we can express $d\mu$ as $d\mu = \sigma(x)dx^1\wedge\cdots\wedge dx^n$. The frequently used volume forms in Finsler geometry are the Busemann–Hausdorff volume form and the Holmes–Thompson volume form. Other useful volume forms are the maximal and minimal volume forms which can be defined as follows. Let

$$dV_{\max} = \sigma_{\max}(x)dx^1 \wedge \cdots \wedge dx^n, \qquad dV_{\min} = \sigma_{\min}(x)dx^1 \wedge \cdots \wedge dx^n$$

with

$$\sigma_{\max}(x) := \max_{y\in T_xM\setminus\{0\}} \sqrt{\det(g_{ij}(x,y))},$$
$$\sigma_{\min}(x) := \min_{y\in T_xM\setminus\{0\}} \sqrt{\det(g_{ij}(x,y))}.$$

Then it is easy to check that the n-forms $dV_{\max}$ and $dV_{\min}$ as well as the function $\nu := \sigma_{\max}/\sigma_{\min}$ are well-defined on M. We call $dV_{\max}$ and $dV_{\min}$ the *maximal volume form* and the *minimal volume form* of (M,F), respectively. We note that both the maximal and the minimal volume forms play a crucial role in comparison techniques in Finsler geometry [W1].

Let $\lambda, \mu : M \to \mathbb{R}$ be defined by

$$\lambda(x) = \max_{y\in T_xM\setminus 0} \frac{F(y)}{F(-y)}, \qquad \mu(x) = \max_{y,z,u\in T_xM\setminus 0} \frac{\mathbf{g}_y(u,u)}{\mathbf{g}_z(u,u)}.$$

They are called respectively the *reversibility at* x and the *uniformity constant at* x [E, R]. It is clear that

$$\lambda(x)^2 \le \mu(x), \qquad \forall x \in M.$$

We have

PROPOSITION 2.1 ([W1]). *Let (M,F) be an n-dimensional Finsler manifold. Then*

(i) *F is Riemannian $\Leftrightarrow \nu = 1 \Leftrightarrow \mu = 1$;*
(ii) *$\nu \le \mu^n$.*

Let (M,F) be a Finsler manifold. For $p \in M$, let $I_p = \{v \in T_pM : F(v) = 1\}$ be the *indicatrix* at p. For $v \in I_p$, the *cut-value* $c(v)$ is defined by

$$c(v) := \sup\{t > 0 : d_F(p, \exp_p(tv)) = t\}.$$

Then, we can define the *tangential cut locus* $\mathbf{C}(p)$ of p by $\mathbf{C}(p) := \{c(v)v : c(v) < \infty, v \in I_p\}$, the *cut locus* $C(p)$ of p by $C(p) = \exp_p \mathbf{C}(p)$, and the *injectivity radius* i_p at p by $i_p = \inf\{c(v) : v \in I_p\}$. It is known that $C(p)$ has zero Hausdorff measure in M. Also, we set $\mathbf{D}_p = \{tv : 0 \le t < c(v), v \in I_p\}$ and $D_p = \exp_p \mathbf{D}_p$. It is known that $\mathbf{D}_p$ is the largest domain which is starlike with respect to the origin of T_pM and such that $\exp_p$ restricted to that domain is a diffeomorphism, and $D_p = M \setminus C(p)$.

Let $B_p(R)$ be the forward geodesic ball of (M, F) centered at p with radius R. By definition, $B_p(R) = r^{-1}([0, R))$, where $r = d_F(p, \cdot) : M \to \mathbb{R}$ is the distance function from p induced by F. Let $\widetilde{g} = \mathbf{g}_{\partial r}$ be the Riemannian metric on $\dot{B}_p(R) = B_p(R) \cap D_p \setminus \{p\}$, where ∂r is the geodesic field with respect to p. We have the following volume comparison results.

THEOREM 2.2 ([W1]). *Let (M, F) be a forward complete Finsler n-manifold.*

(i) *If the flag curvature of M satisfies $\mathbf{K}(V; W) \le c$, then*

$$\mathrm{vol}_{\max}(B_p(R)) \ge \frac{1}{\mu(p)^{n/2}} \mathrm{vol}(\mathbb{B}_c^n(R))$$

for any $R \le i_p$.

(ii) *If the flag curvature of M is nonpositive and the Ricci curvature of M satisfies $\mathbf{Ric}_M \le c < 0$, and if M is simply connected, then*

$$\mathrm{vol}_{\max}(B_p(R)) \ge \frac{\mathrm{vol}_{\widetilde{g}}(B_p(1))}{\mathrm{vol}(\mathbb{B}_c^2(1))} \mathrm{vol}(\mathbb{B}_c^2(R)), \qquad \forall R \ge 1.$$

(iii) *If the Ricci curvature of M satisfies $\mathbf{Ric}_M \ge (n-1)c$, then*

$$\mathrm{vol}_{\min}(B_p(R)) \le \mu(p)^{n/2} \mathrm{vol}(\mathbb{B}_c^n(R)).$$

Here $\mathrm{vol}_{\max}$ and $\mathrm{vol}_{\min}$ are the volume with respect to $dV_{\max}$ and $dV_{\min}$, respectively, and $\mathbb{B}_c^n(R)$ is the geodesic ball of radius R in the (Riemannian) space form of constant curvature c.

THEOREM 2.3 ([W1]). *Let (M, F) be a forward complete and noncompact Finsler manifold with nonnegative Ricci curvature and finite reversibility, then the volume $\mathrm{vol}_{\max}(B_p(R))$ of the forward geodesic ball has at least linear growth:*

$$\mathrm{vol}_{\max}(B_p(R)) \ge c(p)R.$$

3. Universal covering space and fundamental group. Let (M, F) be a Finsler n-manifold, and $f : \widetilde{M} \to M$ be the universal covering space. A homeomorphism $\varphi : \widetilde{M} \to \widetilde{M}$ is called a *deck transformation* of the covering mapping f if $f \circ \varphi = f$. The set $\varGamma$ of deck transformations obviously forms a group under composition. One checks that $\varGamma$ acts properly discontinuously on $\widetilde{M}$. If we endow $\widetilde{M}$ with the pulled-back metric $\widetilde{F} = f^*F$, then $f : (\widetilde{M}, \widetilde{F}) \to (M, F)$ is a local isometry, and it is easy to check that each $\gamma \in \varGamma$ is an isometry, and (M, F) and $(\widetilde{M}, \widetilde{F})$ have the same reversibility and uniformity constants. It is also clear that if (M, F) is (forward) complete, then so is $(\widetilde{M}, \widetilde{F})$ (see [SZ]).

Given $p \in M$, let $\pi_1(M, p)$ be the fundamental group of M based at p, that is, the homotopy classes of loops $\gamma : [0,1] \to M \in C^0$ satisfying $\gamma(0) = \gamma(1) = p$. It is well-known that the deck transformation group Γ is isomorphic to $\pi_1(M, p)$, and the corresponding map is given as follows [C]: fix a point $\widetilde{p} \in f^{-1}(p)$; then given $\gamma \in \Gamma$, all paths joining $\widetilde{p}$ to $\gamma(\widetilde{p})$ are homotopic (since $\widetilde{M}$ is simply connected), and therefore project to a well-defined element of $\pi_1(M, p)$. The map is clearly a homomorphism, and by the homotopy lifting lemma, one can check that it is in fact one-to-one and onto, and thus is an isomorphism. Also, Γ acts transitively on $f^{-1}(p)$ for each $p \in M$. In the following, we shall identify $\pi_1(M, p)$ with the deck transformation group Γ.

Definition 3.1 ([SZ]). Let (M, F) be a forward complete Finsler manifold and $f : (\widetilde{M}, \widetilde{F}) \to (M, F)$ denote its universal covering mapping. Given any point $p \in M$, for each $\gamma \in \Gamma \cong \pi_1(M, p)$, the *geometric norm* $\|\gamma\|$ associated with p is defined by

$$\|\gamma\| = d_{\widetilde{F}}(\widetilde{p}, \gamma(\widetilde{p})),$$

where $\widetilde{p}$ is any point in the fiber $f^{-1}(p)$, and $d_{\widetilde{F}}$ is the distance function on $\widetilde{M}$ induced by $\widetilde{F}$.

It is easy to prove the following

Proposition 3.2 ([SZ]). *The geometric norm $\|\gamma\|$ defined above equals the length of a shortest loop representing $\gamma \in \pi_1(M, p)$, which is a geodesic loop.*

Remark 3.3. It is easy to check that the geometric norm satisfies the triangle inequality, since Γ acts on $(\widetilde{M}, \widetilde{F})$ by isometries, and the above proposition implies that the geometric norm is independent of the choice of point in a fiber. On the other hand, the shortest geodesic loop representing $\gamma \in \pi_1(M, p)$ is just $f \circ \widetilde{\gamma}$, where $\widetilde{\gamma}$ is the shortest geodesic in $\widetilde{M}$ from $\widetilde{p}$ to $\gamma(\widetilde{p})$ (see [SZ]). Note that the shortest geodesic loop representing an element in $\pi_1(M, p)$ may not be smooth at p.

Lemma 3.4. *The set $\Delta(\lambda) = \{\gamma \in \Gamma : \|\gamma\| \le \lambda\}$ is finite for any $\lambda > 0$.*

Proof. Suppose on the contrary that the set $\Delta(\lambda_0) = \{\gamma \in \Gamma : \|\gamma\| \le \lambda_0\}$ is infinite for some $\lambda_0 > 0$. Let $\{\gamma_i\}_{i=1}^{\infty} \subset \Delta(\lambda_0)$ be an infinite sequence, and $\widetilde{q}_i = \gamma_i(\widetilde{p})$. Then $\|\gamma_i\| = d_{\widetilde{F}}(\widetilde{p}, \widetilde{q}_i) \le \lambda_0$, which means that $\{\widetilde{q}_i\}_{i=1}^{\infty} \subset \overline{\widetilde{B}_{\widetilde{p}}(\lambda_0)}$, where $\widetilde{B}_{\widetilde{p}}(\lambda_0)$ is the forward geodesic ball of $(\widetilde{M}, \widetilde{F})$ centered at $\widetilde{p}$ with radius λ_0. Hence $\{\widetilde{q}_i\}_{i=1}^{\infty}$ has a convergent subsequence, and we can assume that $\{\widetilde{q}_i\}_{i=1}^{\infty}$ itself is convergent without loss of generality. As a result, $d_{\widetilde{F}}(\widetilde{p}, \gamma_i^{-1}\gamma_j(\widetilde{p})) = d_{\widetilde{F}}(\widetilde{q}_i, \widetilde{q}_j)$ can be as small as we wish when $i \ne j$ are sufficiently large, or equivalently, the length of the shortest geodesic loop of $\pi_1(M,p)$ corresponding to $\gamma_i^{-1}\gamma_j$ can be as small as we wish when $i \neq j$ are sufficiently large, which implies that $\gamma_i^{-1}\gamma_j = 1$ when $i \neq j$ are sufficiently large, a contradiction. Thus the lemma is proved. ∎

Definition 3.5. The *counting function* $N(\lambda)$ of the fundamental group $\Gamma \cong \pi_1(M,p)$ of (M,F) is defined by

$$N(\lambda) = \sharp\Delta(\lambda) = \sharp\{\gamma \in \Gamma : \|\gamma\| \le \lambda\}.$$

Γ is said to have *exponential growth* if

$$\limsup_{\lambda\to\infty} \frac{\log N(\lambda)}{\lambda} > 0.$$

Γ is said to have *polynomial growth of order* $\le n$ if $N(\lambda) \le \mathrm{const}\cdot\lambda^n$.

Remark 3.6. Let $\Gamma' \subset \pi_1(M,p)$ be any finitely generated subgroup with a set of generators $S = \{\gamma_1,\dots,\gamma_k\}$. The counting function $n(\lambda)$ of Γ' considered in [A, M, SZ] is defined by

$$n(\lambda) = \sharp\{\gamma \in \Gamma' : |\gamma| \le \lambda\},$$

here $|\gamma|$ is the minimum length of γ as a word in $\{\gamma_1,\dots,\gamma_k\}$. The advantage of our definition is that it does not demand that Γ is finitely generated. It is clear by the triangle inequality that

$$\|\gamma\| = d_{\widetilde F}(\widetilde p, \gamma(\widetilde p)) \le |\gamma| \max_{j=1,\dots,k} d_{\widetilde F}(\widetilde p, \gamma_j(\widetilde p)) \le A|\gamma|, \qquad A := \max_{j=1,\dots,k}\|\gamma_j\|,$$

which implies that

$$n(\lambda) \le N(A\lambda). \tag{3.1}$$

On the other hand, if M is compact, then $\Gamma \cong \pi_1(M,p)$ is finitely generated, and one can choose a set of generators by $S = \{\gamma \in \Gamma : \|\gamma\| \le 2D\}$ with $D = \operatorname{diam} M$. For any $\gamma \in \Gamma$ with $\|\gamma\| > 2D$, let $\widetilde\gamma : [0,1] \to \widetilde M$ be the minimal geodesic from $\widetilde p$ to $\gamma(\widetilde p)$. Then $\widetilde\gamma$ must pass through some points of $f^{-1}(p)$, otherwise $f\circ\widetilde\gamma$ would be the shortest geodesic loop representing $\gamma \in \pi_1(M,p)$, and thus

$$d_F\big(p, f\big(\widetilde\gamma\big(\tfrac12\big)\big)\big) = \tfrac12\|\gamma\| > D,$$

a contradiction. Therefore, γ can be decomposed as $\gamma = \gamma_1\cdots\gamma_j$ with $\|\gamma\| = \|\gamma_1\| + \cdots + \|\gamma_j\|$, where $\gamma_i \in S$, $1 \le i \le j$. Consequently,

$$\|\gamma\| \ge j \min_{\gamma_0\in S}\|\gamma_0\| \ge |\gamma| \min_{\gamma_0\in S}\|\gamma_0\| =: \frac{1}{B}\cdot|\gamma|.$$

Therefore,

$$N(\lambda) \le n(B\lambda). \tag{3.2}$$

Now let us recall the definition of fundamental domain. See [C] for the case of Riemannian manifolds and [SZ] for Finsler manifolds.

DEFINITION 3.7. Let $f : \widetilde{M} \to M$ be the universal covering mapping, with deck transformation group Γ. We say $\Omega \subset \widetilde{M}$ is a *fundamental domain* of the covering mapping if

(i) $\gamma(\Omega) \cap \Omega = \emptyset$ for all $\gamma \in \Gamma \setminus \{1\}$;
(ii) $f(\overline{\Omega}) = M$.

Note that (i) is equivalent to $\gamma_1(\Omega) \cap \gamma_2(\Omega) = \emptyset$ when $\gamma_1 \neq \gamma_2$. Since Γ acts transitively on the fibers $f^{-1}(p)$ for each $p \in M$, $f(\overline{\Omega}) = M$ is equivalent to $\bigcup_{\gamma\in\Gamma} \gamma(\overline{\Omega}) = \widetilde{M}$ and $f|_{\gamma(\Omega)} : \gamma(\Omega) \to f(\Omega)$ is a homeomorphism, for each $\gamma \in \Gamma$. For a given forward complete Finsler manifold, one can construct a fundamental domain by using a cut locus as follows. Let $\mathbf{D}_p$ be defined as in §2. Then for each $p \in M$, one has

$$p \mapsto \mathbf{D}_p \subset T_pM \mapsto f_*|_{\widetilde{p}}^{-1}(\mathbf{D}_p) \subset T_{\widetilde{p}}\widetilde{M} \mapsto \widetilde{\exp}_{\widetilde{p}}(f_*|_{\widetilde{p}}^{-1}(\mathbf{D}_p)) =: \widetilde{\Omega}_{\widetilde{p}},$$

which is a fundamental domain in $\widetilde{M}$ (see [SZ]).

4. A key lemma. In this section we shall prove the following key lemma which plays a crucial role in this paper.

LEMMA 4.1. *Let (M, F) be a forward complete Finsler manifold and $f : (\widetilde{M}, \widetilde{F}) \to (M, F)$ denote its universal covering mapping with deck transformation group Γ. Fix $p \in M$ and $\widetilde{p} \in f^{-1}(p)$. Then*

(i) *the counting function of Γ satisfies*

$$N(\lambda) \le \frac{\mathrm{vol}_{\min}(\widetilde{B}_{\widetilde{p}}(\lambda + R))}{\mathrm{vol}_{\min}(B_p(R))}, \quad \forall \lambda, R > 0; \tag{4.1}$$

(ii) *if (M, F) has finite reversibility, that is, $\lambda_F := \max_{x\in M} \lambda(x) < \infty$, then*

$$N(\lambda) \ge \frac{\mathrm{vol}_{\max}\big(\widetilde{B}_{\widetilde{p}}\big(\frac{\lambda}{1+\lambda_F}\big)\big)}{\mathrm{vol}_{\max}\big(B_p\big(\frac{\lambda}{1+\lambda_F}\big)\big)}. \tag{4.2}$$

Here $\widetilde{B}_{\widetilde{p}}(R)$ is the forward geodesic ball in $(\widetilde{M}, \widetilde{F})$ centered at $\widetilde{p}$ with radius R.

Proof. (i) First we see from the properties of a fundamental domain that $f|_{\widetilde{\Omega}_{\widetilde{p}}} : \widetilde{\Omega}_{\widetilde{p}} \to D_p$ is an isometric homeomorphism. Clearly, $f(\widetilde{B}_{\widetilde{p}}(R) \cap \widetilde{\Omega}_{\widetilde{p}}) = B_p(R) \cap D_p$, where $D_p = M \setminus C(p)$, as defined in §2. Since the cut locus $C(p)$ has zero Hausdorff measure, we have

$$\mathrm{vol}_{\min}(\gamma(\widetilde{B}_{\widetilde{p}}(R) \cap \widetilde{\Omega}_{\widetilde{p}})) = \mathrm{vol}_{\min}(\widetilde{B}_{\widetilde{p}}(R) \cap \widetilde{\Omega}_{\widetilde{p}}) = \mathrm{vol}_{\min}(B_p(R)) \tag{4.3}$$

for all $\gamma \in \Gamma$ and $R > 0$. For any $\gamma \in \Delta(\lambda)$ and $\widetilde{x} \in \widetilde{B}_{\widetilde{p}}(R) \cap \widetilde{\Omega}_{\widetilde{p}}$, it is clear that

$$d_{\widetilde{F}}(\widetilde{p}, \gamma(\widetilde{x})) \leq d_{\widetilde{F}}(\widetilde{p}, \gamma(\widetilde{p})) + d_{\widetilde{F}}(\gamma(\widetilde{p}), \gamma(\widetilde{x})) = \|\gamma\| + d_{\widetilde{F}}(\widetilde{p}, \widetilde{x}) < \lambda + R,$$

from which it follows that

$$\bigcup_{\gamma \in \Delta(\lambda)} \gamma(\widetilde{B}_{\widetilde{p}}(R) \cap \widetilde{\Omega}_{\widetilde{p}}) \subset \widetilde{B}_{\widetilde{p}}(\lambda + R). \tag{4.4}$$

Now (4.1) follows from (4.3) and (4.4) together with the fact that $\gamma_1(\widetilde{B}_{\widetilde{p}}(R) \cap \widetilde{\Omega}_{\widetilde{p}}) \cap \gamma_2(\widetilde{B}_{\widetilde{p}}(R) \cap \widetilde{\Omega}_{\widetilde{p}}) = \emptyset$ for any $\gamma_1 \neq \gamma_2$.

(ii) By the properties of a fundamental domain, for any $\widetilde{y} \in \widetilde{B}_{\widetilde{p}}(R)$ there are $\widetilde{x} \in \overline{\widetilde{\Omega}}_{\widetilde{p}}$ and $\gamma \in \Gamma$ such that $\gamma(\widetilde{x}) = \widetilde{y}$. Furthermore, $d_{\widetilde{F}}(\widetilde{p}, \widetilde{x}) = d_F(f(\widetilde{p}), f(\widetilde{x})) = d_F(f(\widetilde{p}), f(\gamma(\widetilde{x}))) = d_F(f(\widetilde{p}), f(\widetilde{y})) \leq d_{\widetilde{F}}(\widetilde{p}, \widetilde{y}) < R$, which implies that $\widetilde{x} \in \widetilde{B}_{\widetilde{p}}(R) \cap \overline{\widetilde{\Omega}}_{\widetilde{p}}$. On the other hand, for $\gamma \in \Gamma$ such that $\gamma(\widetilde{x}) = \widetilde{y}$, we have

$$\begin{aligned} \|\gamma\| = d_{\widetilde{F}}(\widetilde{p}, \gamma(\widetilde{p})) &\leq d_{\widetilde{F}}(\widetilde{p}, \widetilde{y}) + d_{\widetilde{F}}(\gamma(\widetilde{x}), \gamma(\widetilde{p})) \\ &< R + \lambda_F d_{\widetilde{F}}(\gamma(\widetilde{p}), \gamma(\widetilde{x})) < (1 + \lambda_F)R, \end{aligned}$$

and consequently

$$\widetilde{B}_{\widetilde{p}}(R) \subset \bigcup_{\gamma \in \Delta((1+\lambda_F)R)} \gamma(\widetilde{B}_{\widetilde{p}}(R) \cap \overline{\widetilde{\Omega}}_{\widetilde{p}}). \tag{4.5}$$

Similar to (4.3), we have

$$\mathrm{vol}_{\max}(\gamma(\widetilde{B}_{\widetilde{p}}(R) \cap \overline{\widetilde{\Omega}}_{\widetilde{p}})) = \mathrm{vol}_{\max}(\widetilde{B}_{\widetilde{p}}(R) \cap \overline{\widetilde{\Omega}}_{\widetilde{p}}) = \mathrm{vol}_{\max}(B_p(R)),$$

which together with (4.5) implies (4.2). ∎

5. Ricci curvature and topology. Our first result concerns the growth of the fundamental group for Finsler manifolds with nonnegative Ricci curvature.

THEOREM 5.1. *Let (M, F) be a forward complete Finsler n-manifold with nonnegative Ricci curvature. If there exists $p \in M$ such that $\mathrm{vol}_{\min}(B_p(R)) \geq CR^k$ for some constant C and $0 \leq k \leq n$, then $\pi_1(M)$ has polynomial growth of order $\leq n - k$. In particular, the fundamental group of any forward complete noncompact Finsler manifold with nonnegative Ricci curvature and finite uniformity constant must have polynomial growth of order $\leq n - 1$.*

Proof. Fix $\widetilde{p} \in f^{-1}(p)$. For any $R > 0$, by Theorem 2.2(iii) and (4.1) one has

$$N(R) \leq \frac{\mathrm{vol}_{\min}(\widetilde{B}_{\widetilde{p}}(2R))}{\mathrm{vol}_{\min}(B_p(R))} \leq \frac{2^n \mu(p)^{n/2} \,\mathrm{vol}(\mathbb{B}_0^n(1))}{C} R^{n-k},$$

which means that $\Gamma \cong \pi_1(M)$ has polynomial growth of order $\le n-k$. Now suppose that M is noncompact with nonnegative Ricci curvature and finite uniformity constant. Then $\mu_F := \sup_{x\in M}\mu(x) < \infty$, and by the definition of ν, Proposition 2.1 and Theorem 2.3 we see that

$$\mathrm{vol}_{\min}(B_p(R)) \ge \frac{1}{\nu_F}\,\mathrm{vol}_{\max}(B_p(R)) \ge \frac{c(p)}{\mu_F^n}R,$$

where $\nu_F := \sup_{x\in M}\nu(x)$. Thus $\pi_1(M)$ has polynomial growth of order $\le n-1$. ∎

By (3.1) we see that when F is Riemannian, Theorem 5.1 implies the corresponding results in [M, A], while it improves the corresponding results in [S2, SZ] for general Finsler metrics. Combining this with Theorem 1.3 in [A], we have

Theorem 5.2. *Let (M,F) be a forward complete Finsler n-manifold with nonnegative Ricci curvature. If there exists $p\in M$ such that $\mathrm{vol}_{\min}(B_p(R)) \ge CR^k$ for some constant C and $0\le k\le n$, then the first Betti number satisfies $b_1(M)\le n-k$. In particular, $b_1(M)\le n-1$ for any forward complete noncompact Finsler manifold (M,F) with nonnegative Ricci curvature and finite uniformity constant.*

It is also clear that when $k=n$ in Theorem 5.1, $\pi_1(M)$ is finite. We can consider a more general situation. Let (M,F) be a forward complete Finsler manifold with $\mathbf{Ric}_M \ge (n-1)c$. Then by Theorem 2.2(iii),

$$\alpha(p) := \limsup_{R\to\infty}\frac{\mathrm{vol}_{\min}(B_p(R))}{\mathrm{vol}(\mathbb{B}^n_c(R))} \le \mu(p)^{n/2}.$$

Clearly, $\alpha(p)>0$ when $c>0$.

Definition 5.3. Let (M,F) be a forward complete Finsler manifold with $\mathbf{Ric}_M \ge (n-1)c$, $c\le 0$. Then (M,F) is said to have *large volume growth at $p\in M$* if

$$\alpha(p) = \limsup_{R\to\infty}\frac{\mathrm{vol}_{\min}(B_p(R))}{\mathrm{vol}(\mathbb{B}^n_c(R))} > 0.$$

Theorem 5.4. *Let (M,F) be a forward complete Finsler manifold with $\mathbf{Ric}_M \ge (n-1)c$, $c\le 0$. If (M,F) has large volume growth at some $p\in M$, then*

$$\sharp\pi_1(M) \le \mu(p)^{n/2}/\alpha(p), \tag{5.1}$$

so $\pi_1(M)$ is finite. Consequently, if $\alpha(p) > \mu(p)^{n/2}/2$ for some $p\in M$, then M is simply connected.

Proof. By the definition of $\alpha(p)$, we can choose a sequence $\{R_i\}_{i=1}^\infty$ such that

$$\lim_{i\to\infty} \frac{\mathrm{vol}_{\min}(B_p(R_i))}{\mathrm{vol}(\mathbb{B}_c^n(R_i))} = \alpha(p),$$

which together with (4.1) and Theorem 2.2 yields

$$N(\lambda) \le \frac{\mathrm{vol}_{\min}(\widetilde{B}_{\widetilde{p}}(\lambda + R_i))}{\mathrm{vol}_{\min}(B_p(R_i))} \le \frac{\mu(p)^{n/2}\,\mathrm{vol}(\mathbb{B}_c^n(\lambda + R_i))}{\mathrm{vol}_{\min}(B_p(R_i))} \to \frac{\mu(p)^{n/2}}{\alpha(p)} \quad (i \to \infty)$$

for all $\lambda > 0$. Hence we have (5.1). ∎

Remark 5.5. When $c = 0$ and (M, F) is Riemannian, Theorem 5.4 was proved in [A, L].

The following result can be easily verified by using Theorem 2.2 and (4.1).

Theorem 5.6. *Let (M, F) be a compact Finsler manifold with $\mathbf{Ric}_M \ge (n-1)c > 0$. Then*

$$\sharp\pi_1(M) \le \frac{\Lambda\,\mathrm{vol}(\mathbb{S}^n(c))}{\mathrm{vol}_{\min}(M)},$$

where $\mathbb{S}^n(c)$ is the n-sphere of constant curvature c, and

$$\Lambda = \min_{p\in M} \mu(p)^{n/2}.$$

Moreover, M must be simply connected if $\mathrm{vol}_{\min}(M) > \Lambda\,\mathrm{vol}(\mathbb{S}^n(c))/2$.

Lemma 5.7. *Let (M, F) be a compact Finsler manifold with $\chi(M) \ne 0$, where $\chi(M)$ denotes the Euler characteristic. Then there exists $p \in M$ such that $\mu(p) = 1$.*

Proof. Suppose on the contrary that $\mu(p) \ne 1$ for any $p \in M$; this means that the tangent space (T_pM, F_p) is non-Euclidean for each $p \in M$. We may find a unit maximal vector $X_p \in T_pM$ such that

$$\det(g_{ij}(p, X_p)) = \max_{Y_p \in T_pM\setminus\{0\}} \det(g_{ij}(p, Y_p)).$$

Since $g_{ij}(p, Y)$ is smooth on $TM \setminus \{0\}$, we can choose X_p so that it depends smoothly on p, and we obtain a unit vector field on M which implies that $\chi(M) = 0$, a contradiction. So the lemma is proved. ∎

By Theorem 5.6 and Lemma 5.7 we clearly have

Theorem 5.8. *Let (M, F) be a compact Finsler manifold with $\mathbf{Ric}_M \ge (n-1)c > 0$ and $\chi(M) \ne 0$. Then*

$$\sharp\pi_1(M) \le \frac{\mathrm{vol}(\mathbb{S}^n(c))}{\mathrm{vol}_{\min}(M)}.$$

Moreover, M must be simply connected if $\mathrm{vol}_{\min}(M) > \mathrm{vol}(\mathbb{S}^n(c))/2$.

6. Flag curvature and fundamental group

THEOREM 6.1. *Let (M, F) be a forward complete Finsler n-manifold with finite reversibility. If one of the following two conditions holds:*

(i) *the flag curvature of M satisfies $\mathbf{K}(V;W) \leq -a^2 < 0$ and $\mathrm{vol}_{\max}(B_p(R)) \leq c\exp((n-1)bR)$ for some $p \in M$, $c > 0$ and $0 < b < a$;*

(ii) *the flag curvature of M is nonpositive, the Ricci curvature of M satisfies $\mathbf{Ric}_M \leq -a^2 < 0$, and $\mathrm{vol}_{\max}(B_p(R)) \leq c\exp(bR)$ for some $p \in M$, $c > 0$ and $0 < b < a$.*

Then $\pi_1(M)$ has exponential growth.

Proof. We shall assume (i); (ii) can be handled similarly. It is clear by the curvature assumption that the injectivity radius of the universal covering space $(\widetilde{M}, \widetilde{F})$ is infinite, which together with (4.2) and Theorem 2.2 yields

$$\begin{aligned}\frac{\log N(\lambda)}{\lambda} &\geq \frac{\log\big(\mathrm{vol}_{\max}\big(\widetilde{B}_{\widetilde{p}}\big(\frac{\lambda}{1+\lambda_F}\big)\big)\big)}{\lambda} - \frac{\log\big(\mathrm{vol}_{\max}\big(B_p\big(\frac{\lambda}{1+\lambda_F}\big)\big)\big)}{\lambda}\\ &\geq \frac{\log\big(\mathrm{vol}\big(\mathbb{B}^n_{-a^2}\big(\frac{\lambda}{1+\lambda_F}\big)\big)\big)}{\lambda} - \frac{(n-1)b}{1+\lambda_F} - \frac{\frac{n}{2}\log\mu(p) + \log c}{\lambda}.\end{aligned}$$

Recaling that

$$\begin{aligned}\mathrm{vol}(\mathbb{B}^n_{-a^2}(R)) &= \mathrm{vol}(\mathbb{S}^{n-1}(1))\int_0^R \left(\frac{\sinh at}{a}\right)^{n-1} dt\\ &> \frac{\mathrm{vol}(\mathbb{S}^{n-1}(1))}{2^{n-1}a^{n-1}}\int_1^R \exp((n-1)at)(1-\exp(-2at))^{n-1}\,dt\\ &> \big(1-\exp(-2a)\big)^{n-1}\frac{\mathrm{vol}(\mathbb{S}^{n-1}(1))}{(n-1)2^{n-1}a^n}\big(\exp((n-1)aR) - \exp((n-1)a)\big)\end{aligned}$$

for all $R > 1$, we have

$$\frac{\log N(\lambda)}{\lambda} \geq \frac{(n-1)(a-b)}{1+\lambda_F} - \frac{C}{\lambda}, \qquad \forall\lambda > 1+\lambda_F,$$

where C is a constant. Hence,

$$\limsup_{\lambda\to\infty}\frac{\log N(\lambda)}{\lambda} \geq \frac{(n-1)(a-b)}{1+\lambda_F} > 0,$$

so $\pi_1(M)$ has exponential growth. ∎

REMARK 6.2. It is clear by (3.2) that Theorem 6.1 improves the corresponding results in [M, SZ, WX].

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