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A new invariant Kähler metric on relatively compact domains in a complex manifold

by Bo-Yong Chen (Shanghai)

Abstract. We introduce a new invariant Kähler metric on relatively compact domains in a complex manifold, which is the Bergman metric of the L^2 space of holomorphic sections of the pluricanonical bundle equipped with the Hermitian metric introduced by Narasimhan–Simha.

1. Introduction. It is well-known that for a bounded domain in $\mathbb{C}^n$, there are three canonical invariant metrics, i.e., the Carathéodory, Bergman and Kobayashi metrics (for their definitions and basic properties, see [6]). Moreover, if the domain is pseudoconvex, then there exists a complete invariant Kähler–Einstein metric (cf. [1], [10]). For a relatively compact domain in a complex manifold, these metrics might be degenerate (consider $\mathbb{C}^n$ and its compactification $\mathbb{P}^{n+1}$). It is also known that the canonical line bundle is closely related to the Bergman and Kähler–Einstein metrics.

In this paper, we introduce an invariant Kähler metric on complex manifolds with positive canonical bundle, which is in fact the Bergman metric with respect to a certain *weighted* Bergman space. Generally, the Bergman metric with respect to a weighted Bergman space is not invariant under the group of holomorphic transformations. Here we use a special weight originally contained in the work of Narasimhan–Simha [11]. Equipped with this weight, the associated Bergman metric turns out to be invariant.

We focus on the special case of relatively compact domains in complex manifolds. We give a localization principle for the new invariant metric on locally pseudoconvex domains in complex manifolds with positive canonical bundle; consequently, one can reduce the study to the case of bounded domains of holomorphy in $\mathbb{C}^n$. We also give a characterization for the exis-

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tence of this invariant metric on locally pseudoconvex domains in complex manifolds with trivial canonical bundle.

This metric has the advantage of flexibility as regards completeness compared to the standard Bergman metric and is easier to analyze than the Kähler–Einstein metric. In fact, we conjecture that it is complete for any locally pseudoconvex domains in complex manifolds with positive canonical bundle. Although we cannot prove it, we provide a powerful evidence to support this conjecture by showing the completeness of the metric in the "worst" case, i.e., the complement of an effective divisor in a compact complex manifold with ample canonical bundle. It is well-known that if in addition the divisor is ample, then its complement carries a complete (invariant) Kähler–Einstein metric (cf. [7]). A little surprise is that the above two metrics are not equivalent.

The paper is organized as follows: In Section 2, we give the definition of the invariant metric and study its basic properties along the lines of [8]. In Section 3, we give a localization principle for this metric. In Section 4, we study the case of relatively compact Stein domains in a complex manifold with trivial canonical bundle. The last section investigates the asymptotic behavior of the metric on a hypersurface complement in a compact complex manifold with ample canonical bundle.

2. Definition and basic properties. Let M be a complex manifold of dimension n and K_M its canonical line bundle. Let $\Gamma(M, mK_M)$ denote the space of holomorphic sections of $K_M^{\otimes m}$. For every integer $m \geq 1$, we define a continuous pseudonorm $\|\cdot\|_m$ on $\Gamma(M, mK_M)$ by

$$\|s\|_m = \Big\{ \int_M [(-1)^{mn^2/2} s \otimes \bar{s}]^{1/m} \Big\}^{m/2}, \qquad s \in \Gamma(M, mK_M).$$

Consider a volume form τ_m on M defined by

$$\tau_m = \sup_{\|s\|_m = 1} [(-1)^{mn^2/2} s \otimes \bar{s}]^{1/m}.$$

Clearly $1/\tau_m$ is a non-negative Hermitian metric on K_M if τ_m is not identically zero. This metric was introduced by Narasimhan–Simha [11] in order to study the moduli space of compact complex manifolds with ample canonical bundle (for recent applications in complex geometry see Tsuji [14]).

LEMMA 2.1. *The form τ_m is invariant under biholomorphic mappings.*

Proof. Let $f : M \to N$ be a biholomorphic mapping. For any $s \in \Gamma(N, mK_N)$, we have $\|f^* s\|_m = \|s\|_m$ by the transformation formula for integrals. Given $p \in M$, take a sequence $\{s_k\}$ in $\Gamma(N, mK_N)$ such that

$\|s_k\|_m = 1$ and

$$\tau_m(f(p)) = \lim_{k\to\infty} [(-1)^{mn^2/2} s_k(f(p)) \otimes \bar{s}_k(f(p))]^{1/m}.$$

Then

$$\begin{aligned} f^*\tau_m(p) = \tau_m(f(p)) &= \lim_{k\to\infty} [(-1)^{mn^2/2} s_k(f(p)) \otimes \bar{s}_k(f(p))]^{1/m} \\ &= \lim_{k\to\infty} [(-1)^{mn^2/2} f^*s_k(p) \otimes \overline{f^*s_k}(p)]^{1/m} \le \tau_m(p). \end{aligned}$$

The opposite inequality can be obtained similarly.

LEMMA 2.2. *Let M and M' be complex manifolds of complex dimensions n and n' respectively. Then*

$$\tau_m((p, p'), M \times M') = \tau_m(p, M) \otimes \tau_m(p', M').$$

Proof. Take $\{s_k\} \subset \Gamma(M, mK_M)$ and $\{s'_k\} \subset \Gamma(M', mK_{M'})$ such that $\|s_k\|_m = \|s'_k\|_m = 1$ and

$$\begin{aligned} \tau_m(p, M) &= \lim_{k\to\infty} [(-1)^{mn^2/2} s_k(p) \otimes \bar{s}_k(p)]^{1/m}, \\ \tau_m(p, M') &= \lim_{k\to\infty} [(-1)^{mn'^2/2} s'_k(p) \otimes \overline{s'_k}(p)]^{1/m}. \end{aligned}$$

Fubini's theorem implies

$$\|s_k \otimes s'_k\|_m = \|s_k\|_m \cdot \|s'_k\|_m = 1.$$

Thus

$$\begin{aligned} \tau_m((p, p'), M \times M') &\ge \lim_{k\to\infty} [(-1)^{m(n+n')^2/2} s_k \otimes s'_k(p) \otimes \overline{s_k \otimes s'_k}(p)]^{1/m} \\ &= \tau_m(p, M) \otimes \tau_m(p', M'). \end{aligned}$$

On the other hand, for any $S \in \Gamma(M \times M', mK_{M\times M'})$ satisfying $\|S\|_m = 1$, we have

$$\begin{aligned} [(-1)^{m(n+n')^2/2} S \otimes \overline{S}(p, p')]^{1/m} &\le \tau_m(p, M)(-1)^{n'^2/2} \|S(\cdot, p')\|^{2/m}_{m,M} \\ &\le \tau_m(p, M)\tau_m(p', M')\|S\|^{2/m}_{m,M\times M'} \\ &= \tau_m(p, M)\tau_m(p', M'). \end{aligned}$$

Since S is arbitrary, the proof is complete.

Suppose now τ_m is nowhere vanishing. For any integer $k \ge 1$, one can define an inner product of $\Gamma(M, kK_M)$ as follows:

$$(s_1, s_2)_{m,k} = \int_M \frac{(-1)^{kn^2/2} s_1 \otimes \bar{s}_2}{\tau_m^{\otimes(k-1)}}.$$

Let $\|\cdot\|_{m,k}$ denote the induced norm and $H^2_{\tau_m}(M, kK_M)$ the Hilbert space of holomorphic sections of kK_M which are finite with respect to $\|\cdot\|_{m,k}$. Let

$s_1, s_2, \ldots$ be a complete orthonormal basis for $H^2_{\tau_m}(M, kK_M)$. The associated Bergman kernel is

$$B_M(p) = (-1)^{kn^2/2} \sum_j s_j(p) \otimes \bar{s}_j(p).$$

Clearly, B_M is independent of the choice of the basis.

LEMMA 2.3. *B_M is invariant under biholomorphic mappings.*

Proof. Since $f^*\tau_m = \tau_m$ for a biholomorphic mapping f from M to another complex manifold N, one has $\|f^*s\|_{\tau_m} = \|s\|_{\tau_m}$ for all $s \in H^2_{\tau_m}(N, kK_N)$. Note that

$$B_N = \sup[(-1)^{kn^2/2} s \otimes \bar{s}]$$

where the supremum is taken over all $s \in H^2_{\tau_m}(N, kK_N)$ with $\|s\|_{\tau_m} = 1$. The assertion follows by a similar argument to that for Lemma 2.1.

By Lemma 2.2 and a similar argument, we obtain the following product property.

LEMMA 2.4.

$$B_{M\times M'} = B_M \otimes B_{M'}.$$

DEFINITION 2.5. We say that $H^2_{\tau_m}(M, kK_M)$ is *very ample* if the following conditions are satisfied:

(B1) Given any $p \in M$, there exists $s \in H^2_{\tau_m}(M, kK_M)$ such that $s(p) \neq 0$.

(B2) For any holomorphic vector v at p, there exists $s \in H^2_{\tau_m}(M, kK_M)$ such that $s(p) = 0$ and $v(s^*)|_p \neq 0$ where $s = s^* dz_1 \wedge \cdots \wedge dz_n$ in local coordinates.

Write

$$B_M(z) = (-1)^{kn^2/2} B^*_M(z)(dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n)^{\otimes k}$$

where $B^*_M(z)$ is a locally defined function. If B^*_M is nowhere vanishing, we can define a Hermitian form

$$ds^2_M = \sum_{j,k} \frac{\partial^2 \log B^*_M}{\partial z_j \partial \bar{z}_k} \, dz_j \otimes d\bar{z}_k.$$

Clearly, ds^2_M is independent of the choice of the coordinate system, hence is globally defined.

THEOREM 2.6. *If $H^2_{\tau_m}(M, kK_M)$ is very ample, then ds^2_M defines a Kähler metric which is invariant under biholomorphic mappings.*

Proof. Given any $p \in M$ and a holomorphic vector v at p, take a complete orthonormal basis s_j, $j = 1, 2, \ldots$, such that

$$s_2(p) = s_3(p) = \cdots = 0, \quad v(s^*_2)(p) \neq 0, \quad v(s^*_3)(p) = v(s^*_4)(p) = \cdots = 0,$$

where

$$s_j = s_j^*(dz_1 \wedge \cdots \wedge dz_n)^{\otimes k}.$$

Thus

$$B_M^*(p) = |s_1(p)|^2 > 0, \qquad ds_M^2(p; v) = \frac{|v(s_2^*)(p)|^2}{B_M^*(p)} > 0,$$

showing the Hermitian positivity of ds_M^2. By Lemma 2.3, one has

$$B_M^*(p) = B_N^*(f(p))|J_f(p)|^{2k}$$

where J_f is the Jacobian determinant of a biholomorphic mapping $f: M \to N$, and consequently, $f^* ds_N^2 = ds_M^2$.

Lemmas 2.2 and 2.4 imply the following

THEOREM 2.7. $ds^2_{M \times M'} = ds_M^2 + ds_{M'}^2$.

EXAMPLE. Let M be a complex manifold and Ω a relatively compact domain in M. Suppose that K_M is ample on some neighborhood U of $\overline{\Omega}$, i.e., there is an integer m_0 such that $\Gamma(U, m_0 K_M)$ separates points in U and gives a local coordinate system at each point of U. Clearly ds_Ω^2 exists on Ω for any m, k. In particular, it exists on relatively compact domains in a Stein manifold.

REMARK 2.8. Kobayashi's celebrated criterion for completeness of the standard Bergman metric is still valid for this new invariant metric; the proof is exactly the same as in [8].

Recall that a domain Ω in a complex manifold M is called *locally pseudoconvex* if for every $p \in \partial\Omega$, there is a coordinate polydisc Δ^n around p such that $\Omega \cap \Delta^n$ is pseudoconvex in the usual sense. We have the following

BREMERMANN TYPE THEOREM. *Let M be a complex manifold and Ω a relatively compact domain in M. Suppose ds_Ω^2 is a complete Kähler metric on Ω for some m, k. Then Ω is locally pseudoconvex.*

Proof. Since the result is local, we may assume $M = \mathbb{C}^n$. If Ω is not pseudoconvex, then there exists a point $p \in \partial\Omega$ and a neighborhood U of p such that every holomorphic function on Ω extends to $\Omega \cup U$. By the definition of τ_m, τ_m^* extends to a positive continuous function on $\Omega \cap U$, where $\tau_m = (-1)^{n^2/2} \tau_m^* dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$. Since $H^2_{\tau_m}(\Omega, kK_\Omega)$ is identical with the Hilbert space of holomorphic functions on Ω which are square-integrable with the weight $k \log \tau_m^*$, the related Bergman kernel extends to a positive real-analytic function on $\Omega \cup U$, showing ds_Ω^2 cannot be complete at p, a contradiction.

An immediate application is the following

Siegel Type Theorem. *Let M be a complex manifold and Ω a relatively compact domain in M such that ds^2_Ω exists for some m, k. If there exists a properly discontinuous group Γ of holomorphic transformations of Ω such that Ω/Γ is compact, then Ω is locally pseudoconvex.*

Proof. The invariant metric ds^2_Ω descends to a Kähler metric on the compact manifold Ω/Γ, which forces it to be complete on Ω. The above theorem applies.

Remark 2.9. Lárusson [9] has constructed a class of relatively compact domains in $\mathbb{P}^n$ which have compact quotients, but are not locally pseudoconvex.

3. Localization principle. In this section, we give a couple of localization principles.

Theorem 3.1. *Let M be an n-dimensional complex manifold and Ω a relatively compact, locally pseudoconvex domain in M. Suppose that K_M is ample on some neighborhood U of $\overline{\Omega}$. Then for sufficiently large m, k one has, for every $p \in \partial\Omega$,*

$$C_1 ds^2_{\Omega\cap\Delta^n} \le ds^2_\Omega \le C_2 ds^2_{\Omega\cap\Delta^n} \quad \text{on } \Omega \cap \Delta^n_{1/2}.$$

Here Δ^n, $\Delta^n_{1/2}$ denote coordinate polydiscs with center p and radius 1 *and* 1/2 *respectively, and C_1, C_2 are positive constants.*

Proof. Fix a Kähler metric ω on $\overline{\Omega}$ (e.g., ω is generated by $\Gamma(U, m_0 K_M)$). Since Ω is locally pseudoconvex, there is a constant $C_3 > 0$ such that $-\log\delta_\Omega + C_3\omega$ is positive in the sense of currents, according to a theorem of Elencwajg [5]. Here δ_Ω denotes the boundary distance with respect to ω. By a smooth regularization of $-\log\delta_\Omega$, there exists a smooth exhaustion function ψ on Ω such that

$$\psi - \log\psi + C_4\omega$$

gives a complete Kähler metric on Ω for sufficiently large C_4. Given $z_0 \in \Omega \cap \Delta^n_{1/2}$, one can choose $s_1, \ldots, s_l \in \Gamma(U, m_0 K_M)$ such that

$$(-1)^{m_0 n/2} s_j \otimes \bar{s}_j \le C_5 \omega^{\otimes m_0} \quad \text{on } \overline{\Omega}, \qquad \sum s_j \otimes \bar{s}_j \ne 0 \quad \text{on } \Omega \setminus \{z_0\},$$

and

$$\partial\overline{\partial} \sum |s_j^*(z)|^2 \ge \partial\overline{\partial}|z|^2, \quad \forall z \in \Delta^n,$$
$$\log \sum |s_j^*(z)|^2 = \log|z|^2 + O(1) \quad \text{near } 0,$$

where $s_j = s_j^*(dz_1 \wedge \cdots \wedge dz_n)^{\otimes m_0}$ on Δ^n and C_5 is a constant independent of z_0. Clearly, $\tau_{m_0} \ge C\omega$ and its curvature $\Theta_{\tau_{m_0}} > 0$ (it is not known whether

$\Theta_{\tau_{m_0}}$ dominates ω!). Set

$$\varphi = (n+1)\log \frac{(-1)^{m_0 n^2/2}\sum s_j \otimes \overline{s}_j}{\tau_{m_0}^{\otimes m_0}}.$$

Clearly, φ is bounded above by a constant. Set $m_1 = (n+1)m_0$ and define a singular Hermitian metric h^{-1} on $m_1 K_\Omega$ by

$$h = \tau_{m_0}^{\otimes m_1} e^{\varphi}.$$

Then h^{-1} is smooth and positive outside z_0 and $\Theta_{h^{-1}}$ dominates the Euclidean metric on Δ^n. Let χ be a smooth cut-off function such that $\chi|_{(-\infty,2/3]} = 1$ and $\chi|_{[1,\infty)} = 0$. For any local section $s \in H^2_{\tau_{m_0}}(\Omega \cap \Delta^n, (m_1+1)K_{\Delta^n})$, we can solve $\overline{\partial} u = s \otimes \overline{\partial}\chi(|z|)$ by the standard L^2-theory on complete Kähler manifolds (compare [2], [13]) together with the estimate

$$\int_{\Omega} \frac{(-1)^{(m_1+1)n^2/2} u \otimes \overline{u}}{h} \le C_6 \int_{\Omega\cap\Delta^n} \frac{(-1)^{(m_1+1)n^2/2} s \otimes \overline{s}}{\tau_{m_0}^{\otimes(n+1)}}.$$

Since φ is bounded above by a constant, we conclude that $S := \chi(|z|)s - u \in \Gamma(\Omega, (m_1+1)K_\Omega)$ is such that

$$\int_{\Omega} \frac{(-1)^{(m_1+1)n^2/2} S \otimes \overline{S}}{\tau_{m_0}^{\otimes m_1}} \le C_7 \int_{\Omega\cap\Delta^n} \frac{(-1)^{(m_1+1)n^2/2} s \otimes \overline{s}}{\tau_{m_0}^{\otimes m_1}}$$

and $S^*(z_0) = s^*(z_0)$, $(\partial S^*/\partial z_j)(z_0) = (\partial s^*/\partial z_j)(z_0)$ for $j = 1, \ldots, n$, where S^*, s^* are local representations of S, s. Thus the assertion follows.

QUESTION. Is ds^2_Ω complete under the hypothesis of Theorem 3.1?

Next we give a localization principle for holomorphic fiber bundles.

THEOREM 3.2. *Let $\pi : M \to B$ be a holomorphic fiber bundle such that the base B is a compact complex manifold with ample canonical bundle and the typical fiber F is a bounded pseudoconvex domain in $\mathbb{C}^n$. Then for sufficiently large m, k one has, for every $p \in B$ and a coordinate polydisc Δ^l at p,*

$$C_1 ds^2_{\pi^{-1}(\Delta^l)} \le ds^2_M \le C_2 ds^2_{\pi^{-1}(\Delta^l)} \quad \text{on } \pi^{-1}(\Delta^l_{1/2}).$$

Proof. Observe that M carries a complete Kähler metric: as F is a bounded domain of holomorphy, it admits a complete Kähler–Einstein metric ω_F which is invariant under the group of holomorphic transformations of F; one simply chooses the complete metric to be $\omega_F + \omega_B$ with ω_B an arbitrary Kähler metric on B. Similarly, $\tau_m(F)$ is also invariant under the group of holomorphic transformations of F, hence one has the decomposition $\tau_m(M) = \tau_m(B) \otimes \tau_m(F)$ for any $m \ge 1$. A similar application of the L^2-theory on complete Kähler manifolds yields the assertion.

4. More examples on existence. The localization principle provides enough examples for the existence of the invariant metric. In this section, we give more examples.

THEOREM 4.1. *Let M be a complex manifold with trivial canonical bundle* (*e.g. a torus*) *and Ω a relatively compact Stein domain in M. Suppose the following conditions are satisfied*:

(i) *There exists a smooth, strictly plurisubharmonic exhaustion function ψ on Ω such that $C_1 \log 1/\delta_\Omega \le \psi \le C_2 \log 1/\delta_\Omega$ and $\partial\overline{\partial}\psi \ge C_3\omega$, where ω is a fixed Hermitian metric on M and δ_Ω denotes the distance to $\partial\Omega$ with respect to ω.*

(ii) *There is a number $\alpha > 0$ such that $\int_\Omega \delta_\Omega^{-\alpha} dV_\omega < \infty$, where dV_ω denotes the volume form with respect to ω* (*e.g. $\partial\Omega$ is Lipschitz*).

Then ds^2_Ω exists on Ω for sufficiently large m, k.

Theorem 4.1 will be proved by a number of lemmas.

LEMMA 4.2. *For $a > 0$, we define*

$$\mathcal{O}^2_a(\Omega) := \Big\{ f \in \mathcal{O}(\Omega) : \int_\Omega |f|^2 \delta^a_\Omega \, dV_\omega < \infty \Big\}$$

where $\mathcal{O}(\Omega)$ is the sheaf of holomorphic functions on Ω. Then for sufficiently large a, $\mathcal{O}^2_a(\Omega)$ separates points and gives local coordinate systems in Ω. Furthermore, there exist positive numbers a_1, C such that

$$\sup_{f\in \mathcal{O}^2_a(\Omega)} \frac{|f(p)|^2}{\int_\Omega |f|^2 \delta^a_\Omega \, dV_\omega} \ge C\delta_\Omega(p)^{-a_1}, \quad \forall p \in \Omega.$$

Proof. By compactness of $\overline{\Omega}$, for every $p \in \Omega$ we have a coordinate polydisc Δ^n around p such that ω is equivalent to the Euclidean metric on Δ^n with the implicit constants independent of p. By hypothesis (i), we have

$$\partial\overline{\partial}(C\psi + 2(n+1)\chi(|z|)\log|z|) + \mathrm{Ricci}(\omega) > \omega$$

for sufficiently large C. Equipping the anti-canonical bundle $K_M^{\otimes(-1)}$ with the Hermitian metric

$$h = \omega^n e^{-C\psi - 2(n+1)\chi(|z|)\log|z|},$$

and applying the L^2-existence theorem [4] for $K_M^{\otimes(-1)}$-valued $(n,1)$-forms with respect to not necessarily complete metrics in a similar way as above, we conclude that those f in $\mathcal{O}(\Omega)$ satisfying

$$\int_\Omega |f|^2 e^{-C\psi}\, dV_\omega < \infty$$

separate points and give local coordinate systems in Ω. The first assertion follows immediately from $\psi \asymp \log 1/\delta_\Omega$. To show the second assertion, we see that the above argument in fact gives a localization principle for the Bergman kernel of the Hilbert space of holomorphic functions on Ω which are square-integrable with weight $C\psi$, thus one can reduce to the case of bounded pseudoconvex domains in $\mathbb{C}^n$. On the other hand, Demailly's theorem [3] implies that this Bergman kernel always dominates $e^{C\psi}$. This completes the proof.

LEMMA 4.3.

$$f \in \mathcal{O}_a^2(\Omega) \ \Rightarrow\ \sup_{p\in\Omega} |f(p)|^2 \delta_\Omega(p)^{2n+a} < \infty.$$

Proof. Since the assertion is local, we may assume Ω is a bounded domain in $\mathbb{C}^n$ and ω is the Euclidean metric. By the sub-mean-value principle, we have, for arbitrary $p \in \Omega$,

$$|f(p)|^2 \le \frac{1}{\mathrm{Vol}\{B(p;\delta_\Omega(p))\}} \int\limits_{B(p;\delta_\Omega(p)))} |f|^2\,dV \le C_n \delta_\Omega(p)^{-2n-a} \int\limits_\Omega |f|^2 \delta_\Omega^a\,dV$$

where $B(p;r)$ denotes the Euclidean ball with center p and radius r. We are done.

Let us fix a nowhere vanishing holomorphic section s_0 of K_M.

LEMMA 4.4. *If $m \gg a$, then for any $f \in \mathcal{O}_a^2(\Omega)$ we have*

$$s := f s_0^{\otimes m} \in \Gamma(\Omega, mK_\Omega)$$

and

$$\|s\|_m = \Big\{ \int\limits_\Omega [(-1)^{mn^2/2} s \otimes \bar{s}]^{1/m} \Big\}^{m/2} < \infty.$$

Furthermore, τ_m dominates $\delta_\Omega^{-a_1/m}$ where a_1 is as in Lemma 4.2.

Proof. By Lemma 4.3, there is a constant $C > 0$ depending only on the L^2 norm of f in $\mathcal{O}_a^2(\Omega)$ such that $|f|^2 \le C\delta_\Omega^{-2n-a}$ on Ω. By hypothesis (ii) we see that the conclusion holds when

$$m \ge \frac{2n+a}{\alpha}.$$

The second assertion follows from Lemma 2.2.

Proof of Theorem 4.1. Fix $m \gg a \gg 1$ such that the conclusions of Lemmas 4.2 and 4.4 hold. By Lemma 4.4, we claim that for any $f \in \mathcal{O}_a^2(\Omega)$,

$$s := f s_0^{\otimes k} \in H^2_{\tau_m}(\Omega, kK_\Omega)$$

provided

$$\frac{(k-1)a_1}{m} \ge a \quad \text{or} \quad k \ge \frac{am}{a_1} + 1.$$

Therefore, Lemma 4.2 implies the existence of ds_Ω^2.

CONJECTURE. Hypothesis (ii) in Theorem 4.1 is not necessary.

Remark 4.5. It is also interesting to consider relatively compact domains in a complex manifold whose canonical bundle is neither positive nor trivial. For instance, Nemirovskiĭ [12] has constructed a smooth Levi-flat Stein domain in the product of a compact Riemann surface and a one-dimensional torus which is biholomorphically equivalent to the product of a punctured Riemann surface (remove finitely many points from the original Riemann surface) and an annulus. By Theorems 2.7 and 3.1, this domain carries an invariant metric if the genus of the Riemann surface is larger than 1.

5. Asymptotic behavior. In this section, we study the asymptotic behavior of ds^2_Ω when $\partial\Omega$ is relatively simple, namely, we consider the domain $\Omega = M - D$ where M is a compact complex manifold and D is an effective divisor with only simple normal crossings.

Theorem 5.1. *Let M be a compact complex manifold with ample canonical line bundle and D an effective divisor with only simple normal crossings. Then for sufficiently large m, k, ds^2_Ω is a complete Kähler metric on Ω. Furthermore, the distance is equivalent to $-\log\delta_\Omega$ where δ_Ω is the boundary distance with respect to a Hermitian metric on M.*

Remark 5.2. If in addition D is ample, then Ω admits an invariant Kähler–Einstein metric whose distance is equivalent to $\log|\log\delta_\Omega|$ (cf. [7]). Thus ds^2_Ω is not equivalent to the Kähler–Einstein metric.

Proof of Theorem 5.1. First fix a positive smooth Hermitian metric h of K_M such that the curvature $\omega := \Theta(h)$ gives a Kähler metric on M. Write $D = D_1 + \cdots + D_N$ where the irreducible components D_i are smooth and intersect transversely. Let σ_i be a holomorphic section of $[D_i]$ defining D_i and set $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_N$. Let $\|\cdot\|$ denote the norm with respect to a Hermitian metric for $[D_i]$ and also the norm for the product $[D] = [D_1] \otimes \cdots \otimes [D_N]$ associated to the induced metric. Assume $\|\sigma\| < 1$ for simplicity. It is easy to verify that

$$C\omega - \log(-\log\|\sigma\|)$$

defines a complete Kähler metric on Ω for large $C > 0$. Setting

$$H^2_h(\Omega, mK_\Omega) = \left\{ s \in \Gamma(\Omega, mK_\Omega) : \|s\|^2_{h,m} = \int_\Omega \frac{(-1)^{mn^2/2} s \otimes \overline{s}}{h^{\otimes(m-1)}\|\sigma\|^{-1/2}} < \infty \right\}.$$

Let $B_{\Omega,h,m}$ denote the Bergman kernel of $H^2_h(\Omega, mK_\Omega)$, i.e.,

$$B_{\Omega,h,m} = \sup_{\|s\|_{h,m}=1} [(-1)^{mn^2/2} s \otimes \overline{s}].$$

Lemma 5.3. *There is an integer $m_0 > 1$ such that for all $m \geq m_0$, $B_{\Omega,h,m}$ is a smooth Hermitian metric on mK_Ω such that*

$$C'_m\|\sigma\|^{-2} \leq B_{\Omega,h,m}/h^{\otimes m} \leq C''_m\|\sigma\|^{-2}.$$

Proof. By compactness of M, there is an integer $m_0 > 0$ such that for any $m \geq m_0$ and any $p \in \Omega$, there is a coordinate polydisc (Δ^n, z) with $z(p) = 0$ such that the singular Hermitian metric

$$h_{m-1} := h^{\otimes(m-1)} \|\sigma\|^{-1/2} e^{-2(n+1)\chi(|z|)\log|z|}$$

is smooth and positive on $\Omega - \{p\}$ and dominates $\partial\overline{\partial}|z|^2$ on $\Omega \cap \Delta^n$. A standard application of the L^2-theory shows that the localization principle holds for $B_{\Omega,h,m}$. Note that for any $p \in D$, there is a coordinate polydisc Δ^n centered at p such that

$$\Omega \cap \Delta^n = (\Delta^*)^k \times \Delta^{n-k} \tag{1}$$

for some $1 \leq k \leq N$ where Δ^* denotes the punctured disc. It is clear that

$$\begin{aligned} s = s^*(dz_1 \wedge \cdots \wedge dz_n)^{\otimes m} &\in H^2_h(\Omega \cap \Delta^n, mK_\Omega) \\ &\Leftrightarrow\ c_{\alpha_1 \cdots \alpha_n} = 0 \text{ if } \alpha_i < -1 \text{ for some } 1 \leq i \leq k, \end{aligned}$$

where

$$s^* = \sum c_{\alpha_1 \cdots \alpha_n} z_1^{\alpha_1} \cdots z_n^{\alpha_n}$$

is the Laurent expansion. By the localization principle, the assertion follows.

Now fix such an $m_0 > 1$. Let τ_{m_0} be defined as in Section 2. As $\|\sigma\|^{-2/m_0}$ is integrable on Ω, Lemma 5.3 implies $\tau_{m_0}/h \geq C\|\sigma\|^{-2/m_0}$. Let Δ^n be a coordinate polydisc around a boundary point p as in (1) and let $z^* \in \Delta^n \cap \Omega$ be any point. Take $s \in \Gamma(\Omega, m_0 K_\Omega)$ such that

$$\|s\|_{m_0} = \Big\{ \int_\Omega [(-1)^{m_0 n^2/2} s \otimes \overline{s}]^{1/m_0} \Big\}^{m_0/2} = 1$$

and

$$\tau_{m_0}(z^*) \leq 2[(-1)^{m_0 n^2/2} s \otimes \overline{s}(z^*)]^{1/m_0}.$$

Considering the Laurent expansion of s^*, we conclude that there exist positive integers λ_i, $i = 1, \ldots, k$, such that

$$\tau_{m_0}(z^*) \asymp \prod_{1 \leq i \leq k} \|\sigma_i\|^{-2\lambda_i/m_0}(z^*). \tag{2}$$

LEMMA 5.4. *There exists an integer $k_0 \geq 2$ such that $H^2_{\tau_{m_0}}(\Omega, k_0 K_\Omega)$ is very ample and the ratio of its Bergman kernel and $h^{\otimes k_0}$ dominates $\|\sigma\|^{-2}$.*

Proof. The localization principle also holds for the Bergman kernel of the Hilbert space

$$\mathcal{H}_k := \left\{ s \in \Gamma(\Omega, kK_\Omega) : \int_\Omega \frac{(-1)^{kn^2/2} s \otimes \overline{s}}{\tau_{m_0} \otimes h^{\otimes(k-1)}} < \infty \right\}$$

provided k sufficiently large, and its Bergman kernel dominates $\|\sigma\|^{-2}$ according to (2). Since $\mathcal{H}_k \subset H^2_{\tau_{m_0}}(\Omega, kK_\Omega)$, we are done.

End of proof of Theorem 5.1. Lemma 5.4 implies the existence of ds^2_Ω. Consider again the coordinate polydisc Δ^n chosen as in (1). Given any $z^* \in \Omega \cap \Delta^n$, suppose that the Bergman kernel at z^* equals $[(-1)^{k_0 n^2/2} s \otimes \overline{s}(z^*)]^{1/k_0}$ for some $s \in H^2_{\tau_{m_0}}(\Omega, k_0 K_\Omega)$ with unit norm. As above, we conclude from (2) and Lemma 5.4 that there are positive integers $l_1, \ldots, l_k$ such that

$$|s^*(z^*)| \asymp \prod_{1\le i\le k} \|\sigma_i\|^{-l_i}(z^*),$$

$$|(\partial s^*/\partial z_i)(z^*)| \asymp \|\sigma_1\|^{-l_1} \cdots \|\sigma_i\|^{-l_i-1} \cdots \|\sigma_k\|^{-l_k}(z^*) \qquad \forall 1 \le i \le k.$$

The assertion follows immediately from the extremal property of the invariant metric.

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Department of Mathematics
Tongji University
Shanghai 200092, P.R. China
E-mail: boychen@mail.tongji.edu.cn

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