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Disc formulas for the weighted Siciak–Zahariuta extremal function

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Dedicated to Professor Józef Siciak on the occasion of his 75th birthday

Abstract. We prove a disc formula for the weighted Siciak–Zahariuta extremal function $V_{X,q}$ for an upper semicontinuous function q on an open connected subset X in $\mathbb{C}^n$. This function is also known as the weighted Green function with logarithmic pole at infinity and weighted global extremal function.

Introduction. If X is a subset of $\mathbb{C}^n$ and $q : X \to \overline{\mathbb{R}} = [-\infty, \infty]$ is a function, then the *weighted Siciak–Zahariuta extremal function* $V_{X,q}$ *with respect to* q is defined as

$$V_{X,q} = \sup\{u \in \mathcal{L};\ u \le q \text{ on } X\}$$

where $\mathcal{L}$ denotes the Lelong class of all plurisubharmonic functions u on $\mathbb{C}^n$ of minimal growth, i.e., functions u satisfying $u(z) \le \log^+ \|z\| + c_u$, $z \in \mathbb{C}^n$, for some constant c_u. The *Siciak–Zahariuta extremal function* V_X corresponds to the case $q = 0$. The functions V_X and $V_{X,q}$ were first introduced by Siciak in the fundamental paper [8] where he proved his celebrated approximation theorem in several complex variables. The theorem states that for every compact subset X of $\mathbb{C}^n$ such that V_X is continuous, every holomorphic function f on some neighbourhood of X can be approximated uniformly on X by polynomials P_ν of degree less than or equal to ν in such a way that

$$\limsup_{\nu\to\infty} \big(\sup_{z\in X} |f(z) - P_\nu(z)|\big)^{1/\nu} = \varrho < 1$$

if and only if f has a holomorphic extension to the sublevel set $\{z \in \mathbb{C}^n;\ V_X(z) < -\log \varrho\}$.

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The purpose of this paper is to extend the methods of Lárusson and Sigurdsson [3] in order to prove disc envelope formulas for $V_{X,q}$. Our main result is the following

THEOREM 1. *Let X be an open connected subset of $\mathbb{C}^n$ and q be an upper semicontinuous function on X. Then for every $z \in \mathbb{C}^n$,*

$$V_{X,q}(z) = \inf\Big\{ -\sum_{a\in f^{-1}(H_\infty)} \log|a| + \int_{\mathbb{T}} q\circ f\,d\sigma;\quad f \in \mathcal{O}(\overline{\mathbb{D}},\mathbb{P}^n),\ f(\mathbb{T})\subset X,\ f(0)=z\Big\}.$$

Here $\mathbb{P}^n$ is the complex projective space viewed in the usual way as the union of the affine space $\mathbb{C}^n$ and the hyperplane at infinity H_∞, $\mathbb{D}$ and $\mathbb{T}$ are the open unit disc and the unit circle in $\mathbb{C}$, and σ is the normalized arc length measure on $\mathbb{T}$.

Our approach is the following. Based on the observation (see Guedj and Zeriahi [1]) that a function u is in the Lelong class if and only if $(z_0,\dots,z_n)\mapsto u(z_1/z_0,\dots,z_n/z_0)+\log|z_0|$ extends as a plurisubharmonic function from $\mathbb{C}^{n+1}\setminus\{z_0=0\}$ to $\mathbb{C}^{n+1}\setminus\{0\}$, we derive a fundamental inequality $u(z)\le J_q(f)$ for any closed analytic disc mapping the origin to z and the unit circle into X. This inequality defines a disc functional J_q associated to q. Then we define *good* sets of analytic discs with respect to q and observe that Poletsky's theorem implies a disc formula for $V_{X,q}$. From this formula we deduce that $V_{X,q}$ is the envelope of J_q with respect to the class of all closed analytic discs mapping the unit circle into X. This result gives the theorem above.

Notation and some basic results. An *analytic disc* in a manifold Y is a holomorphic map $f:\mathbb{D}\to Y$ from the unit disc $\mathbb{D}$ in $\mathbb{C}$ into Y. We denote the set of all analytic discs in Y by $\mathcal{O}(\mathbb{D},Y)$. A *disc functional* on Y is a map $H:\mathcal{A}\to\overline{\mathbb{R}}$ defined on some subset $\mathcal{A}$ of $\mathcal{O}(\mathbb{D},Y)$ with values in the extended real line $\overline{\mathbb{R}}=[-\infty,\infty]$. The *envelope* $E_{\mathcal{B}}H:Y\to\overline{\mathbb{R}}$ of H with respect to the subclass $\mathcal{B}$ of $\mathcal{A}$ is defined by

$$E_{\mathcal{B}}H(x)=\inf\{H(f);\ f\in\mathcal{B},\ f(0)=x\},\qquad x\in Y.$$

We let $\mathcal{A}_Y$ denote the set of all closed analytic discs in Y, i.e., analytic discs that extend to holomorphic maps in some neighbourhood of the closed unit disc $\overline{\mathbb{D}}$, and for a subset S of Y we let $\mathcal{A}_Y^S$ denote the set of all discs in $\mathcal{A}_Y$ which map the unit circle $\mathbb{T}$ into S.

We let $\mathbb{P}^n$ denote the complex projective space with the natural projection $\pi:\mathbb{C}^{n+1}\setminus\{0\}\to\mathbb{P}^n$, $(z_0,\dots,z_n)\mapsto[z_0:\dots:z_n]$, and we identify $\mathbb{C}^n$ with the subspace of $\mathbb{P}^n$ consisting of all $[z_0:\dots:z_n]$ with $z_0\neq 0$. The hyperplane at infinity H_∞ in $\mathbb{P}^n$ is the projection of $Z_0 \setminus \{0\}$ where Z_0 is the hyperplane in $\mathbb{C}^{n+1}$ defined by the equation $z_0 = 0$.

It is an easy observation that a function $u \in \mathcal{PSH}(\mathbb{C}^n)$ is in the Lelong class $\mathcal{L}$ if and only if the function

$$\widetilde{z} = (z_0, \dots, z_n) \mapsto u \circ \pi(\widetilde{z}) + \log|z_0| = u(z_1/z_0, \dots, z_n/z_0) + \log|z_0| \tag{1}$$

extends as a plurisubharmonic function from $\mathbb{C}^{n+1} \setminus Z_0$ to $\mathbb{C}^{n+1} \setminus \{0\}$. If we denote this extension by v, take $f = [f_0 : \dots : f_n] \in \mathcal{A}_{\mathbb{P}^n}$ with $f(0) = z \in \mathbb{C}^n$, $f(\mathbb{T}) \subset \mathbb{C}^n$, and set $\widetilde{f} = (f_0, \dots, f_n) \in \mathcal{A}_{\mathbb{C}^{n+1}\setminus\{0\}}$, then by subharmonicity of $v \circ \widetilde{f}$ we get

$$u(z) + \log|f_0(0)| = v \circ \widetilde{f}(0) \le \int_{\mathbb{T}} v \circ \widetilde{f}\, d\sigma = \int_{\mathbb{T}} u \circ f\, d\sigma + \int_{\mathbb{T}} \log|f_0|\, d\sigma. \tag{2}$$

Since $f(\mathbb{T}) \subset \mathbb{C}^n$, the set $f(\mathbb{D})$ has finitely many intersection points with H_∞, which means that f_0 has finitely many zeros in $\mathbb{D}$. We write

$$f_0(\zeta) = \prod_{a \in f^{-1}(H_\infty)} \left(\frac{\zeta - a}{1 - \bar{a}\zeta}\right)^{m_{f_0}(a)} g_0(\zeta)$$

where $m_{f_0}(a)$ denotes the multiplicity of a as a zero of f_0, and g_0 is holomorphic and without zeros in some neighbourhood of $\overline{\mathbb{D}}$. We have

$$\log|f_0(0)| = \sum_{a \in f^{-1}(H_\infty)} m_{f_0}(a) \log|a| + \log|g_0(0)|, \tag{3}$$

and since the product has modulus 1 on $\mathbb{T}$ and $\log|g_0|$ is harmonic in some neighbourhood of $\overline{\mathbb{D}}$, we have

$$\int_{\mathbb{T}} \log|f_0|\, d\sigma = \int_{\mathbb{T}} \log|g_0|\, d\sigma = \log|g_0(0)|. \tag{4}$$

By combining (3) and (4) with (2) we arrive at the inequality

$$u(z) \le -\sum_{a \in f^{-1}(H_\infty)} m_{f_0}(a) \log|a| + \int_{\mathbb{T}} u \circ f\, d\sigma. \tag{5}$$

As in [3] we define the disc functional

$$J : \mathcal{O}(\mathbb{D}, \mathbb{P}^n) \to \overline{\mathbb{R}}_+ = [0, \infty], \qquad J(f) = -\sum_{a \in f^{-1}(H_\infty)} m_{f_0}(a) \log|a|,$$

where we take $J(f) = 0$ if $f^{-1}(H_\infty) = \emptyset$. If q is Borel measurable, then we add a mean value term to J and define J_q by

$$J_q : \mathcal{O}(\mathbb{D}, \mathbb{P}^n) \cap C(\overline{\mathbb{D}}, \mathbb{P}^n) \to \overline{\mathbb{R}}, \qquad J_q(f) = J(f) + \int_{\mathbb{T} \cap f^{-1}(X)} q \circ f\, d\sigma.$$

If $f^{-1}(H_\infty)$ is an infinite set the sum is understood as the infimum over all finite subsets, which is well defined since the terms are all negative. In the case when $J(f) = \infty$ and the integral is $-\infty$ we define $J_q(f) = \infty$. If $f(\mathbb{T}) \subset X$, then the sum is finite. For the constant disc k_x, $\overline{\mathbb{D}} \ni \zeta \mapsto x \in X$, we have $J(k_x) = 0$, and hence $J_q(k_x) = q(x)$.

The inequality (5) implies that for every $u \in \mathcal{L}$ with $u \le q$ on X and every $f \in \mathcal{A}_{\mathbb{P}^n}$ with $f(0) = z$ we have

$$u(z) \le J_q(f) + \int_{\mathbb{T}\setminus f^{-1}(X)} u \circ f \, d\sigma.$$

If $f(\mathbb{T}) \subset X$, then the second term on the right hand side vanishes. If we take the supremum over all $u \in \mathcal{L}$ with $u \le q$ on X on the left hand side and the infimum over all $f \in \mathcal{B}$ for some subclass $\mathcal{B} \subseteq \mathcal{A}_{\mathbb{P}^n}^X$ on the right hand side, then we arrive at the inequality

$$V_{X,q}(z) \le E_{\mathcal{A}_{\mathbb{P}^n}^X} J_q(z) \le E_{\mathcal{B}} J_q(z), \qquad z \in \mathbb{C}^n.$$

We will prove that the first inequality is actually an equality:

THEOREM 2. *Let X be an open connected subset of $\mathbb{C}^n$ and $q : X \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function. Then $V_{X,q} = E_{\mathcal{A}_{\mathbb{P}^n}^X} J_q$, i.e., for every $z \in \mathbb{C}^n$ we have*

$$V_{X,q}(z) = \inf\Big\{ -\sum_{a \in f^{-1}(H_\infty)} m_{f_0}(a) \log|a| + \int_{\mathbb{T}} q \circ f \, d\sigma; \\ f \in \mathcal{A}_{\mathbb{P}^n},\ f(\mathbb{T}) \subset X,\ f(0) = z \Big\}.$$

Observe that the formula in Theorem 1 is the same except for the multiplicities. In order to show that Theorem 1 follows from Theorem 2, we first observe that the upper semicontinuity of q implies that for every $\varepsilon > 0$ and every $f \in \mathcal{A}_{\mathbb{P}^n}^X$ there exists a continuous function $\widetilde{q} \ge q$ on X such that $\int_{\mathbb{T}} \widetilde{q} \circ f \, d\sigma < \int_{\mathbb{T}} q \circ f \, d\sigma + \varepsilon$. By Proposition 1 in [3], every $f \in \mathcal{A}_{\mathbb{P}^n}$ can be approximated uniformly on $\overline{\mathbb{D}}$ by $g \in \mathcal{A}_{\mathbb{P}^n}$ such that all the zeros of g_0 are simple, $g(0) = f(0)$, and $J(g) = J(f)$. Since $\widetilde{q}$ is continuous we can choose g such that $\int_{\mathbb{T}} \widetilde{q} \circ g \, d\sigma < \int_{\mathbb{T}} \widetilde{q} \circ f \, d\sigma + \varepsilon$. This gives $J_q(g) \le J_{\widetilde{q}}(g) < J_q(f) + 2\varepsilon$ and we conclude that the infima in Theorems 1 and 2 are equal.

Good sets of analytic discs. We modify the definition from [3] of *good sets* of analytic discs by saying that a subset $\mathcal{B}$ of $\mathcal{A}_{\mathbb{P}^n}$ is *good with respect to the function q* if:

(i) $f(\mathbb{T}) \subset X$ for every $f \in \mathcal{B}$,
(ii) for every $z \in \mathbb{C}^n$, there is a disc in $\mathcal{B}$ with centre z,
(iii) for every $x \in X$, the constant disc at x is in $\mathcal{B}$, and
(iv) the envelope $E_{\mathcal{B}} J_q$ is upper semicontinuous on $\mathbb{C}^n$ and has minimal growth, that is, $E_{\mathcal{B}} J_q - \log^+ \|\cdot\|$ is bounded above on $\mathbb{C}^n$.

Condition (i) implies that $u(z) \le J_q(f)$ for every $u \in \mathcal{L}$ with $u \le q$ and $f \in \mathcal{B}$ with $f(0) = z$; (ii) implies that $E_{\mathcal{B}}J_q(z) < \infty$ for every $z \in \mathbb{C}^n$; (iii) implies that $E_{\mathcal{B}}J_q(x) \le q(x)$ for all $x \in X$; and (iv) implies that $V_{X,q}$ is the largest plurisubharmonic function on $\mathbb{C}^n$ dominated by $E_{\mathcal{B}}J_q$.

Poletsky's theorem states that for every upper semicontinuous function $\psi : Y \to \mathbb{R} \cup \{-\infty\}$ on a complex manifold Y, and every $x \in Y$, we have

$$\sup\{u(x);\, u \in \mathcal{PSH}(Y),\, u \le \psi\} = \inf\Big\{ \int_{\mathbb{T}} \psi \circ h \, d\sigma;\, h \in \mathcal{A}_Y,\, h(0) = x \Big\}.$$

See Poletsky [6], Lárusson and Sigurdsson [4, 5], and Rosay [7]. As a consequence we get a disc formula for $V_{X,q}$:

THEOREM 3. *Let X be an open subset of $\mathbb{C}^n$, $q : X \to \overline{\mathbb{R}}$ be a Borel measurable function, and $\mathcal{B}$ be a good class of analytic discs with respect to q. Then*

$$V_{X,q}(z) = \inf\Big\{ \int_{\mathbb{T}} E_{\mathcal{B}}J_q \circ h \, d\sigma;\, h \in \mathcal{A}_{\mathbb{C}^n},\, h(0) = z \Big\}, \qquad z \in \mathbb{C}^n.$$

The remaining proof. Assume that $q : X \to \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous. From now on we choose $\mathcal{B}$ to be the set of all analytic discs in $\mathbb{P}^n$ which are either a constant disc in X or of the form

$$f_{z,w,r} : \zeta \mapsto w + \frac{\|z-w\| + r\zeta}{r + \|z-w\|\zeta} \, \frac{r}{\|w-z\|} \, (z-w)$$

where $z \in \mathbb{C}^n$, $w \in X \setminus \{z\}$ and $r < \min\{\|z-w\|, d(w, \partial X)\}$.

Observe that $f_{z,w,r}$ maps $\overline{\mathbb{D}}$ into the projective line through z and w, $\mathbb{T}$ is mapped to the circle with centre w and radius r, 0 is mapped to z, and $-r/\|z-w\|$ is mapped into H_∞. The conditions on z, w and r ensure that $f_{z,w,r}(\mathbb{T}) \subset X$ and we have the formula

$$J_q(f_{z,w,r}) = \log(\|z-w\|/r) + \int_{\mathbb{T}} q \circ f_{z,w,r} \, d\sigma. \tag{6}$$

It is obvious that conditions (i)–(iii) in the definition of a good set are satisfied. By (iii) we have $E_{\mathcal{B}}J_q(x) \le q(x)$ for all $x \in X$, and since q is upper semicontinuous, this implies that $E_{\mathcal{B}}J_q$ is upper bounded on every compact subset of X. If we fix $w \in X$ and $r < d(w, \partial X)$, then it follows from (6) that $E_{\mathcal{B}}J_q$ is upper bounded on every compact subset of $\mathbb{C}^n$ and is of minimal growth. The upper semicontinuity of $E_{\mathcal{B}}J_q$ follows from

LEMMA 1. *Assume that $q : X \to \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous. For every $z_0 \in \mathbb{C}^n$ and every $\alpha \in \mathbb{R}$ such that $E_{\mathcal{B}}J_q(z_0) < \alpha$ there exist $w_0 \in \mathbb{C}^n$, $r_0 > 0$, and a neighbourhood U of z_0 such that $0 < r_0 < \min\{\|z - w_0\|, d(w_0, \partial X)\}$ and $J_q(f_{z,w_0,r_0}) < \alpha$ for all $z \in U$.*

Proof. Let $f \in \mathcal{B}$ be such that $f(0) = z_0$ and $J_q(f) < \alpha$. If f is of the form f_{z_0,w_0,r_0} for some $w_0 \in \mathbb{C}^n$ and $0 < r_0 < \min\{d(w_0, \partial X), \|z_0 - w_0\|\}$, then we can choose a continuous function $\widetilde{q} \geq q$ on X such that $J_{\widetilde{q}}(f_{z_0,w_0,r_0}) < \alpha$. The continuity of $\widetilde{q}$ implies that there exists a neighbourhood U of z_0 such that $r_0 < \|z - w_0\|$ and $J_{\widetilde{q}}(f_{z,w_0,r_0}) < \alpha$ for all $z \in U$. Since $J_q \leq J_{\widetilde{q}}$ the statement holds in this case.

Assume now that f is the constant disc z_0. Then $z_0 \in X$ and $J_q(f) = q(z_0) < \alpha$. Since q is upper semicontinuous, there exists $0 < \delta < d(z_0, \partial X)$ such that $q(z) < \alpha$ for all $z \in B(z_0, \delta)$, the ball with centre z_0 and radius δ. Then for every z and w in $B(z_0, \delta/2)$ and $0 < r < \min\{\|z - w\|, \delta/2\}$ we have $\int_{\mathbb{T}} q \circ f_{z,w,r}\, d\sigma < \alpha$. Now choose $w_0 \in B(z_0, \delta/2)$ and $0 < r_0 < \min\{\|z_0 - w_0\|, \delta/2\}$ such that $J_q(f_{z_0,w_0,r_0}) = \log(\|z_0 - w_0\|/r_0) + \int_{\mathbb{T}} q \circ f_{z_0,w_0,r_0}\, d\sigma < \alpha$. The statement now follows as in the first part of the proof. ∎

If $q : X \to \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous and $q_j : X \to \mathbb{R}$ is a decreasing sequence of continuous functions converging to q, then it is obvious that $V_{X,q_j} \searrow V_{X,q}$. It also immediately follows that $J_{q_j}(f) \searrow J_q(f)$ for every $f \in \mathcal{A}^X_{\mathbb{P}^n}$ and as a consequence we get $E_{\mathcal{A}^X_{\mathbb{P}^n}} J_{q_j} \searrow E_{\mathcal{A}^X_{\mathbb{P}^n}} J_q$. This shows that for the proof of Theorem 2 we may assume that q is continuous.

In the previous section we have seen that $V_{X,q} \leq E_{\mathcal{A}^X_{\mathbb{P}^n}} J_q$ and that $V_{X,q}$ is the largest plurisubharmonic function on $\mathbb{C}^n$ dominated by $E_{\mathcal{B}} J_q$. Hence, Theorem 2 is a direct consequence of Theorem 3 and the following

LEMMA 2. *Let X be an open connected subset of $\mathbb{C}^n$, $q : X \to \mathbb{R}$ be continuous, and $\mathcal{B}$ be as above. For every $h \in \mathcal{A}_{\mathbb{C}^n}$, every continuous function $v \geq E_{\mathcal{B}} J_q$ on $\mathbb{C}^n$, and every $\varepsilon > 0$, there exists $g \in \mathcal{A}^X_{\mathbb{P}^n}$ with $g(0) = h(0)$ and*

$$J_q(g) \leq \int_{\mathbb{T}} v \circ h\, d\sigma + \varepsilon.$$

The proof is exactly the same as the proof of the Lemma in [3] with J_q in place of J. We only have to note that if we choose $\varphi : \mathbb{C}^{n+1} \to \mathbb{R}$ with $\varphi(z) = \log|z_0|$, let $f = [f_0 : \cdots : f_n] \in \mathcal{A}_{\mathbb{P}^n}$, and let $\widetilde{f} = (f_0, \ldots, f_n) \in \mathcal{A}_{\mathbb{C}^{n+1}\setminus\{0\}}$ be a lifting of f, then

$$J_q(f) = \int_{\mathbb{T}} (\varphi \circ \widetilde{f} + q \circ \pi \circ \widetilde{f})\, d\sigma - \varphi(\widetilde{f}(0)),$$

and that the last part of the proof holds with $\varphi + q \circ \pi$ in place of φ.

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