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Meromorphic solutions of q-shift difference equations

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Abstract. We establish a q-shift difference analogue of the logarithmic derivative lemma. We also investigate the value distributions of q-shift difference polynomials and the growth of solutions of complex q-shift difference equations.

1. Introduction. A function $f(z)$ is called *meromorphic* if it is analytic in the complex plane except at isolated poles. The *order* $\sigma(f)$ and *hyper-order* $\sigma_2(f)$ are defined by

$$\sigma(f) := \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \quad \sigma_2(f) := \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r}.$$

The *logarithmic density* of a set F_n is defined as follows:

$$\limsup_{r\to\infty} \frac{1}{\log r} \int\limits_{[1,r]\cap F_n} \frac{1}{t}\, dt.$$

In what follows, we assume that the reader is familiar with the basic notation and results of Nevanlinna theory [12, 15].

The well-known logarithmic derivative lemma [7] states that

$$m\left(r, \frac{f'(z)}{f(z)}\right) = o(T(r,f)) = S(r,f) \tag{1.1}$$

outside a possible set of finite linear measure. It is a very useful tool in dealing with uniqueness problems for meromorphic functions. Recently, two important results have been established which are similar to the logarithmic derivative lemma. They can be used to investigate the properties of solutions of difference equations. We state them as follows:

Theorem A ([9, Theorem 2.1]). *Let f be a meromorphic function of finite order, and let $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r,f) \tag{1.2}$$

outside of a possible exceptional set with finite logarithmic measure.

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We also know that Theorem A has been improved by Halburd, Korhonen and Tohge [10]. They proved (1.2) is also true when f is a meromorphic function of hyper-order $\sigma_2(f) < 1$.

Theorem B ([1, Theorem 1.1]). *Let $f(z)$ be a nonconstant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = S(r, f)$$

on a set of logarithmic density 1.

In this paper, a *q-shift* of $f(z)$ is defined by $f(qz+\eta)$, and *q-shift difference polynomials* are defined by

$$G(z) := \sum_{\lambda \in J} b_\lambda(z) \prod_{j=1}^{\tau_\lambda} f(q_{\lambda,j} z + \delta_{\lambda,j})^{\mu_{\lambda,j}}, \tag{1.3}$$

where $\delta_{\lambda,j} \in \mathbb{C}$, at least one $q_{\lambda,j}$ is nonzero, $b_\lambda(z)$ are small functions with respect to $f(z)$, and J is a subset of $\{1, \ldots, n\}$. The *degree* of G is defined

$$d(G) := \max_{\lambda \in J} \sum_{j=1}^{\tau_\lambda} \mu_{\lambda,j}.$$

It is natural to ask the following question when considering generalizations of Theorems A and B.

Question 1.1. Let q, η be fixed complex constants. Under what assumptions on $f(z)$, do we have the following q-shift difference analogue of (1.1):

$$m\left(r, \frac{f(qz+\eta)}{f(z)}\right) = S(r, f)? \tag{1.4}$$

In this paper, we will answer the above question and give some applications of the result. First, we consider the proximity function of $f(qz+\eta)/f(z)$. Then, using the above result, we consider the value distributions of q-shift difference polynomials, which can be seen as q-shift difference analogues of results given by Hayman [11]. Finally, we investigate the growth of meromorphic solutions of q-shift difference equations.

2. q-shift difference analogue of the logarithmic derivative lemma

Theorem 2.1. *Let $f(z)$ be a nonconstant zero-order meromorphic function and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(qz+\eta)}{f(z)}\right) = S(r, f)$$

on a set of logarithmic density 1.

For the proof of Theorem 2.1, we need the following lemma.

LEMMA 2.2 ([13]). *If $T : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function such that*

$$\limsup_{r\to\infty} \frac{\log T(r)}{\log r} = 0, \tag{2.1}$$

then the set $E := \{r : T(C_1 r) \geq C_2 T(r)\}$ has logarithmic density 0 for all $C_1, C_2 > 1$.

Proof of Theorem 2.1. From the definitions of $T(r, f)$, we have

$$T(r, f(qz)) = T(|q|r, f) + O(1). \tag{2.2}$$

If $|q| \leq 1$, then

$$T(|q|r, f) \leq T(r, f) + O(1).$$

If $|q| > 1$, from Lemma 2.2 we have

$$T(|q|r, f) < C_0 T(r, f) + O(1),$$

where $C_0 > 1$ on a set of logarithmic density 1. Thus, we get

$$S(r, f(qz)) = o(T(r, f)) = S(r, f)$$

on a set of logarithmic density 1. From Theorems A and B,

$$\begin{aligned} m\left(r, \frac{f(qz+\eta)}{f(z)}\right) &= m\left(r, \frac{f(qz)}{f(z)}\, \frac{f(qz+\eta)}{f(qz)}\right) \\ &\leq m\left(r, \frac{f(qz)}{f(z)}\right) + m\left(r, \frac{f(qz+\eta)}{f(qz)}\right) = S(r, f) \end{aligned}$$

on a set of logarithmic density 1. We have completed the proof.

For finite nonzero-order meromorphic functions, the conclusion of Theorem 2.1 is not true, as can be seen from the following example.

EXAMPLE 2.3. Let $f(z) = e^z$. Then

$$m\left(r, \frac{f(2z+1)}{f(z)}\right) = em(r, f) = eT(r, f).$$

Similar to the proof of [8, Theorem 3.1] or [1, Theorem 2.1], we can also get the following theorem analogous to the Clunie Lemma [3]. It can be used to investigate the value distribution of zero-order meromorphic solutions of nonlinear q-shift difference equations.

THEOREM 2.4. *Let f be a nonconstant zero-order meromorphic solution of*

$$f(z)^n P(z, f) = Q(z, f),$$

where $P(z, f)$ and $Q(z, f)$ are q-shift difference polynomials in f. If the degree of $Q(z, f)$ as a polynomial in $f(z)$ and its q-shifts is at most n, then

$$m(r, P(z, f)) = S(r, f)$$

on a set of logarithmic density 1.

We also get the following result, which is a difference counterpart to the standard result due to Mohon'ko and Mohon'ko [19]. We omit the proof, which is similar to that of [8, Theorem 3.2].

THEOREM 2.5. *Let f be a nonconstant zero-order meromorphic solution of*

$$P(z,f)=0,$$

where $P(z,f)$ is a q-shift difference polynomial in $f(z)$. If $P(z,a)\not\equiv 0$ for a small function $a(z)$ of $f(z)$, then

$$m\left(r,\frac{1}{f-a}\right)=S(r,f)$$

on a set of logarithmic density 1.

3. Value distributions of q-shift difference polynomials. Using the above theorems, we can investigate the value distributions of q-shift difference polynomials, and we obtain the following results, which can be seen as difference versions of classical results given by Hayman in [11].

THEOREM 3.1. *Let f be a zero-order transcendental meromorphic function with finitely many poles, $n\in\mathbb{N}$, $q\in\mathbb{C}\setminus\{0\}$, $\eta\in\mathbb{C}$, and $R(z)$ be a rational function. Then the difference polynomial $f(z)^n f(qz+\eta)-R(z)$ has infinitely many zeros in the complex plane.*

REMARK 3.2. The condition of zero order cannot be replaced with finite order: if $f(z)=e^{-z}$, $q=-n$ and $\eta=0$, then $f(z)^n f(qz+\eta)-R(z)=1-R(z)$ has only finitely many zeros.

THEOREM 3.3. *Let f be a zero-order transcendental meromorphic function with finitely many poles, $n\geq 2$, $q\in\mathbb{C}\setminus\{0\}$, $\eta\in\mathbb{C}$, and $R(z)$ be a rational function. Then $f(z)^n+f(qz+\eta)-f(z)-R(z)$ has infinitely many zeros.*

REMARK 3.4. The conclusion of Theorem 3.3 is not true for finite-order meromorphic functions: if $f(z)=1-e^z$, $q=n=2$ and $\eta=0$, then $f(z)^n+f(qz+\eta)-f(z)-1=-e^z$ has no zeros. The condition $n\geq 2$ also cannot be replaced by $n\geq 1$: if $f(z)=e^z+z$, q,η are any constants, and $R(z)=qz+\eta$, then $f(qz+\eta)-R(z)=e^{qz+\eta}$ has no zeros.

REMARK 3.5. We have recently studied the case $q=1$ of the above theorems in [18].

Proof of Theorems 3.1 and 3.3. Assume that $f(z)^n f(qz+\eta)-R(z)$ has finitely many zeros. Since $f(z)$ has zero order and finitely many poles, from the Hadamard factorization theorem we have

$$f(z)^n f(qz+\eta)-R(z)=R_1(z), \tag{3.1}$$

where $R_1(z)$ is a rational function. Thus, from the Valiron–Mohon'ko theorem, we obtain

$$T(r, f(qz+\eta)) = nT(r, f) + O(\log r). \tag{3.2}$$

From Theorem 2.4 and (3.1), we get

$$m(r, f(qz+\eta)) = S(r, f) = S(r, f(qz+\eta))$$

on a set of logarithmic density 1, which contradicts the fact that f has finitely many poles and is a transcendental meromorphic function. Thus, we have completed the proof of Theorem 3.1. Similarly, we can get the proof of Theorem 3.3.

If we remove the condition that f has finitely many poles, then we can get the following results.

THEOREM 3.6. *Let f be a zero-order transcendental meromorphic function, $n \geq 6$, $q \in \mathbb{C}\backslash\{0\}$, $\eta \in \mathbb{C}$. Then the difference polynomial $f(z)^n f(qz+\eta) - R(z)$, where $R(z)$ is a nonzero rational function, has infinitely many zeros in the complex plane.*

REMARK 3.7. The condition that $R(z)$ is nonzero cannot be removed. Let $p \in \mathbb{C}\backslash\{0\}$ and $f(z) = \prod_{j=0}^{\infty}(1-p^j z)^{-1}$. Then $f(pz) = (1-z)f(z)$. Since f has infinitely many poles and no zeros, $f(z)^n f(pz)$ has only one zero. We should remark that

$$m\left(r, \frac{1}{f}\right) = \frac{1}{-2\log|p|}(\log r)^2(1+o(1)),$$

and $1/f$ is a transcendental entire function. Thus, the order of f is 0.

THEOREM 3.8. *Let f be a zero-order transcendental meromorphic function, $n \geq 8$, $q \in \mathbb{C} \setminus \{0\}$, and $\eta \in \mathbb{C}$. Then the difference polynomial $f(z)^n + a(z)f(qz+\eta) - a(z)f(z) - R(z)$, where $a(z)$, $R(z)$ are nonzero rational functions, has infinitely many zeros.*

The proofs of Theorems 3.6 and 3.8 are similar. Here, we just give the proof of Theorem 3.8.

Proof of Theorem 3.8. Since f is a zero-order transcendental meromorphic function, by Lemma 2.2 and [2, Theorem 2.1], we obtain

$$\begin{aligned} T(r, f(qz+\eta)) &= T(r, f(q(z+\eta/q))) = T(|q|r, f(z+\eta/q)) + O(1) \\ &\leq (1+\varepsilon)T(r, f(z+\eta/q)) + O(1) \\ &= (1+\varepsilon)T(r, f(z)) + S(r, f) \end{aligned} \tag{3.3}$$

on a set of logarithmic density 1, where $0 < \varepsilon < 1/3$. If $|q| \leq 1$, ε can be removed.

Let us denote

$$\psi := \frac{a(z)f(z) - a(z)f(qz+\eta) + R(z)}{f(z)^n}.$$

If $\psi-1$ has infinitely many zeros, then $f(z)^n+a(z)f(qz+\eta)-a(z)f(z)-R(z)$ has infinitely many zeros. We will show that

$$T(r,\psi) \geq (n-2-\varepsilon)T(r,f) + S(r,f). \tag{3.4}$$

Using the first main theorem and (3.3), we observe that

$$\begin{aligned} T(r,f(z)^n) &= T\left(r, \psi \cdot \frac{1}{a(z)f(z) - a(z)f(qz+\eta) + R(z)}\right) + O(1) \\ &\leq T(r,\psi) + T(r, a(z)f(z) - a(z)f(qz+\eta) + R(z)) + O(1). \\ &\leq T(r,\psi) + (2+\varepsilon)T(r,f) + S(r,f). \end{aligned} \tag{3.5}$$

From (3.5), we easily obtain the inequality (3.4). We will estimate the zeros and poles of ψ:

$$\overline{N}(r,\psi) \leq \overline{N}(r,f(qz+\eta)) + \overline{N}(r,1/f) + S(r,f), \tag{3.6}$$

and

$$\overline{N}(r,1/\psi) \leq \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{a(z)f(z) - a(z)f(qz+\eta) + R(z)}\right). \tag{3.7}$$

Using the second main theorem, (3.6) and (3.7), we get

$$\begin{aligned} (n-2-\varepsilon)T(r,f) &\leq T(r,\psi) + S(r,f) \\ &\leq \overline{N}(r,\psi) + \overline{N}(r,1/\psi) + \overline{N}\left(r, \frac{1}{\psi-1}\right) + S(r,f) \\ &\leq \overline{N}(r,f(qz+\eta)) + \overline{N}(r,1/f) + \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{\psi-1}\right) \\ &\quad + \overline{N}\left(r, \frac{1}{a(z)f(z) - a(z)f(qz+\eta) + R(z)}\right) + S(r,f) \\ &\leq (5+2\varepsilon)T(r,f) + \overline{N}\left(r, \frac{1}{\psi-1}\right) + S(r,f). \end{aligned} \tag{3.8}$$

Since $n \geq 8$, (3.8) implies that $\psi - 1$ has infinitely many zeros, completing the proof.

REMARK 3.9. It seems that the number n in Theorems 3.6 and 3.8 can be reduced, but we have not succeeded in doing that. Some further results on the value distribution of q-difference operators $f(qz+\eta) - f(z)$ can be found in [4].

4. Solutions of q-shift difference equations. There exist some fruitful results in complex differential equations theory [15]. Recently, meromorphic solutions of complex difference equations have become a subject of great interest, due to the apparent role of the existence of such solutions of finite order for the integrability of difference equations. The following result can be found in [14].

THEOREM C. *Let $\eta_1, \dots, \eta_n$ be nonzero complex numbers. If the difference equation*

$$\sum_{i=1}^{n} f(z+\eta_i) = R(z, f(z)) = \frac{a_0(z) + \sum_{i=1}^{p} a_i(z) f^i(z)}{b_0(z) + \sum_{j=1}^{q} b_j(z) f^j(z)} \tag{4.1}$$

with rational coefficients $a_i(z)$, $b_j(z)$ admits a finite-order meromorphic solution $f(z)$, then $\max\{p, q\} \le n$.

The same conclusion holds with $\sum_{i=1}^{n} f(z+\eta_i)$ replaced by $\prod_{i=1}^{n} f(z+\eta_i)$.

QUESTION 4.1. Can we get a similar result when $f(z+\eta_i)$ is replaced by $f(q_i z + \eta_i)$, where $q_i \neq 1$?

The following example shows that the answer is negative.

EXAMPLE 4.2. $f(z) = e^{\pi i z}$ is a finite order solution of the q-shift difference equation $f(2z+2) + f(3z+4) = f(z)^3 + f(z)^2$. It is also a solution of $f(2z+2) \cdot f(3z+4) = f(z)^5$. In both equations, the degree of the left hand side is less than that of the right hand side.

In this part, we investigate the growth of meromorphic solutions of linear q-shift difference equations. Similar results can be found in [2, 17].

THEOREM 4.3. *Let $\eta_0, \dots, \eta_n$ be nonzero complex constants, and $A_0(z)$, $\dots, A_n(z)$ be entire functions of finite order such that $\sigma = \max_{0 \le i \le n} \sigma(A_i)$, and exactly one has finite type strictly greater than the others. Then every meromorphic solution f of*

$$A_n(z) f(q_n z + \eta_n) + \cdots + A_1(z) f(q_1 z + \eta_1) + A_0(z) f(q_0 z + \eta_0) = 0 \tag{4.2}$$

satisfies $\sigma(f) \ge \sigma$.

Proof. If f is an entire function of order $\sigma(f)$ and type $\tau_f < \infty$, then for any $\varepsilon > 0$,

$$M(r, f) = O(e^{(\tau_f + \varepsilon) r^{\sigma(f)}}).$$

From the conditions of Theorem 4.3, without loss of generality, we may assume that

$$\sigma(A_0) = \cdots = \sigma(A_k) = \sigma$$

and

$$\max_{k+1 \le l \le n} \sigma(A_l) = \mu < \sigma, \qquad \tau = \max_{1 \le j \le k} \{\tau(A_j)\} < \tau(A_0).$$

Suppose that $f(z)$ is a meromorphic solution of (4.2) and $\sigma(f) < \sigma$. Choose $\varepsilon > 0$ small enough to satisfy $\mu + \varepsilon < \sigma$, $\sigma(f) + \varepsilon < \sigma$ and pick ς satisfying $\tau < \varsigma < \varsigma + 3\varepsilon < \tau(A_0)$. Dividing (4.2) by $f(q_0 z + \eta_0)$, we get

$$M(r, A_0) \le e^{r^{\sigma(f)+\varepsilon}}(O(e^{(\tau+\varepsilon)r^{\sigma}}) + O(e^{r^{\mu+\varepsilon}})) = O(e^{(\varsigma+2\varepsilon)r^{\sigma}})$$

outside of a possible exceptional set of finite logarithmic measure. Hence $\tau(A_0) \le \varsigma + 2\varepsilon$, a contradiction. Thus, we have completed the proof.

For the nonhomogeneous q-shift difference equation

$$A_n(z)f(q_n z+\eta_n)+\cdots+A_1(z)f(q_1 z+\eta_1)+A_0(z)f(q_0 z+\eta_0) = H(z), \tag{4.3}$$

it is easy to see that if $\sigma(H(z)) > \max \sigma(A_i)$, then $\sigma(f) \ge \sigma(H(z)) > \sigma(A_i)$. The following examples show that there is no exact relation between $\sigma(f)$ and $\sigma(A_i)$ when $\sigma(H(z)) \le \max \sigma(A_i)$, $i = 1, \dots, n$.

Example 4.4. Consider the nonhomogeneous difference equation

$$(e^{z+1} + 1)f(z + 1) - ef(z + 2) = 1. \tag{4.4}$$

It is easy to see that $f(z) = e^{-z}$ solves (4.4) and $\sigma(f) = 1 = \sigma(e^{z+1} + 1)$.

Example 4.5. Consider the nonhomogeneous difference equation

$$e^{-z}f(z) + e^{z^2}f(2z + 1) = e^{z^2+2z+1} + 1. \tag{4.5}$$

Then $f(z) = e^{z}$ solves (4.5) and $\sigma(f) = 1 < \sigma(e^{z^2})$.

Example 4.6. Consider the nonhomogeneous difference equation

$$e^{-2z-1}f(z + 1) - e^{-4z-4}f(z + 2) = e^{-2z-1}(z + 1) - e^{-4z-4}(z + 2). \tag{4.6}$$

It is easy to see that $f(z) = e^{z^2} + z$ solves (4.6) and $\sigma(f) = 2 > \sigma(e^{-2z-1})$.

We now continue to investigate the growth of solutions of nonlinear q-shift difference equations of certain forms, using an idea from [16]. We need the following lemmas.

Lemma 4.7 ([5]). *Let $p(z) = a_k z^k + a_{k-1}z^{k-1} + \cdots + a_1 z + a_0$, $a_k \ne 0$, be a nonconstant polynomial of degree k and let f be a transcendental meromorphic function. Given $0 < \varrho < |a_k|$, denote $\zeta = |a_k| + \varrho$ and $\eta = |a_k| - \varrho$. Then, given $\varepsilon > 0$, we have*

$$(1 - \varepsilon)T(\eta r^k, f) \le T(r, f \circ p) \le (1 + \varepsilon)T(\zeta r^k, f)$$

for all r large enough.

Lemma 4.8 ([6]). *Let $\psi : [r_0, \infty) \to (0, \infty)$ be positive and bounded in every finite interval, and suppose that $\psi(\mu r^m) \le A\psi(r) + B$ for all r large enough, where $\mu > 0$, $m > 1$, $A > 1$ and B are real constants. Then*

$$\psi(r) = O((\log r)^{\alpha}),$$

where $\alpha = \log A/\log m$.

THEOREM 4.9. *Let $c \neq 0$ be a complex constant, and f be a transcendental meromorphic solution of the equation*

$$R(cz+\eta, f(cz+\eta)) = f(p(z)), \tag{4.7}$$

where $p(z)$ is a polynomial of degree $k \geq 2$, and $R(z,y)$ is a rational function in z, y with rational coefficients such that $R(z,y)$ is irreducible in y. If $\deg_f R = n \geq k$, then

$$T(r,f) = O((\log r)^{\alpha+\varepsilon}),$$

where $\alpha = \log n/\log k$.

Proof. Firstly, we replace z by $(z-\eta)/c$ in (4.7); then applying the Valiron–Mohon'ko theorem to the left hand side of (4.7), and Lemma 4.7, we obtain

$$\begin{aligned} nT(r,f) + S(r,f) = T(r,R) = T\left(r, f\left(p\left(\frac{z-\eta}{c}\right)\right)\right) &= T(r, f(q(z))) \\ &\geq (1-\varepsilon)T(\mu r^k, f), \end{aligned} \tag{4.8}$$

where $q(z)$ is a polynomial such that $\deg p(z) = \deg q(z) = k$. Since we may assume that r is large enough to satisfy

$$n(1+\varepsilon)T(r,f) \geq (1-\varepsilon)T(\mu r^k, f)$$

outside of a possible exceptional set of finite linear measure, we conclude that for every $\lambda > 1$ there exists an $r_0 > 0$ such that

$$n(1+\varepsilon)T(\lambda r, f) \geq (1-\varepsilon)T(\mu r^k, f) \tag{4.9}$$

for all $r \geq r_0$. Denoting $t = \lambda r$, (4.9) can be written as

$$T\left(\frac{\mu}{\lambda^k} t^k, f\right) \leq \frac{n(1+\varepsilon)}{1-\varepsilon} T(t,f).$$

Then we apply Lemma 4.8 to conclude that $T(r,f) = O((\log r)^{\alpha+\varepsilon})$, and

$$\alpha + \varepsilon = \frac{\log[n(1+\varepsilon)/(1-\varepsilon)]}{\log k} = \frac{\log n}{\log k} + o(1).$$

THEOREM 4.10. *Suppose that f is a transcendental meromorphic solution of*

$$f(cz+\eta) = R(z, f(z)) = \frac{\sum_{i=0}^{p} a_i(z) f(z)^i}{\sum_{j=0}^{q} b_j(z) f(z)^j}, \tag{4.10}$$

where the coefficients $a_i(z)$, $b_j(z)$ are rational, and $|c| > 1$. Assume that $R(z,f)$ is irreducible in f, and $a_p(z) b_q(z) \not\equiv 0$. If $p > q+1$, then $\lambda(r, 1/f) \geq \log|m|/\log|c|$, provided that f has infinitely many poles.

Proof. Set $m = p-q \geq 2$. Choose a pole z_0 of f of multiplicity $\nu \geq 1$ such that z_0 is neither a zero nor a pole of any coefficient of $R(z,f)$. Then the right hand side of (4.10) has a pole of multiplicity $m\nu$ at z_0. Therefore, there exists a pole of $f(z)$ of multiplicity $m\nu$ at $cz_0 + \eta$. Then, we may continue inductively to construct a pole $c^k z_0 + c^{k-1}\eta + \cdots + c\eta + \eta$ of f with multiplicity $m^k\nu \to \infty$ as $k \to \infty$, unless the process terminates because a zero or a pole of a coefficient of $R(z, f)$ appears for some $c^k z_0 + c^{k-1}\eta + \cdots + c\eta + \eta$. Since the coefficients of $R(z, f)$ are rational, they have just finitely many zeros and poles. Therefore, we may avoid this situation happening by a suitable choice of z_0. Let $r_0 = |c^k|\,|z_0| + |c^{k-1}\eta| + \cdots + |c\eta| + |\eta|$ and let $k \to \infty$. It is clear that, for k large enough, $n(r_0, f) \geq m^k\nu$. Thus, for each sufficiently large r, there exists $k \in \mathbb{N}$ such that $r \in [|c^k|\,|z_0| + |c^{k-1}\eta| + \cdots + |c\eta| + |\eta|, |c^{k+1}|\,|z_0| + k|c^{k-1}\eta|]$. We deduce that

$$n(r, f) \geq \nu m^{\frac{\log r - \log|cz_0| - \log 2}{\log|c|}} \quad \text{or} \quad n(r, f) \geq \nu m^{\frac{\log r - \log|\eta| - \log 2 + \log|c|}{\log|c|}}$$

for all $r \geq r_0$, and we immediately obtain the conclusion.

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