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Concerning the energy class $\mathcal{E}_p$ for $0 < p < 1$

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Abstract. The energy class $\mathcal{E}_p$ is studied for $0 < p < 1$. A characterization of certain bounded plurisubharmonic functions in terms of $\mathcal{F}_p$ and its pluricomplex p-energy is proved.

1. Introduction. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded hyperconvex domain, i.e., there exists a bounded plurisubharmonic function $\varphi : \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega : \varphi(z) < c\}$ is compact in Ω for every $c \in (-\infty, 0)$. In this article our class of test functions will be the convex cone $\mathcal{E}_0$ $(= \mathcal{E}_0(\Omega))$ consisting of all bounded plurisubharmonic functions φ defined on Ω such that $\lim_{z\to\xi} \varphi(z) = 0$ for every $\xi \in \partial\Omega$, and $\int_\Omega (dd^c\varphi)^n < \infty$, where $(dd^c\,\cdot\,)^n$ is the complex Monge–Ampère operator.

Assume that u is a plurisubharmonic function defined on Ω and $[\varphi_j]_{j=1}^\infty$, $\varphi_j \in \mathcal{E}_0$, is a decreasing sequence which converges pointwise to u on Ω as $j \to \infty$. If there can be no misinterpretation a sequence $[\,\cdot\,]_{j=1}^\infty$ will be denoted by $[\,\cdot\,]$. For $p > 0$ fixed, consider the following assertions:

(1) $\sup_j \int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n < \infty$,

(2) $\sup_j \int_\Omega (dd^c\varphi_j)^n < \infty$.

If the sequence $[\varphi_j]$ can be chosen such that (1) holds, then we say that u belongs to $\mathcal{E}_p$, and if (2) holds, then u belongs to $\mathcal{F}$. Finally, if both (1) and (2) are satisfied, then $u \in \mathcal{F}_p$. For $p = 0$, we say by convention that $u \in \mathcal{F}$. The energy classes $\mathcal{F}_p$ and $\mathcal{E}_p$ are two of the so called *Cegrell classes.* For $p \geq 1$, the classes $\mathcal{F}_p$ and $\mathcal{E}_p$ were introduced and extensively studied in [4] and here we will study them for $0 < p < 1$. For further information about the Cegrell classes see e.g. [4, 6, 7] and the references therein. It follows from [4]

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that any function in $\mathcal{E}_p$ is in $\mathcal{E}$ and hence by [6] the operator $(dd^c\,\cdot)^n$ is well defined on $\mathcal{E}_p$, $p \geq 0$ (see [6] for the definition of $\mathcal{E}$).

Now, let $e_p(u)$ be defined by

$$e_p(u) = \int_\Omega (-u)^p (dd^c u)^n$$

for $p > 0$. The integral $e_p(u)$ is the *pluricomplex p-energy* of the function u. As in [4, 11] the pluricomplex p-energy will be used to study $\mathcal{E}_p$. In [11], Persson proved that if $p \geq 1$ and $u_0, u_1, \ldots, u_n \in \mathcal{E}_0$, then

$$\int_\Omega (-u_0)^p dd^c u_1 \wedge \cdots \wedge dd^c u_n \leq D_{n,p} e_p(u_0)^{p/(p+n)} e_p(u_1)^{1/(p+n)} \cdots e_p(u_n)^{1/(p+n)}$$

(see also [8]), where $D_{n,p}$ is a constant depending only on n and p. This Hölder type inequality is a fundamental tool in [4]. In Section 2, we will extend this estimate to $p > 0$; as a direct consequence, it follows that $\mathcal{F}_p$ and $\mathcal{E}_p$ are convex cones (Corollary 2.4). The aim of this article is to prove the following characterization of the Dirichlet problem: Let $n \geq 1$, $p > 0$, and μ a non-negative measure (not necessarily of finite total mass). Then there exists a unique function $u \in \mathcal{E}_p$ such that $(dd^c u)^n = \mu$ if, and only if, there exists a constant $A > 0$ such that

$$\int_\Omega (-\varphi)^p \, d\mu \leq A e_p(\varphi)^{p/(n+p)}$$

for every $\varphi \in \mathcal{E}_0$ (Theorem 3.6). For $p \geq 1$ this was proved in [4]. A related Dirichlet problem for the case $p = 0$ was considered in [6].

In Section 4, we will prove, as an application of the framework induced by the energy classes, that $u \in \mathcal{E}_0$ if, and only if,

(1) $u \in \mathcal{F}_p$ for every $p \geq 0$,
(2) $\lim_{z \to \xi} u(z) = 0$ for every $\xi \in \partial\Omega$,
(3) $\sup_{p>0} e_p(u)^{1/p} < \infty$.

We end this article by constructing two examples which motivate this characterization.

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2. A Hölder type inequality. We will proceed as in [11] by using Lemma 2.1 below as a counterpart of Lemma 5.1 in [11].

LEMMA 2.1. *Let* $u, v \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$, $\lim_{z\to\xi} u(z) = \lim_{z\to\xi} v(z) = 0$ *for every* $\xi \in \partial\Omega$ *and* T *be a positive closed current of bidegree* $(n-1, n-1)$.

For $0 < p < 1$,

$$\int_\Omega (-u)^p dd^c v \wedge T \le p^{-\frac{1}{1-p}} \Big(\int_\Omega (-u)^p dd^c u \wedge T\Big)^{\frac{p}{p+1}} \Big(\int_\Omega (-v)^p dd^c v \wedge T\Big)^{\frac{1}{p+1}}.$$

Proof. Let $0 < p < 1$ and $w = -(-v)^p$. Then $w \in \mathcal{PSH}(\Omega) \cap L^\infty(\Omega)$ and $\lim_{z\to\xi} w(z) = 0$ for every $\xi \in \partial\Omega$. We have

$$\begin{aligned}(2.1)\qquad \int_\Omega (-u)^p dd^c v \wedge T &= -\int_\Omega (-u)^p (dd^c(-w)^{1/p}) \wedge T \\ &= -\frac{1}{p}\int_\Omega (-u)^p (-w)^{1/p-1} dd^c(-w) \wedge T \\ &\quad - \frac{1-p}{p^2}\int_\Omega (-u)^p (-w)^{1/p-2} d(-w) \wedge d^c(-w) \wedge T \\ &\le \frac{1}{p}\int_\Omega (-u)^p (-w)^{1/p-1} dd^c w \wedge T = \frac{1}{p}\int_\Omega (-u)^p (-v)^{1-p} dd^c w \wedge T.\end{aligned}$$

Hölder's inequality yields

$$\begin{aligned}(2.2)\qquad \int_\Omega (-u)^p dd^c v \wedge T &\le \frac{1}{p}\Big[\int_\Omega (-u) dd^c w \wedge T\Big]^p \Big[\int_\Omega (-v) dd^c w \wedge T\Big]^{1-p} \\ &= \frac{1}{p}\Big[\int_\Omega (-w) dd^c u \wedge T\Big]^p \Big[\int_\Omega (-w) dd^c v \wedge T\Big]^{1-p} \\ &= \frac{1}{p}\Big[\int_\Omega (-v)^p dd^c u \wedge T\Big]^p \Big[\int_\Omega (-v)^p dd^c v \wedge T\Big]^{1-p}.\end{aligned}$$

By combining inequalities (2.1) and (2.2) we get

$$\begin{aligned}\int_\Omega (-u)^p dd^c v \wedge T &\le \frac{1}{p}\Big[\int_\Omega (-v)^p dd^c u \wedge T\Big]^p \Big[\int_\Omega (-v)^p dd^c v \wedge T\Big]^{1-p} \\ &\le \frac{1}{p^{1+p}}\Big[\int_\Omega (-u)^p dd^c v \wedge T\Big]^{p^2} \Big[\int_\Omega (-u)^p dd^c u \wedge T\Big]^{p(1-p)} \\ &\quad \times \Big[\int_\Omega (-v)^p dd^c v \wedge T\Big]^{1-p}.\end{aligned}$$

Thus, the desired inequality is achieved. ■

THEOREM 2.2. *Let $u_0, u_1, \ldots, u_n \in \mathcal{E}_0$ and $p > 0$. Assume that X is a non-empty set, $n \ge 1$ an integer and that $F : X^{n+1} \to \mathbb{R}$ is a function which is symmetric in the last n variables. If there exists a constant $C > 0$ such that*

$$F(u_0, u_1, \ldots, u_n) \le C F(u_0, u_0, u_2, \ldots, u_n)^{\frac{p}{p+1}} F(u_1, u_1, u_2, \ldots, u_n)^{\frac{1}{p+1}},$$

then

$$F(u_0,u_1,\dots,u_n) \le C^{\alpha(n,p)} F(u_0,\dots,u_0)^{\frac{p}{p+1}} F(u_1,\dots,u_1)^{\frac{1}{p+1}} \cdots F(u_n,\dots,u_n)^{\frac{1}{p+1}},$$

where $\alpha(n,p)$ *is given by*

$$\begin{cases} \alpha(1,p)=1, \\ \alpha(n,p)=\alpha(n-1,p)+\dfrac{(p+1)(p+n-1)}{p(p+n)}\Big(1+\dfrac{\alpha(n-1,p)}{p+1}\Big). \end{cases}$$

Moreover, *if* $C\ge 1$, *then*

$$\alpha(n,p)=(p+2)\Big(\frac{p+1}{p}\Big)^{n-1}-(p+1).$$

Proof. Cf. Theorem 4.1 in [11]. ∎

Let $p>0$. The *mutual pluricomplex p-energy* $(u_0,\dots,u_n)_p$ is defined by

$$(u_0,\dots,u_n)_p=\int_\Omega (-u_0)^p dd^c u_1\wedge\cdots\wedge dd^c u_n.$$

For $p\ge 1$, Theorem 2.3 below was proved in [11]. If $p=0$, then (2.3) can be interpreted as Corollary 5.6 in [6].

THEOREM 2.3. *Let* $p>0$ *and* $u_0,u_1,\dots,u_n\in\mathcal{E}_0$. *Then*

$$(u_0,\dots,u_n)_p \le D_{n,p} e_p(u_0)^{p/(p+n)} e_p(u_1)^{1/(p+n)}\cdots e_p(u_n)^{1/(p+n)}, \tag{2.3}$$

where

$$D_{n,p}=\begin{cases} p^{-\alpha(n,p)/(1-p)} & \text{if } 0<p<1, \\ 1 & \text{if } p=1, \\ p^{p\alpha(n,p)/(p-1)} & \text{if } p>1, \end{cases}$$

and $\alpha(n,p)=(p+2)\big(\frac{p+1}{p}\big)^{n-1}-(p+1)$.

Proof. Let $0<p<1$ and $(u_0,u_1,\dots,u_n)_p=F(u_0,u_1,\dots,u_n)$ in Theorem 2.2. The proof then follows from Lemma 2.1 and Theorem 2.2. ∎

COROLLARY 2.4. *For any* $p\ge 0$, *the classes* $\mathcal{F}_p$ *and* $\mathcal{E}_p$ *are convex cones.*

Proof. This follows as in [4] by using Theorem 2.3. ∎

If $q>p>0$, then $\mathcal{F}_q\subset\mathcal{F}_p$, by Hölder's inequality. We will end this section by explaining why a similar result for $\mathcal{E}_p$ is not possible. Let $q>p>0$ be fixed. Then it follows from Example 2.3 in [5] that $\mathcal{E}_p\setminus\mathcal{E}_q$ is non-empty. Example 2.6 below shows that $\mathcal{E}_q\setminus\mathcal{E}_p$ is non-empty as well. First note that if $u_1,\dots,u_k\in\mathcal{E}_0$, then

$$e_p(u_1+\cdots+u_k)\ge\sum_{j=1}^k e_p(u_j). \tag{2.4}$$

We will also need the following lemma.

LEMMA 2.5. *Let $p>0$ and $u,v\in\mathcal{E}_0$. Then $e_p(u+v)\to e_p(u)$ as $e_p(v)\to 0$.*

Proof. Let $0<p<1$. Hölder's inequality together with (2.3) and the fact that $(-u-v)^p\le(-u)^p+(-v)^p$ yields

$$e_p(u)\le e_p(u+v)\le e_p(u)+C\sum_{j=0}^{n}e_p(u)^{\frac{j}{p+n}}e_p(v)^{\frac{p+n-j}{p+n}} \tag{2.5}$$

and the case $0<p<1$ is proved. Assume now that $p\ge 1$. Using Minkowski's inequality we get

$$e_p(u+v)^{1/p}\le\Big[\int_\Omega(-u)^p(dd^c(u+v))^n\Big]^{1/p}+\Big[\int_\Omega(-v)^p(dd^c(u+v))^n\Big]^{1/p}. \tag{2.6}$$

Employing (2.3) to estimate

$$\int_\Omega(-u)^p(dd^cu)^{n-j}\wedge(dd^cv)^j\quad\text{for } j=1,\dots,n$$

and

$$\int_\Omega(-v)^p(dd^cu)^{n-j}\wedge(dd^cv)^j\quad\text{for } j=0,\dots,n$$

together with (2.6) completes this proof. ∎

REMARK. It follows from the estimate (2.5) and Example 3.11 in [4] that $(\bigcap_{p>0}\mathcal{E}_p)\setminus\mathcal{F}\neq\emptyset$.

EXAMPLE 2.6. Let $q>p>0$ and $g=g(z,z_0)$ be the pluricomplex Green function with pole $z_0\in\Omega$. Define $v_j=j^p\max(g,1/j^{p+n})\in\mathcal{E}_0$. Then $e_p(v_j)=(2\pi)^n$ and $e_q(v_j)=(2\pi)^nj^{n(p-q)}$, hence $\lim_{j\to\infty}e_q(v_j)=0$. Therefore, Lemma 2.5 implies that there exist integers s_j such that the decreasing sequence defined by $u_k=v_{s_1}+\dots+v_{s_k}$ converges pointwise to a function $u\in\mathcal{E}_q$. Inequality (2.4) implies that $e_p(u_k)\ge k(2\pi)^n$. Thus, $u\notin\mathcal{E}_p$.

3. The Dirichlet problem

LEMMA 3.1. *Let $p\ge 0$ and $\mathcal{K}\in\{\mathcal{F}_p,\mathcal{E}_p\}$. If $u\in\mathcal{K}$ and $v\in\mathcal{PSH}(\Omega)$, $v\le 0$, then $\max(u,v)\in\mathcal{K}$.*

Proof. For the case $p=0$ cf. [6] and for the case $p\ge 1$ see [4]. Let $0<p<1$ and $u\in\mathcal{E}_p$. Then by definition there exists a decreasing sequence $[u_j]$, $u_j\in\mathcal{E}_0$, which converges pointwise to u as $j\to\infty$, and $\sup_j e_p(u_j)<\infty$. Set $w_j=\max(u_j,v)$. Then $[w_j]$, $w_j\in\mathcal{E}_0$, is a decreasing sequence which converges pointwise to $\max(u,v)$ as $j\to\infty$, and $\sup_j e_p(w_j)\le\sup_j e_p(u_j)<\infty$, hence $\max(u,v)\in\mathcal{E}_p$. If $u\in\mathcal{F}_p$, then we additionally need to prove that $\sup_j\int_\Omega(dd^cw_j)^n<\infty$. But $u_j\le w_j$, which implies that $\sup_j\int_\Omega(dd^cw_j)^n\le\sup_j\int_\Omega(dd^cu_j)^n<\infty$. ∎

For $p \geq 1$, Lemma 3.2 below was proved in [4]. By using Theorem 2.3 together with Lemma 3.1 the proof of Lemma 5.4 in [4] is also valid for the case $0 < p < 1$.

LEMMA 3.2. *Let $p > 0$. If $\psi \in \mathcal{PSH}(\Omega) \cap C(\overline{\Omega})$, $\psi < 0$, and $u \in \mathcal{E}_p$, then*

$$\chi_A (dd^c u)^n = \chi_A (dd^c \max(u, \psi))^n,$$

where χ_A is the characteristic function of the set $A = \{z \in \Omega : u > \psi\}$.

LEMMA 3.3. *Let $p \geq 0$. If $u, v \in \mathcal{E}_p$ are such that $u \leq v$, then*

$$\int_\Omega (-\varphi)(dd^c v)^n \leq \int_\Omega (-\varphi)(dd^c u)^n$$

for every $\varphi \in \mathcal{PSH}(\Omega)$ with $\varphi \leq 0$.

Proof. First assume that $\varphi \in \mathcal{E}_0$. Then integration by parts (see [6]) implies that

$$\int_\Omega (-\varphi)(dd^c u)^n = \int_\Omega (-u)(dd^c \varphi) \wedge (dd^c u)^{n-1}; \tag{3.1}$$

but $-u \geq -v$ by assumption and therefore

$$\int_\Omega (-u)(dd^c \varphi) \wedge (dd^c u)^{n-1} \geq \int_\Omega (-v)(dd^c \varphi) \wedge (dd^c u)^{n-1}. \tag{3.2}$$

By using integration by parts once again we get

$$\int_\Omega (-v)(dd^c \varphi) \wedge (dd^c u)^{n-1} = \int_\Omega (-\varphi)(dd^c v) \wedge (dd^c u)^{n-1} \tag{3.3}$$

and therefore $\int_\Omega (-\varphi)(dd^c u)^n \geq \int_\Omega (-\varphi)(dd^c v) \wedge (dd^c u)^{n-1}$ by (3.1)–(3.3). Continuing in a similar manner using integration by parts and the assumption $u \leq v$ yields the desired inequality when $\varphi \in \mathcal{E}_0$. The general case then follows from Theorem 2.1 in [6] together with the monotone convergence theorem. ∎

For $p = 0$, Theorem 3.4 below was proved in [6] (Theorem 5.15) and for $p \geq 1$ it follows from the proof of Theorem 6.2 in [4]. Here we will use the method of [4] to achieve the result for $0 < p < 1$.

THEOREM 3.4. *Let $p \geq 0$. If $u \in \mathcal{E}$ and $v \in \mathcal{E}_p$ are such that $(dd^c v)^n \leq (dd^c u)^n$, then $u \leq v$.*

Proof. Assume that $0 < p < 1$ and let $h \in \mathcal{E}_0 \cap C(\overline{\Omega})$, not identically 0. For each $m \geq 1$, Lemmas 3.1 and 3.2 imply that

$$(dd^c \max(v, mh))^n = \chi_{\{v > mh\}} (dd^c v)^n + \chi_{\{v \leq mh\}} (dd^c \max(v, mh))^n.$$

Kołodziej's theorem (see [10], and also Proposition 6.1 in [4]) implies that there exists $g_m \in \mathcal{E}_0$ such that $(dd^c g_m)^n = \chi_{\{v \leq mh\}} (dd^c \max(v, mh))^n$.

Thus, $(dd^c(u+g_m))^n \geq (dd^c \max(v, mh))^n$. Theorem 5.15 in [6] shows that $\max(v, mh) \geq u + g_m$, hence

$$v = \limsup_{m\to\infty} \max(v, mh) \geq u + \limsup_{m\to\infty} g_m. \tag{3.4}$$

Let $w_m = \sup_{j\geq m} g_j$. Then $w_m^* \in \mathcal{E}_0$, where w^* denotes the upper semicontinuous regularization of the function w. Moreover, $[w_m]$ is a decreasing sequence which converges pointwise to $\limsup_{m\to\infty} g_m$ as $m \to \infty$. Fix $m \geq 1$ and let $j \geq m$. Lemma 3.3 and the fact that $\max(v, jh) \leq g_j \leq w_m^*$ imply that

$$\begin{aligned} e_p(w_m^*) &\leq m^p \int_\Omega (-h)^p (dd^c w_j^*)^n \leq m^p \int_\Omega (-h)^p (dd^c g_j)^n \\ &= \left(\frac{m}{j}\right)^p \int_\Omega (-jh)^p \chi_{\{v\leq jh\}} (dd^c \max(v, jh))^n \\ &\leq \left(\frac{m}{j}\right)^p \sup_{j\geq m} e_p(\max(v, jh)) < \infty \end{aligned}$$

and therefore $w_m^* = 0$. Hence, $\limsup_{m\to\infty} g_m = \lim_{m\to\infty} w_m = 0$ almost everywhere and by inequality (3.4) it follows that $v \geq u$. ∎

The next corollary was proved in [1] for $p \geq 1$ and $p = 0$. Using exactly the same methods together with Theorem 3.4 yields the first statement. The second statement follows from Example 3.7 in [1].

COROLLARY 3.5. *If $u \in \bigcup_{p\geq 0} \mathcal{E}_p$, then $\limsup_{z\to\xi} u(z) = 0$ for every $\xi \in \partial\Omega$. Moreover, for each $p \geq 0$ there exists a function $v \in \mathcal{E}_p$ such that $\liminf_{z\to\xi} v(z) = -\infty$ for every $\xi \in \partial\Omega$.*

We now prove a characterization of the Dirichlet problem in $\mathcal{E}_p$ for $p > 0$. For $p \geq 1$ this was proved in [4, Theorem 6.2].

THEOREM 3.6. *Let $p > 0$ and μ a non-negative measure. Then there exists a unique function $u \in \mathcal{E}_p$ such that $(dd^c u)^n = \mu$ if, and only if, there exists a constant $A > 0$ such that*

$$\int_\Omega (-\varphi)^p d\mu \leq A e_p(\varphi)^{p/(n+p)} \tag{3.5}$$

for every $\varphi \in \mathcal{E}_0$.

Proof. Let $0 < p < 1$. Assume that there exists a unique $u \in \mathcal{E}_p$ such that $(dd^c u)^n = \mu$. There exists a sequence $[u_j]$, $u_j \in \mathcal{E}_0$, which converges pointwise on Ω to u as $j \to \infty$, and $\lim_{j\to\infty} e_p(u_j) = e_p(u) < \infty$ (Lemma 2.1 in [7]). Let $\varphi \in \mathcal{E}_0$. Then Theorem 2.3 implies that there exists a constant $C > 0$

such that $\int_\Omega (-\varphi)(dd^c u_j)^n \le C e_p(\varphi)^{p/(p+n)} e_p(u_j)^{1/(p+n)}$ and therefore

$$\int_\Omega (-\varphi)^p d\mu \le \liminf_{j\to\infty} \int_\Omega (-\varphi)^p (dd^c u_j)^n \le C e_p(u)^{1/(p+n)} e_p(\varphi)^{p/(p+n)} \le A e_p(\varphi)^{p/(n+p)}.$$

For the converse assume that there exists a constant $A > 0$ such that (3.5) holds. In particular this assumption implies that μ vanishes on pluripolar sets and so Theorem 5.11 in [6] shows that there exist functions $\phi \in \mathcal{E}_0$ and $0 \le f \in L^1_{\mathrm{loc}}((dd^c\phi)^n)$ such that $\mu = f(dd^c\phi)^n$. Kołodziej's theorem (see [10], [4, Proposition 6.1]) implies that there exist $u_j \in \mathcal{E}_0$ such that $(dd^c u_j)^n = \min(f, j)(dd^c\phi)^n$. Hence, $\sup_j e_p(u_j) < A^{(n+p)/p} < \infty$ and therefore there exists $u \in \mathcal{E}_p$ such that $(dd^c u)^n = \mu$. Uniqueness follows from Theorem 3.4. ∎

Using Theorem 3.6 together with the methods of [2] we obtain

Corollary 3.7. *Let $n \ge 1$ and $\psi \in \mathcal{PSH}(\Omega)$ with $\lim_{z\to\xi} \psi(z) = 0$ for every $\xi \in \partial\Omega$, and $\varphi \in L^q((dd^c\psi)^n)$, $\varphi \ge 0$, $1 < q < \infty$. Then there exists a unique function $u \in \mathcal{E}_{n(q-1)}$ such that $(dd^c u)^n = \varphi(dd^c\psi)^n$. Moreover, if $\int_\Omega (dd^c\psi)^n < \infty$, then $u \in \mathcal{F}_{n(q-1)}$.*

4. A characterization of bounded plurisubharmonic functions. The following well-known lemma is an elementary exercise in L^p-theory.

Lemma 4.1. *Let $q > 1$ and assume that u in $\mathcal{E}_q$ is not identically 0. Then*

$$\lim_{p\to\infty} e_p(u)^{1/p} = \inf\Big\{\alpha \in \mathbb{R} : \Big[\int_\Omega \chi_{\{-u>\alpha\}} (dd^c u)^n\Big] = 0\Big\}.$$

Proof. Set $M = \inf\{\alpha \in \mathbb{R} : \int_\Omega \chi_{\{-u>\alpha\}}(dd^c u)^n = 0\}$. Without loss of generality we can assume that $M > 0$. Take $0 < \widetilde{M} < M$. If $A = \{z \in \Omega : |u(z)| > \widetilde{M}\}$ and $C_1 = \int_\Omega \chi_A (dd^c u)^n$, then $C_1 > 0$ and

$$\infty > C_2 = \int_\Omega (-u)^q (dd^c u)^n \ge \int_A (-u)^q (dd^c u)^n \ge \widetilde{M}^q C_1.$$

For $p > q$, it then follows that $e_p(u)^{1/p} \ge (\int_A (-u)^p (dd^c u)^n)^{1/p} \ge \widetilde{M} C_1^{1/p}$. Thus

$$\liminf_{p\to\infty} e_p(u)^{1/p} \ge M, \tag{4.1}$$

since $0 < \widetilde{M} < M$ was chosen arbitrarily. Moreover, for $p > q$ we have

$$e_p(u)^{1/p} = \Big(\int_\Omega (-u)^q (-u)^{p-q} (dd^c u)^n\Big)^{1/p} \le C_2^{1/p} M^{1-q/p}.$$

Hence

$$\limsup_{p\to\infty} e_p(u)^{1/p} \le M. \tag{4.2}$$

Inequalities (4.1) and (4.2) complete the proof. ∎

THEOREM 4.2. *A function u belongs to $\mathcal{E}_0$ if, and only if,*

(1) $u \in \mathcal{F}_p$ *for every* $p \ge 0$,
(2) $\lim_{z\to\xi} u(z) = 0$ *for every* $\xi \in \partial\Omega$,
(3) $\sup_{p>0} e_p(u)^{1/p} < \infty$.

Proof. Without loss of generality assume that $u(z) < 0$ for each $z \in \Omega$. Let $u \in \mathcal{E}_0$. Then properties (1) and (2) follow from the definition of $\mathcal{E}_0$ and $\mathcal{F}_p$. The function u is bounded by assumption and therefore $e_p(u)^{1/p} \le C_1(\int_\Omega (dd^c u)^n)^{1/p}$, where $C_1 \ge 0$ is a constant. Thus, $\sup_{p>0} e_p(u)^{1/p} < \infty$, since $\lim_{p\to\infty}(\int_\Omega (dd^c u)^n)^{1/p} = 1$.

For the converse, assume that u is a function satisfying (1)–(3). Let M be as in Lemma 4.1. Then $M < \infty$ by (3). Moreover $M > 0$, since $u < 0$ by assumption. Let $A = \{z \in \Omega : u(z) < -M\}$. The set A is open, since u is upper semicontinuous, $\int_A (dd^c u)^n = 0$ and $-u \le M$ on $\Omega \setminus A$.

Now assume that u is unbounded, and let $\varepsilon > 0$ be such that $\varepsilon|z|^2 < M$ on Ω. Set $v(z) = \max(u(z), \varepsilon|z|^2 - 2M)$. Then $v \in \mathcal{F}_p \cap L^\infty(\Omega)$ for each $p \ge 0$. As u is unbounded, the set $\{u < v\} = \{u < \varepsilon|z|^2 - 2M\}$ is non-empty and open. Lemma 4.4 in [4] implies that $\int_{\{u<v\}} (dd^c v)^n \le \int_{\{u<v\}} (dd^c u)^n \le \int_A (dd^c u)^n = 0$, since $\{u < v\} \subset A$, but

$$\int_{\{u<v\}} (dd^c v)^n = \int_{\{u<v\}} (dd^c(\varepsilon|z|^2 - 2M))^n = C\lambda(\{u < v\}) > 0,$$

where λ is the Lebesgue measure and C is a constant depending only on n and ε. This is a contradiction, which implies that u is bounded. Thus $u \in \mathcal{E}_0$. ∎

EXAMPLE 4.3. Let $B = B(0,1)$ be the unit ball in $\mathbb{C}^n$ and $[a_k]$ a sequence in B such that $a_k \to \zeta$ for some $\zeta \in \partial B$. Let $T_{a_k} = T_k$ be the automorphism of B which maps a_k to 0, i.e.,

$$T_k(z) = T_{a_k}(z) = \frac{1}{|a_k|^2} \frac{\sqrt{1-|a_k|^2}(\langle z, a_k\rangle a_k - |a_k|^2 z) + a_k(|a_k|^2 - \langle z, a_k\rangle)}{1 - \langle z, a_k\rangle},$$

where $\langle x, y\rangle = \sum_{j=1}^n x_j \bar{y}_j$ is the usual inner product in $\mathbb{C}^n$. The real Jacobian of T_k at $z \in B$ is given by

$$|T_k'(z)|^2 = \frac{F(z, a_k)}{|1 - \langle z, a_k\rangle|^{4n}},$$

where F is a bounded function. Moreover for all compact subsets K we have $\max_{z\in K}|T_k'(z)|^2 \leq C_1$, where C_1 is a constant not depending on k. Define $\varphi_j(z) = 2^{-j}\max(\log|T_j(z)|, -1)$. Then $\varphi_j \in \mathcal{PSH}(B)\cap L^\infty(B)$, $\lim_{z\to\xi}\varphi_j(z) = 0$ for every $\xi \in \partial B$, and

$$\begin{aligned}\int_B (dd^c\varphi_j)^n &= \int_B \left(dd^c \frac{1}{2^j}\max(\log|T_j|, -1)\right)^n \\ &= \frac{1}{2^{jn}}\int_B (dd^c\max(\log|T_j|, -2^j))^n \\ &= \frac{1}{2^{jn}}\int_B |T_k'|^2(dd^c\max(\log|z|, -2^j))^n \\ &\leq \frac{1}{2^{jn}}(2\pi)^n \max_{\overline{B}(0,e^{-1})}|T_k'|^2 \leq C_2\frac{1}{2^{jn}},\end{aligned}$$

where C_2 is a constant not depending on j. Set

$$u_k(z) = \max\left(\sum_{j=1}^{k}\frac{1}{2^j}\log|T_j(z)|, -1\right).$$

Then $u_k \in \mathcal{PSH}(B)\cap L^\infty(B)$, $\lim_{z\to\xi}u_k(z) = 0$ for every $\xi\in\partial B$ and $u_k \geq \sum_{j=1}^k\varphi_j$. The comparison principle (see e.g. [3]) together with Lemma 2.5 in [9] shows that $u_k \in \mathcal{E}_0$. The function u defined by

$$u(z) = \max\left(\sum_{j=1}^{\infty}\frac{1}{2^j}\log|T_j(z)|, -1\right)$$

belongs to $\mathcal{F}\cap L^\infty(B)$ and therefore to $\mathcal{F}_p$ for all $p\geq 0$. But $u\notin\mathcal{E}_0$, since $\liminf_{z\to\zeta}u(z) \leq \lim_{j\to\infty}u(a_j) = -1$. ∎

EXAMPLE 4.4. Let $B = B(0,1)\subseteq\mathbb{C}^2$ and let $[a_j]$ and $[b_j]$, $0 < a_j, b_j < 1$, be decreasing sequences which converge to 0 as $j\to\infty$. For each $j\in\mathbb{N}$, define $\varphi_j : B\to\mathbb{R}\cup\{-\infty\}$ by $\varphi_j(z) = a_j\max(\log|z|, \log b_j)$. Then $\varphi_j \in \mathcal{PSH}(B)\cap L^\infty(B)$ and $\lim_{z\to\xi}\varphi_j(z) = 0$ for every $\xi\in\partial B$. Moreover

$$dd^c\varphi_j\wedge dd^c\varphi_k = \begin{cases}(2\pi)^2a_j^2d\sigma_{b_j} & \text{if } j = k,\\ (2\pi)^2a_ja_kd\sigma_{\max(b_j,b_k)} & \text{otherwise,}\end{cases}$$

where $d\sigma_r$ is the normalized Lebesgue measure on $\partial B(0,r)$, hence $\varphi_j\in\mathcal{E}_0$ and therefore the function $u_k : B\to\mathbb{R}$ defined by $u_k = \sum_{j=1}^k\varphi_j$ is in $\mathcal{E}_0$. The functions u_k are radially symmetric, i.e., $u_k(|z|) = u_k(z)$, and

$$(4.3)\qquad \int_B (-u_k)^p (dd^c u_k)^2 = \int_B (-u_k)^p \Big(dd^c \sum_{j=1}^k \varphi_j\Big)^2$$
$$= \sum_{j,l=1}^k \int_B (-u_k)^p dd^c\varphi_j \wedge dd^c\varphi_l = \sum_{j,l=1}^k (-u_k(\max(b_j,b_l)))^p (2\pi)^2 a_j a_l$$
$$\leq \sum_{j,l=1}^k (-u_k(b_j))^{p/2} (-u_k(b_l))^{p/2} (2\pi)^2 a_j a_l = (2\pi)^2 \Big(\sum_{j=1}^k (-u_k(b_j))^{p/2} a_j\Big)^2.$$

Let $z \in B$ be such that $|z| = b_j$. Then

$$\varphi_k(z) = \begin{cases} a_k \log b_k & \text{if } k \leq j, \\ a_k \log b_j & \text{otherwise} \end{cases}$$

and therefore $\sum_{k=1}^\infty \varphi_k(z) = \sum_{k=1}^j a_k \log b_k + \log b_j \sum_{k=j+1}^\infty a_k = c_j$. Assume now that the sequences $[a_j]$ and $[b_j]$ are chosen such that

(1) $\displaystyle\sum_{j=1}^\infty a_j < \infty,$

(2) $\displaystyle\sum_{j=1}^\infty a_j \log b_j = -\infty,$

(3) $\displaystyle\sum_{j=1}^\infty (-c_j)^{p/2} a_j < \infty.$

Let $u : B \to \mathbb{R} \cup \{-\infty\}$ be defined by $u = \lim_{k\to\infty} u_k$. Then u is plurisubharmonic, since it is the limit of a decreasing sequence of plurisubharmonic functions and $u(1/2, 0) > -\infty$. Assumption (1) implies that $\int_B (dd^c u)^2 < \infty$ and from inequality (4.3) and assumption (3) it follows that

$$\sup_k \int_B (-u_k)^p (dd^c u_k)^2 < \infty.$$

Hence $u \in \mathcal{F}_p$ for each $p \geq 0$. But assumption (2) yields $u(0) = -\infty$. Let now the sequences $[a_j]$ and $[b_j]$ be defined by $a_j = 1/2^j$ and $b_j = e^{-2^j/j}$. These sequences decrease to 0 as $j \to \infty$, and by straightforward calculations, they satisfy assumptions (1)–(3). Hence, the function defined on B by

$$u(z) = \sum_{j=1}^\infty \frac{1}{2^j} \max(\log|z|, \log e^{-2^j/j}) = \sum_{j=1}^\infty \max\Big(\frac{1}{2^j}\log|z|, -\frac{1}{j}\Big)$$

belongs to $\mathcal{F}_p$ for every $p \geq 0$, and $\lim_{z\to\xi} u(z) = 0$ for every $\xi \in \partial B$. But $u \notin \mathcal{E}_0$, since u is unbounded. ∎

References

[1] P. Åhag, *The complex Monge–Ampère operator on bounded hyperconvex domains*, Ph.D. thesis, Umeå Univ., 2002.

[2] P. Åhag and R. Czyż, *On the Dirichlet problem in the Cegrell classes*, Ann. Polon. Math. 84 (2004), 273–279.

[3] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), 1–40.

[4] U. Cegrell, *Pluricomplex energy*, ibid. 180 (1998), 187–217.

[5] —, *Two examples in pluripotential theory*, Mid Sweden Univ. research report 14 (2000).

[6] —, *The general definition of the complex Monge–Ampère operator*, Ann. Inst. Fourier (Grenoble) 54 (2004), 159–179.

[7] U. Cegrell, S. Kołodziej and A. Zeriahi, *Subextension of plurisubharmonic functions with weak singularities*, Math. Z. 250 (2005), 7–22.

[8] U. Cegrell and L. Persson, *An energy estimate for the complex Monge–Ampère operator*, Ann. Polon. Math. 67 (1997), 95–102.

[9] U. Cegrell and J. Wiklund, *A Monge–Ampère norm for delta-plurisubharmonic functions*, Math. Scand. 97 (2005), 201–216.

[10] S. Kołodziej, *The range of the complex Monge–Ampère operator, II*, Indiana Univ. Math. J. 44 (1995), 765–782.

[11] L. Persson, *A Dirichlet principle for the complex Monge–Ampère operator*, Ark. Mat. 37 (1999), 345–356.

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