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Uniqueness theorems for meromorphic functions concerning fixed points

by Xiu-Qing Lin (Ningde) and Wei-Chuan Lin (Fuzhou)

Abstract. This paper is devoted to the study of uniqueness of meromorphic functions sharing only one value or fixed points. We improve some related results due to J. L. Zhang [Comput. Math. Appl. 56 (2008), 3079–3087] and M. L. Fang [Comput. Math. Appl. 44 (2002), 823–831], and we supplement some results given by M. L. Fang and X. H. Hua [J. Nanjing Univ. Math. Biquart. 13 (1996), 44–48] and by C. C. Yang and X. H. Hua [Ann. Acad. Sci. Fenn. Math. 22 (1997), 395–406].

1. Introduction and main results. In this paper, a meromorphic function always means a function which is meromorphic in the whole complex plane. Let $f(z)$ be a nonconstant meromorphic function. We shall use the standard notations of Nevanlinna's value distribution theory such as $T(r,f)$, $N(r,f)$, $\overline{N}(r,f)$ and $m(r,f)$ (see [10, 14]). The notation $S(r,f)$ stands for any quantity satisfying

$$S(r,f) = o\{T(r,f)\}$$

as $r \to +\infty$, possibly outside a set of finite linear measure. A meromorphic function $\alpha(z)$ is called a *small function* of $f(z)$ provided that $T(r,\alpha) = S(r,f)$. As usual, we say that two meromorphic functions f and g *share the small function* α *IM* (ignoring multiplicity) when $f(z)-\alpha(z)$ and $g(z)-\alpha(z)$ have the same zeros. If $f(z)-\alpha(z)$ and $g(z)-\alpha(z)$ have the same zeros with the same multiplicity, then we say that f and g *share* α *CM* (counting multiplicity). In particular, when $\alpha(z) = z$, we also say that f and g *have the same fixed points* if f and g share z CM.

Let p be a positive integer, and let $a \in \mathbb{C}\cup\{\infty\}$. We denote by $N_p\big(r, \frac{1}{f-a}\big)$ the counting function of the zeros of $f-a$, where an a-point with multiplicity m is counted m times if $m \le p$ and p times if $m > p$.

Hayman [9], Clunie [4] and Chen and Fang [3] proved the following result.

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Theorem A. *Let f be a transcendental meromorphic function, and $n \geq 1$ an integer. Then $f^n f' = 1$ has infinitely many zeros.*

Fang and Hua [7] and Yang and Hua [13] obtained a unicity theorem corresponding to the above result.

Theorem B. *Let f and g be nonconstant meromorphic functions, and $n \geq 11$ an integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, or $f(z) \equiv t g(z)$ for a constant t such that $t^{n+1} = 1$.*

On the other hand, concerning the value distribution of differential polynomials in f, Hennekemper [8], Chen [2] and Wang [12] proved the following theorem.

Theorem C. *Let f be a transcendental entire function, and let n, k be positive integers with $n \geq k+1$. Then $(f^n)^{(k)} = 1$ has infinitely many zeros.*

Fang [6] proved the following unicity theorem corresponding to the above result.

Theorem D. *Let f and g be nonconstant entire functions, and let n, k be positive integers with $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM, then either $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$, or $f(z) \equiv t g(z)$ for a constant t such that $t^n = 1$.*

Naturally, one can ask whether there exist results for meromorphic functions corresponding to Theorems C and D respectively. Recently, a result similar to Theorem D appeared in [1, Theorem 2]; unfortunately, the proof there contains an incorrect detail. (See the final section in [15].)

In [15], Zhang obtained the following result concerning fixed points of differential polynomials for entire functions.

Theorem E. *Let f and g be nonconstant entire functions, and let n, k be positive integers with $n > 2k+4$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM, then either*

(1) *$k = 1$, $f(z) = c_1 e^{cz^2}$ and $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are constants satisfying $4(c_1 c_2)^n (nc)^2 = -1$, or*
(2) *$f \equiv tg$ for a constant t such that $t^n = 1$.*

In this paper, we get the following theorems for meromorphic functions improving Theorems D, E and C, which are of interest in themselves. Also we supplement Theorem B.

Theorem 1.1. *Let f and g be nonconstant meromorphic functions, and let n, k be positive integers. Suppose that $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 1 CM.*

(1) *If* $N(r, f) \neq S(r, f)$ *and* $n > 3k+8$ *then* $f \equiv tg$ *for a constant* t *such that* $t^n = 1$.

(2) *If* $f \neq \infty$ *and* $n \geq \frac{5}{2}k+6$, *then either* $f(z) = c_1 e^{cz}$ *and* $g(z) = c_2 e^{-cz}$, *where* c_1, c_2 *and* c *are constants satisfying* $(-1)^k (c_1c_2)^n (nc)^{2k} = 1$, *or* $f(z) \equiv tg(z)$ *for a constant* t *such that* $t^n = 1$.

Corollary. *Let* g *be a nonconstant meromorphic function and* f *be an entire function, and let* n, k *be positive integers such that* $n \geq \frac{5}{2}k + 6$. *If* $(f^n)^{(k)}$ *and* $(g^n)^{(k)}$ *share* 1 *CM, then either* $f(z) = c_1 e^{cz}$ *and* $g(z) = c_2 e^{-cz}$, *where* c_1, c_2 *and* c *are constants satisfying* $(-1)^k (c_1c_2)^n (nc)^{2k} = 1$, *or* $f(z) \equiv tg(z)$ *for a constant* t *such that* $t^n = 1$.

Theorem 1.2. *Let* f *and* g *be nonconstant meromorphic functions, and let* n, k *be positive integers. Suppose that* $(f^n)^{(k)}$ *and* $(g^n)^{(k)}$ *share* z *CM.*

(1) *If* $N(r, f) \neq S(r, f)$ *and* f *has infinitely many poles, and if* $n > 3k+8$, *then* $f \equiv tg$ *for a constant* t *such that* $t^n = 1$.

(2) *If* $N(r, f) = S(r, f)$ *and* f *has finitely many poles, and if* $n > 2k + 4$ *and* $g \neq \infty$, *then the conclusion of Theorem E holds.*

In order to prove the above results, we shall first prove the following two theorems.

Theorem 1.3. *Let* f *be a transcendental meromorphic function, and let* n, k *be positive integers. If either*

(i) $k \geq 2$ *and* $n > 2$, *or*

(ii) $k = 1$ *and* $n > 1$,

then $(f^n)^{(k)} = 1$ *has infinitely many zeros.*

Theorem 1.4. *Let* f *be a transcendental meromorphic function, and* n, k *be positive integers with* $n > k + 2$. *Then* $(f^n)^{(k)}$ *has infinitely many fixed points.*

2. Lemmas. For the proof of our results, we need the following lemmas.

Lemma 2.1 (see [16, 11]). *Let* f *be a nonconstant meromorphic function, and* p, k *be positive integers. Then*

$$N_p(r, 1/f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 1/f) + S(r, f), \tag{2.1}$$

$$N_p(r, 1/f^{(k)}) \leq N_{p+k}(r, 1/f) + k\overline{N}(r, f) + S(r, f), \tag{2.2}$$

$$N(r, 1/f^{(k)}) \leq N(r, 1/f) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.2 (see [9]). *Let* f *be a nonconstant meromorphic function and* n *be a positive integer. Suppose* $P(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0$, *where* a_i *are meromorphic functions such that* $T(r, a_i) = S(r, f)$ $(i = 0, 1, \ldots, n)$

and $a_n \not\equiv 0$. *Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

LEMMA 2.3 (see [10]). *Let f be a nonconstant meromorphic function. Then for each positive integer k,*

$$\frac{f^{(k)}}{f} = \left(\frac{f'}{f}\right)^k + \frac{k(k-1)}{2}\left(\frac{f'}{f}\right)^{k-2}\left(\frac{f'}{f}\right)' + \frac{k(k-1)(k-2)}{6}\left(\frac{f'}{f}\right)^{k-3}\left(\frac{f'}{f}\right)'' + P_{k-2}\left(\frac{f'}{f}\right),$$

where $P_{k-2}(f'/f)$ is a polynomial in f'/f and its derivatives with constant coefficients and of total degree $\leq k-2$.

LEMMA 2.4 (see [10]). *Suppose that f is a nonconstant meromorphic function and k is a positive integer. Then*

$$T(r, f) \leq \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - 1}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f).$$

LEMMA 2.5 (see [12]). *Suppose that f is a transcendental meromorphic function, $k \geq 3$ is an integer and $\varepsilon > 0$. Then*

$$(k-2)\overline{N}(r, f) + N(r, 1/f) \leq 2\overline{N}(r, 1/f) + N(r, 1/f^{(k)}) + \varepsilon T(r, f) + S(r, f).$$

LEMMA 2.6 (see [14, 13]). *Let F and G be nonconstant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds:*

(i) $T(r, F) \leq N_2(r, 1/F) + N_2(r, 1/G) + N_2(r, F) + N_2(r, G) + S(r, F) + S(r, G)$, *the same inequality holding for $T(r, G)$;*
(ii) $F \equiv G$;
(iii) $FG \equiv 1$.

LEMMA 2.7 (see [10]). *Suppose that f is a nonconstant meromorphic function, and $k \geq 2$ is an integer. If*

$$N(r, f) + N(r, 1/f) + N(r, 1/f^{(k)}) = S(r, f'/f),$$

then $f = e^{az+b}$, where $a \neq 0$ and b are constants.

LEMMA 2.8. *Suppose f and g are nonconstant meromorphic functions, and n, k are positive integers. Let $F = (f^n)^{(k)}, G = (g^n)^{(k)}$, and suppose there exists a nonzero constant c such that $F = G + c$.*

(i) *If $\overline{N}(r, f) = S(r, f)$, then $n \leq 2(k+1)$.*
(ii) *If $N(r, f) \neq S(r, f)$, then $n \leq 3(k+1)$.*

Proof. By the second fundamental theorem, we get

$$T(r,F) \le \overline{N}(r,F) + \overline{N}(r,1/F) + \overline{N}\left(r,\frac{1}{F-c}\right) + S(r,F) \tag{2.3}$$
$$= \overline{N}(r,F) + \overline{N}(r,1/F) + \overline{N}(r,1/G) + S(r,F).$$

Applying (2.2) to the function g^n for $p=1$, we get

$$\overline{N}(r,1/G) \le k\overline{N}(r,g) + N_{k+1}(r,1/g^n) + S(r,g) \tag{2.4}$$
$$\le k\overline{N}(r,g) + (k+1)\overline{N}(r,1/g) + S(r,g).$$

By Lemma 2.2 and applying (2.1) to the function f^n for $p=1$, we get

$$nT(r,f) = T(r,f^n) + S(r,f) \tag{2.5}$$
$$\le T(r,(f^n)^{(k)}) - \overline{N}(r,1/(f^n)^{(k)}) + N_{k+1}(r,1/f^n) + S(r,f)$$
$$\le T(r,F) - \overline{N}(r,1/F) + (k+1)\overline{N}(r,1/f) + S(r,f).$$

It follows from (2.3)–(2.5) that

$$nT(r,f) \le \overline{N}(r,f) + k\overline{N}(r,g) + (k+1)\overline{N}(r,1/g) + (k+1)\overline{N}(r,1/f)$$
$$+ S(r,f) + S(r,g).$$

Similarly,

$$nT(r,g) \le \overline{N}(r,g) + k\overline{N}(r,f) + (k+1)\overline{N}(r,1/f) + (k+1)\overline{N}(r,1/g)$$
$$+ S(r,f) + S(r,g).$$

The above two inequalities yield

$$n(T(r,f)+T(r,g))$$
$$\le (k+1)(\overline{N}(r,f)+\overline{N}(r,g)) + 2(k+1)(\overline{N}(r,1/f)+\overline{N}(r,1/g))$$
$$+ S(r,f) + S(r,g)$$
$$\le (k+1)(\overline{N}(r,f)+\overline{N}(r,g)) + 2(k+1)(T(r,f)+T(r,g))$$
$$+ S(r,f) + S(r,g).$$

From this and the condition $F = G + c$, we easily obtain the desired result. ■

LEMMA 2.9. *Let f and g be nonconstant meromorphic functions, and let n,k be positive integers. If $(f^n)^{(k)} \equiv (g^n)^{(k)}$, and either*

(i) $\overline{N}(r,f) = S(r,f)$ *and* $n > 2(k+1)$, *or*
(ii) $N(r,f) \ne S(r,f)$ *and* $n > 3(k+1)$,

then $f \equiv tg$, where t is a constant satisfying $t^n = 1$.

Proof. Since $(f^n)^{(k)} \equiv (g^n)^{(k)}$, by integration we get

$$(f^n)^{(k-1)} \equiv (g^n)^{(k-1)} + c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \ne 0$ and either (i) or (ii) holds, then applying Lemma 2.8 we always get a contradiction. Hence $c_{k-1} = 0$. Repeating the same process $k-1$ times, we arrive at

$$f^n \equiv g^n.$$

Thus $f \equiv tg$, where t is a constant satisfying $t^n = 1$. ∎

LEMMA 2.10. *Let f and g be transcendental meromorphic functions with finitely many poles, and let n, k be positive integers with $n > 2k+2$. If*

$$(f^n)^{(k)}(g^n)^{(k)} = \varphi(z), \tag{2.6}$$

where $\varphi(z) = z^2$ or $\varphi(z) \equiv 1$, then:

(i) $f \neq 0,\ g \neq 0$;
(ii) $f = e^\alpha/P$ *and* $g = e^\beta/Q$, *where* α, β, P, Q *are polynomials and* $\alpha, \beta \not\equiv \text{const}$;
(iii) *if* $g \neq \infty$, *then* $f = e^\alpha/P$ *and* $g = ce^{-\alpha}$, *where c is a nonzero constant and α, P are given by* (ii).

Proof. In fact, suppose that f has a zero z_0 with multiplicity m. Then z_0 must be a zero of $(f^n)^{(k)}$ with multiplicity $nm-k$. Since $nm-k \geq n-k > 2$ and $\deg \varphi \leq 2$, by (2.6) we deduce that z_0 must be a pole of g (with multiplicity q, say), thus $(nm-k)-(nq+k) \leq 2$, i.e., $n(m-q) \leq 2k+2$. This is impossible since $n > 2k+2$. So $f \neq 0$, similarly $g \neq 0$, and (i) holds.

Now we may suppose that

$$f(z) = \frac{e^{\alpha(z)}}{P(z)}, \qquad g(z) = \frac{e^{\beta(z)}}{Q(z)}, \tag{2.7}$$

where α, β are nonconstant entire functions and P, Q are polynomials.

First we consider the case when $k \geq 2$. From (2.6) and the assumption,

$$N(r, 1/(f^n)^{(k)}) = N(r, (g^n)^{(k)}/\varphi) \leq N(r, (g^n)^{(k)}) + N(r, 1/\varphi) = O(\log r),$$

which yields

$$N(r, 1/f^n) + N(r, f^n) + N(r, 1/(f^n)^{(k)}) = O(\log r). \tag{2.8}$$

Noting that

$$T(r, (f^n)'/f^n) = T(r, nf'/f) = T(r, n(\alpha' - P'/P)), \tag{2.9}$$

if α is a transcendental entire function, by (2.8), (2.9) and applying Lemma 2.7 we get $f = e^{az+b}$, where $a \neq 0$ and b are constants, which contradicts (2.7). Hence α must be a polynomial, and similarly β is also a polynomial.

Now we consider the case when $k = 1$. Using the theorem on the characteristic and the order, we know that $\sigma(f) = \sigma(f^n) = \sigma((f^n)^{(k)})$, where $\sigma(f)$ denotes the order of f (see [14, Theorem 1.21 and Corollary]). Now in view of (2.6) and (2.7) we see that α and β are either both transcendental entire functions or both polynomials. From (2.6) and (2.7), we get

$$n^2 e^{n(\alpha+\beta)}(\alpha' - P'/P)(\beta' - Q'/Q) = (PQ)^n \varphi. \tag{2.10}$$

It follows that both $\alpha' - P'/P$ and $\beta' - Q'/Q$ have only finitely many zeros and poles. If α and β are transcendental entire functions, set

$$\alpha' - \frac{P'}{P} = \frac{h_1}{h_2}e^{\delta}, \qquad \beta' - \frac{Q'}{Q} = \frac{h_3}{h_4}e^{\gamma}, \tag{2.11}$$

where δ, γ are nonconstant entire functions, and h_i $(i = 1, 2, 3, 4)$ are nonzero polynomials. From this and (2.10), we have

$$n^2 e^{n(\alpha+\beta)+\delta+\gamma} h_1 h_2 = (PQ)^n h_3 h_4 \varphi.$$

Thus $e^{n(\alpha+\beta)+\delta+\gamma} \equiv$ const. Differentiating this yields

$$n(\alpha' + \beta') + \delta' + \gamma' \equiv 0. \tag{2.12}$$

Substituting (2.11) into (2.12), we get

$$n\left(\frac{P'}{P} + \frac{h_1}{h_2}e^{\delta}\right) + \delta' = -n\left(\frac{Q'}{Q} + \frac{h_3}{h_4}e^{\gamma}\right) - \gamma'. \tag{2.13}$$

Since $T(r, \delta') = S(r, e^{\delta})$ and $T(r, \gamma') = S(r, e^{\gamma})$, (2.13) implies that

$$S(r, e^{\delta}) = S(r, e^{\gamma}) =: S(r). \tag{2.14}$$

Let

$$\omega = -n\left(\frac{P'}{P} + \frac{Q'}{Q}\right) - (\delta' + \gamma').$$

Then $T(r, \omega) = S(r)$ by (2.14), and (2.13) can be written as

$$\frac{h_1}{h_2}e^{\delta} + \frac{h_3}{h_4}e^{\gamma} = \frac{\omega}{n}.$$

If $\omega \not\equiv 0$, by the second fundamental theorem and the above equality, we get

$$\begin{aligned} T(r, e^{\delta}) &= T\left(r, \frac{\frac{h_1}{h_2}e^{\delta}}{\omega}\right) + S(r) \\ &\leq \overline{N}\left(r, \frac{\frac{h_1}{h_2}e^{\delta}}{\omega}\right) + \overline{N}\left(r, \frac{1}{\frac{\frac{h_1}{h_2}e^{\delta}}{\omega}}\right) + \overline{N}\left(r, \frac{1}{\frac{\frac{h_1}{h_2}e^{\delta}}{\omega} - \frac{1}{n}}\right) + S(r) \\ &= \overline{N}\left(r, \frac{1}{\frac{\frac{h_3}{h_4}e^{\gamma}}{\omega}}\right) + S(r) = S(r), \end{aligned}$$

which is a contradiction by (2.14). Therefore $\omega \equiv 0$, i.e.,

$$\frac{h_1}{h_2}e^{\delta} + \frac{h_3}{h_4}e^{\gamma} \equiv 0. \tag{2.15}$$

This together with (2.11) yields

$$\alpha' + \beta' = \frac{P'}{P} + \frac{Q'}{Q}.$$

Since α, β are entire functions, the above equality shows $\alpha' + \beta' \equiv 0$. It follows from this and (2.12) that $\delta' + \gamma' \equiv 0$. This and (2.15) imply that both δ and γ are constants, which contradicts (2.11). Hence α and β are polynomials, and (ii) is proved.

If $g \neq \infty$, then from (i) and (ii), we have

$$f = e^{\alpha}/P, \quad g = e^{\beta}, \tag{2.16}$$

where α, β, P are polynomials and $\alpha, \beta \not\equiv$ const.

By (2.16) and applying Lemma 2.3 to the function f^n and g^n respectively, we get

$$(f^n)^{(k)} = R_1(\alpha', \alpha'', \ldots, \alpha^{(k)}, P)e^{n\alpha}, \quad (g^n)^{(k)} = R_2(\beta', \beta'', \ldots, \beta^{(k)})e^{n\beta},$$

where R_1 is a differential polynomial in $\alpha', \alpha'', \ldots, \alpha^{(k)}$ with coefficients which are rational functions in P and its derivatives, and R_2 is a differential polynomial in $\beta', \beta'', \ldots, \beta^{(k)}$ with constant coefficients. Obviously, R_1 is a rational function and R_2 is a polynomial. Together with (2.6) this yields

$$R_1 R_2 e^{n(\alpha+\beta)} = \varphi,$$

so $\alpha + \beta \equiv$ const. From this and (2.16), we get (iii) immediately, which completes the proof of Lemma 2.10. ∎

Lemma 2.11 (see [5]). *Suppose that f is a nonconstant meromorphic function, and $k \geq 2$ is an integer. If $ff^{(k)} \neq 0$, then $f = e^{az+b}$ or $f = (Az+B)^{-m}$, where a ($\neq 0$), b, A ($\neq 0$), B are constants and m is a positive integer.*

Lemma 2.12 (see [13]). *Let f and g be nonconstant meromorphic functions and $n \geq 6$. If $f^n f' g^n g' = 1$, then $g = c_1 e^{cz}$ and $f = c_2 e^{-cz}$, where c_1, c_2 and c are constants and $(c_1 c_2)^{n+1} c^2 = -1$.*

3. Proofs of results

Proof of Theorem 1.3. Set $F = f^n$. First, we consider the case when $k \geq 2$. By Lemmas 2.2 and 2.4, we have

$$\begin{aligned} nT(r,f) &= T(r,F) + S(r,f) \\ &\leq \overline{N}(r,F) + N(r,1/F) \\ &\quad + N\left(r, \frac{1}{F^{(k)}-1}\right) - N\left(r, \frac{1}{F^{(k+1)}}\right) + S(r,f). \end{aligned}$$

On the other hand, applying Lemma 2.5, we get

$$(k-1)\overline{N}(r,F)+N(r,1/F) \le 2\overline{N}(r,1/F)+N(r,1/F^{(k+1)})+\varepsilon T(r,F)+S(r,F),$$

for any given positive number ε. The above two inequalities give

$$(n-\varepsilon)T(r,f) \le 2\overline{N}\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{F^{(k)}-1}\right)+S(r,f)$$
$$= 2\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{(f^n)^{(k)}-1}\right)+S(r,f).$$

From this, we see that $(f^n)^{(k)}-1$ has infinitely many zeros when $n>2$.

Next, we suppose that $k=1$. Then $(f^n)^{(k)}=nf^{n-1}f'$, from this and by Theorem A, we can easily obtain the desired result.

This completes the proof of Theorem 1.3.

Proof of Theorem 1.4. Since f is a transcendental meromorphic function, by the second fundamental theorem for small functions, we have

$$T(r,(f^n)^{(k)}) \le \overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)+\overline{N}\left(r,\frac{1}{(f^n)^{(k)}-z}\right)$$
$$+\overline{N}(r,(f^n)^{(k)})+S(r,(f^n)^{(k)})$$
$$=\overline{N}\left(r,\frac{1}{(f^n)^{(k)}}\right)+\overline{N}\left(r,\frac{1}{(f^n)^{(k)}-z}\right)+\overline{N}(r,f)+S(r,f).$$

Applying Lemma 2.1 to the function f^n with $p=1$, we get

$$\overline{N}(r,1/(f^n)^{(k)}) \le T(r,(f^n)^{(k)})-T(r,f^n)+N_{k+1}(r,1/f^n)+S(r,f).$$

From Lemma 2.2 and the above two inequalities, we deduce that

$$nT(r,f) \le \overline{N}\left(r,\frac{1}{(f^n)^{(k)}-z}\right)+\overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f^n}\right)+S(r,f)$$
$$\le \overline{N}\left(r,\frac{1}{(f^n)^{(k)}-z}\right)+\overline{N}(r,f)+(k+1)\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)$$
$$\le (k+2)T(r,f)+\overline{N}\left(r,\frac{1}{(f^n)^{(k)}-z}\right)+S(r,f).$$

This shows $(f^n)^{(k)}$ has infinitely many fixed points when $n>k+2$, which completes the proof of Theorem 1.4.

Proof of Theorem 1.2. Set

$$F=(f^n)^{(k)}/z, \qquad G=(g^n)^{(k)}/z. \tag{3.1}$$

The condition that $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share z CM implies that F and G share the value 1 CM, and by Theorem 1.4 we see that either both f and g are transcendental meromorphic functions or both are rational functions.

Next we consider the following two cases:

Case 1: $N(r,f) \neq S(r,f)$. Since f has infinitely many poles, we know that both f and g are transcendental meromorphic functions. Applying Lemma 2.6 to F and G, it follows that there are three subcases to consider.

Subcase 1:

$$T(r,F) \le N_2(r,1/F) + N_2(r,1/G) + N_2(r,F) + N_2(r,G) + S(r,F) + S(r,G). \tag{3.2}$$

Obviously,

$$N_2(r,F) \le 2\overline{N}(r,f) + S(r,f), \qquad N_2(r,G) \le 2\overline{N}(r,g) + S(r,g).$$

By Lemma 2.1, we have

$$\begin{aligned} N_2(r,1/F) &\le T(r,F) - nT(r,f) + N_{k+2}(r,1/f^n) + S(r,f),\\ N_2(r,1/G) &\le k\overline{N}(r,g) + N_{k+2}(r,1/g^n) + S(r,g). \end{aligned}$$

Combining (3.2) and the last four inequalities, we obtain

$$\begin{aligned} nT(r,f) &\le N_{k+2}(r,1/f^n) + N_{k+2}(r,1/g^n) + (k+2)\overline{N}(r,g)\\ &\quad + 2\overline{N}(r,f) + S(r,f) + S(r,g)\\ &\le (k+2)(\overline{N}(r,1/f) + \overline{N}(r,1/g)) + (k+2)\overline{N}(r,g)\\ &\quad + 2\overline{N}(r,f) + S(r,f) + S(r,g). \end{aligned}$$

Similarly,

$$\begin{aligned} nT(r,g) &\le (k+2)(\overline{N}(r,1/f) + \overline{N}(r,1/g)) + (k+2)\overline{N}(r,f)\\ &\quad + 2\overline{N}(r,g) + S(r,f) + S(r,g). \end{aligned}$$

The above two inequalities yield

$$\begin{aligned} n(T(r,f) + T(r,g)) &\le (2k+4)(\overline{N}(r,1/f) + \overline{N}(r,1/g))\\ &\quad + (k+4)(\overline{N}(r,f) + \overline{N}(r,g))\\ &\quad + S(r,f) + S(r,g), \end{aligned} \tag{3.3}$$

which contradicts the assumption $n > 3k + 8$.

Subcase 2: $FG \equiv 1$, i.e.,

$$(f^n)^{(k)}(g^n)^{(k)} \equiv z^2. \tag{3.4}$$

By an argument similar to the proof of (i) in Lemma 2.10, we have

$$f \neq 0, \qquad g \neq 0. \tag{3.5}$$

This together with (3.4) yields

$$\begin{aligned} nN(r,g)+k\overline{N}(r,g) &= N(r,(g^n)^{(k)}) \le N(r,1/(f^n)^{(k)}) \\ &\le N(r,1/f^n)+k\overline{N}(r,f)+S(r,f) \\ &= k\overline{N}(r,f)+S(r,f). \end{aligned}$$

Similarly, we have

$$nN(r,f)+k\overline{N}(r,f) \le k\overline{N}(r,g)+S(r,g).$$

The above two inequalities yield

$$N(r,f)+N(r,g)=S(r,f)+S(r,g). \tag{3.6}$$

Also (3.4) implies $S(r,(f^n)^{(k)}) = S(r,(g^n)^{(k)})$, thus $S(r,f) = S(r,g)$. By (3.6) this shows that $N(r,f)=S(r,f)$, a contradiction too.

Subcase 3: $F \equiv G$, i.e., $(f^n)^{(k)} = (g^n)^{(k)}$. Then by Lemma 2.9, we obtain $f \equiv tg$ for a constant t.

Case 2. $N(r,f) = S(r,f)$. First, we suppose that f and g are transcendental meromorphic functions. Similar to Case 1, using Lemma 2.6, if (3.2) holds, we can get (3.3), which together with the condition $g \neq \infty$ and $n > 2k+4$ yields a contradiction. Next we only consider the following two subcases:

Subcase 1: $FG \equiv 1$, i.e.,

$$(f^n)^{(k)}(g^n)^{(k)} \equiv z^2. \tag{3.7}$$

By assumption and Lemma 2.10, we have

$$f=e^{\alpha}/P, \qquad g=ce^{-\alpha}, \tag{3.8}$$

where α ($\not\equiv$ const) and P are polynomials of degree d (>0) p respectively, and c is a nonzero constant.

Applying Lemma 2.3 to the function f^n, we obtain

$$(f^n)^{(k)} = f^n[\gamma^k + c_1\gamma^{k-2}\gamma' + c_2\gamma^{k-3}\gamma'' + \cdots + H_{k-2}(\gamma)] \tag{3.9}$$

where

$$\gamma=(f^n)'/f^n = n(P\alpha'-P')/P, \quad c_1=k(k-1)/2, \quad c_2=k(k-1)(k-2)/6$$

and $H_{k-2}(\gamma)$ is a differential polynomial in γ with constant coefficients and of total degree $\le k-2$. By computing, we have

$$\gamma^k = n^k\frac{(P\alpha'-P')^k}{P^k} =: \frac{R_0}{P^k}, \qquad \deg R_0 = k(p+d-1);$$

$$\gamma^{k-2}\gamma' = n^{k-2}\frac{(P\alpha'-P')^{k-2}(P^2\alpha''-PP''+P'^2)}{P^k} =: \frac{R_1}{P^k},$$

$$\deg R_1 \le (k-2)(p+d-1)+(2p+d-2) = k(p+d-1)-d;$$

$$\gamma^{k-3}\gamma'' = n^{k-3}\frac{(P\alpha' - P')^{k-3}(P^3\alpha''' - P^2P''' + 3PP'P'' - 2P'^3)}{P^k} =: \frac{R_2}{P^k},$$
$$\deg R_2 \le (k-3)(p+d-1) + (3p+d-3) = k(p+d-1) - 2d;$$

etc. From these and (3.9), we see that

$$(f^n)^{(k)} = f^n \cdot \frac{R}{P^k}, \tag{3.10}$$

where R is a polynomial of degree $k(p+d-1)$. Similarly,

$$(g^n)^{(k)} = g^n Q,$$

where Q is a polynomial of degree $k(d-1)$. This together with (3.7), (3.8) and (3.10) yields

$$e^{n\alpha} \cdot e^{-n\alpha} \cdot \frac{RQ}{P^{n+k}} \equiv z^2.$$

Hence, we have

$$k(p+d-1) + k(d-1) = (n+k)p + 2,$$

i.e.,

$$2k(d-1) - 2 = np. \tag{3.11}$$

On the other hand, considering $g \ne \infty$, $f \ne 0$, by (3.7) and (3.10), we see that $R/(P^k \cdot z^2)$ has no zeros, thus $\deg R \le \deg(P^k z^2)$, therefore $k(p+d-1) \le kp + 2$, i.e.,

$$k(d-1) \le 2. \tag{3.12}$$

Combining (3.12) and (3.11), we get $np \le 2$, so $p = 0$; then by (3.11), we have $k = 1$ and $d = 2$, and from (3.7) and (3.8) we easily deduce that

$$f(z) = c_1 e^{cz^2}, \quad g(z) = c_2 e^{-cz^2},$$

where c_1, c_2 and c are constants satisfying $4(c_1c_2)^n(nc)^2 = -1$.

SUBCASE 2: $F \equiv G$, i.e., $(f^n)^{(k)} \equiv (g^n)^{(k)}$. By Lemma 2.9, we get $f \equiv tg$, where t is a constant satisfying $t^n = 1$.

Next, we consider the case when f and g are rational functions. By the condition $N(r,f) = S(r,f)$ and $g \ne \infty$, we see that both f and g are polynomials. Then there exists a nonzero constant c such that

$$(f^n)^{(k)} - z = c((g^n)^{(k)} - z). \tag{3.13}$$

If $c \ne 1$, taking derivatives on both sides of (3.13) gives

$$(f^n)^{(k+1)} = c(g^n)^{(k+1)} + 1 - c.$$

By Lemma 2.8 and the above equality, we get $n \le 2(k+1)$, a contradiction. Hence $c = 1$, and (3.13) shows $(f^n)^{(k)} = (g^n)^{(k)}$. Applying Lemma 2.9, we obtain $f = tg$, where t is a constant satisfying $t^n = 1$.

This completes the proof of Theorem 1.2.

Proof of Theorem 1.1. Set

$$F = (f^n)^{(k)}, \qquad G = (g^n)^{(k)}. \tag{3.14}$$

Then F and G share 1 CM, and by Theorem 1.3 we see that either both f and g are transcendental meromorphic functions or both are rational functions. We consider the following two cases:

CASE 1: $N(r, f) \neq S(r, f)$. By an argument similar to the proof of Theorem 1.2, we get $F \equiv G$, i.e., $(f^n)^{(k)} \equiv (g^n)^{(k)}$. From Lemma 2.9, we obtain $f = tg$, where t is a constant satisfying $t^n = 1$.

CASE 2: $f \neq \infty$. Applying Lemma 2.6 to F and G, it follows that there are three subcases to consider.

SUBCASE 1:

$$T(r,F) \le N_2(r,1/F) + N_2(r,1/G) + N_2(r,F) + N_2(r,G) + S(r,F) + S(r,G).$$

Similar to the proof of Theorem 1.2, we get

$$nT(r,f) \le (k+2)(\overline{N}(r,1/f) + \overline{N}(r,1/g)) + (k+2)\overline{N}(r,g) + 2\overline{N}(r,f) + S(r,f) + S(r,g) \tag{3.15}$$

and

$$nT(r,g) \le (k+2)(\overline{N}(r,1/f) + \overline{N}(r,1/g)) + (k+2)\overline{N}(r,f) + 2\overline{N}(r,g) + S(r,f) + S(r,g). \tag{3.16}$$

If $T(r,f) \le T(r,g)$, since $f \neq \infty$, from (3.16) we get

$$\begin{aligned} nT(r,g) &\le (k+2)(T(r,f) + T(r,g)) + 2T(r,g) + S(r,g) \\ &\le (2k+6)T(r,g) + S(r,g), \end{aligned}$$

which contradicts the assumption $n \ge \frac{5}{2}k + 6$.

If $T(r,g) \le T(r,f)$, then (3.16) gives

$$nT(r,g) \le (k+2)(T(r,f) + T(r,g)) + 2T(r,g) + S(r,f).$$

Thus

$$T(r,g) \le \frac{k+2}{n-(k+4)} T(r,f) + S(r,f). \tag{3.17}$$

On the other hand, (3.15) gives

$$nT(r,f) \le (k+2)(T(r,f) + T(r,g)) + (k+2)T(r,g) + S(r,f).$$

From this and (3.17), we get

$$[n-(k+2)]T(r,f) \le 2(k+2)T(r,g) + S(r,f) \le \frac{2(k+2)^2}{n-(k+4)} T(r,f) + S(r,f),$$

which implies

$$n-(k+2) \le \frac{2(k+2)^2}{n-(k+4)},$$

so $[n-(k+3)]^2 \le 2k^2+8k+9 < (\frac{3}{2}k+3)^2$, which contradicts $n \ge \frac{5}{2}k+6$ too.

SUBCASE 2: $F \equiv G$, i.e., $(f^n)^{(k)} \equiv (g^n)^{(k)}$. By Lemma 2.9, we obtain $f = tg$, where t is a constant satisfying $t^n = 1$.

SUBCASE 3: $FG \equiv 1$, i.e.,

$$(f^n)^{(k)} \cdot (g^n)^{(k)} \equiv 1. \tag{3.18}$$

By Lemma 2.12, we only need to consider the case $k \ge 2$. Since $f \ne \infty$, from (3.18) we have $(g^n)^{(k)} \ne 0$. On the other hand, similar to the proof of (i) in Lemma 2.10, we get $f \ne 0$, $g \ne 0$, and then $g^n(g^n)^{(k)} \ne 0$. Applying Lemma 2.11, we obtain $g = e^{az+b}$ or $g = (Az+B)^{-m}$, where a ($\ne 0$), b, A ($\ne 0$), B are constants, and m is a positive integer. If $g = (Az+B)^{-m}$, then both f and g are rational functions. Assuming $f \ne 0$ and $f \ne \infty$, we get $f \equiv$ const, a contradiction. Hence $g = e^{az+b}$. Together with (3.18), we see that $\sigma(r,f) = \sigma(r,g) = 1$, where $\sigma(r,f)$ denotes the order of f. Again noting $f \ne 0$ and $f \ne \infty$, we have $f = e^{\alpha}$, where α is a polynomial of degree 1. From these and (3.18), we easily get $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$.

This completes the proof of Theorem 1.1.

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Xiu-Qing Lin
Department of Mathematics
Ningde Normal University
Ningde 352100
Fujian Province, P.R. China
E-mail: lxqnd4407@yahoo.cn

Wei-Chuan Lin
Department of Mathematics
Fujian Normal University
Fuzhou 350007
Fujian Province, P.R. China
E-mail: wclin936@163.com

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