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On continuous solutions to linear hyperbolic systems

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Abstract. We study the conditions under which the Cauchy problem for a linear hyperbolic system of partial differential equations of the first order in two independent variables has a unique continuous solution (not necessarily Lipschitz continuous). In addition to obvious continuity assumptions on coefficients and initial data, the sufficient conditions are the bounded variation of the left eigenvectors along the characteristic curves.

1. Introduction. It is known [4] that the Cauchy problem for a hyperbolic system

$$\frac{\partial u}{\partial t} + A(t,x)\,\frac{\partial u}{\partial x} = b(t,x) + B(t,x)u, \tag{1}$$

$$u(0,x) = u^0(x), \qquad x \in [\alpha,\beta], \tag{2}$$

with C^k coefficients, $k \ge 1$, has a unique C^k solution provided that $u^0 \in C^k([\alpha,\beta])$. A similar result has been shown in [2] for Lipschitz continuous solutions: if all coefficients are Lipschitz continuous then $u(t,x)$ is Lipschitz continuous in (t,x) if in addition u^0 is Lipschitz continuous. In [1] it has been shown that Lipschitz continuity can be in principle also replaced, to some extent, by absolute continuity. To have uniqueness one assumes however the Lipschitz continuity in x of the eigenvalues of A. The question arises whether we can still replace the assumption of Lipschitz and absolute continuity by a weaker one to assure only the continuity of the solution.

This work concerns the Cauchy problem (1)–(2) where u is an n-dimensional column vector function of two variables t and x. System (1) is hyperbolic, which means that the matrix A has real eigenvalues $\{\xi_k(t,x)\}_{k=1,\dots,p}$, $p \le n$ (with multiplicities m_k) and the corresponding eigenvectors span the n-dimensional space. We assume that the multiplicities m_k, $k = 1\dots,p$, are constant (not depending on (t,x)) and the inequality $\xi_1 < \dots < \xi_p$ holds for all $(t,x) \in [0,T] \times [\alpha_0,\beta_0]$, where $[\alpha,\beta] \subset (\alpha_0,\beta_0)$. Let $\{L_j\}_{j=1,\dots,n}$ be

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left linearly independent eigenvectors of A corresponding to the eigenvalues $\{\xi_k\}$. The matrix A is of the form $A = L^{-1}\, D\, L$, where $D = \mathrm{diag}[\xi_1, \ldots, \xi_n]$ and L is the nonsingular $n \times n$ matrix whose rows are the left eigenvectors $L_1, \ldots, L_n$.

We will prove the existence and uniqueness of continuous solutions provided that the left eigenvectors have bounded variation along characteristic curves. Our purpose is to show the existence and uniqueness of a generalized solution of system (1)–(2). By a continuous generalized solution we understand a function satisfying an integral system obtained from the differential system by integration along characteristic curves. In the proof we use the contraction mapping principle.

2. Characteristic curves and generalized solution. Throughout this section we assume that $L \in C^1([0,T] \times [\alpha_0, \beta_0])$. Multiplying (1) on the left by L,

$$L\frac{\partial u}{\partial t} + DL\frac{\partial u}{\partial x} = Lb + LBu,$$

and introducing the new unknown vector function (Riemann invariants)

$$r(t,x) = L(t,x) \cdot u(t,x) \tag{3}$$

we transform problem (1)–(2) to the following one:

$$\frac{\partial r}{\partial t} + D\,\frac{\partial r}{\partial x} = Lb + \left[LB + \left(\frac{\partial L}{\partial t} + D\frac{\partial L}{\partial x}\right)\right]u, \tag{4}$$

$$r^0(x) = r(0,x) = L(0,x) \cdot u^0(x), \qquad x \in [\alpha, \beta]. \tag{5}$$

Obviously the function u on the right hand side of (4) can be expressed by r, namely $u = L^{-1}r$.

The characteristic curve $x = x_k(t; \overline{t}, \overline{x})$ of the kth family passing through the point $(\overline{t}, \overline{x})$ is the solution of the equation

$$\frac{dx}{dt} = \xi_k(t,x) \tag{6}$$

which satisfies the initial condition

$$x_k(t; \overline{t}, \overline{x})|_{t=\overline{t}} = \overline{x}. \tag{7}$$

If the function $\xi_k(t,x)$ is continuous and satisfies the Lipschitz condition with respect to x then, by the Picard theorem, there is only one curve x_k passing through the point $(\overline{t}, \overline{x})$, which is continuously differentiable with respect to t. Since $\xi_k(t,x)$ is bounded on $[0,T] \times [\alpha_0, \beta_0]$, the curve represented by the solution $x = x_k(t; \overline{t}, \overline{x})$, $(\overline{t}, \overline{x}) \in (0,T) \times (\alpha_0, \beta_0)$, exists for all $t \in [0,T]$ unless it intersects the lateral boundaries $[0,T] \times \{\alpha_0\} \cup [0,T] \times \{\beta_0\}$ of $[0,T] \times [\alpha_0, \beta_0]$.

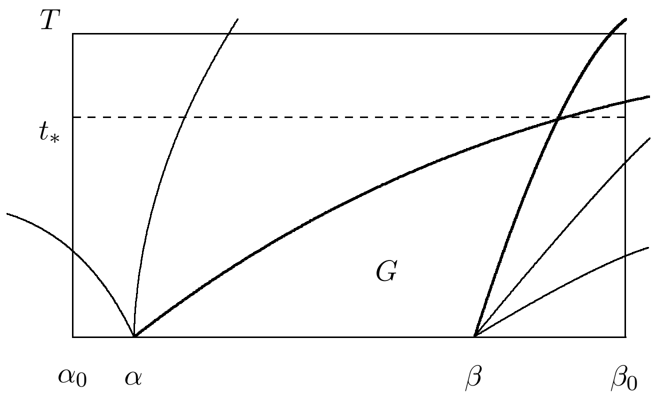
Set

$$G = \{(t,x) \in [0,\widetilde{T}] \times [\alpha_0,\beta_0] : X(t) \le x \le Y(t)\} \tag{8}$$

where

$$\frac{dX(t)}{dt} = \max_{k=1,\dots,p} \{\xi_k(t,X)\}, \qquad X(0)=\alpha,$$
$$\frac{dY(t)}{dt} = \min_{k=1,\dots,p} \{\xi_k(t,Y)\}, \qquad Y(0)=\beta, \qquad \alpha<\beta.$$

The time $\widetilde{T}$ is defined in the following way: if $X(t)$ and $Y(t)$ have the (first) intersection point at time $t_* \le T$, $X(t_*) = Y(t_*)$, or if $X(t)$ or $Y(t)$ intersect (for the first time) the lateral boundaries of $[0,T] \times [\alpha_0,\beta_0]$ at time t_1, t_2 (respectively) then $\widetilde{T} = \min\{t_1,t_2,t_*\}$, otherwise $\widetilde{T} = T$.



Notice that G has the property that every characteristic curve starting from $(\overline{t},\overline{x}) \in G$ is fully contained in G for $0 \le t \le \widetilde{T}$ and $x_k(0;\overline{t},\overline{x}) \in [\alpha,\beta]$, $k=1,\dots,n$.

The differential operators appearing in equation (4) are in fact the directional derivatives along the characteristic curves. Indeed, for any differentiable function $f(t,x)$ we have

$$\frac{d}{dt} f(t,x_k(t;\overline{t},\overline{x})) = \frac{\partial}{\partial t} f(t,x) + \xi_k \frac{\partial}{\partial x} f(t,x).$$

Therefore (4) becomes

$$\begin{aligned} \frac{dr_k(t,x_k(t;\overline{t},\overline{x}))}{dt} &= L_k(t,x_k(t;\overline{t},\overline{x})) \cdot b(t,x_k(t;\overline{t},\overline{x})) \\ &\quad + L_k(t,x_k(t;\overline{t},\overline{x})) \cdot B(t,x_k(t;\overline{t},\overline{x})) \cdot u(t,x_k(t;\overline{t},\overline{x})) \\ &\quad + \frac{d\,L_k(t,x_k(t;\overline{t},\overline{x}))}{dt} \cdot u(t,x_k(t;\overline{t},\overline{x})) \end{aligned} \tag{9}$$

for $k=1,\dots,n$. Here $d\,L_k/dt$ denotes the vector $[d\,L_{k1}/dt,\dots,d\,L_{kn}/dt]$.

We define a linear mapping

$$\mathcal{P}_t : C^0([0,\widetilde{T}] \times [\alpha_0,\beta_0]) \to C^0([0,\widetilde{T}] \times [0,\widetilde{T}] \times [\alpha_0,\beta_0])$$

acting on vector functions $f = (f_1,\dots,f_n)^T$ by

$$(\mathcal{P}f)_k(t,\overline{t},\overline{x}) = f_k(t, x_k(t;\overline{t},\overline{x})), \quad k = 1,\dots,n. \tag{10}$$

Since we have

$$\sup_{(t,\bar{t},\bar{x})\in[0,\widetilde{T}]\times[0,\widetilde{T}]\times[\alpha_0,\beta_0]} |f_k(t,x_k(t;\overline{t},\overline{x}))| = \sup_{(t,x)\in[0,\widetilde{T}]\times[\alpha_0,\beta_0]} |f_k(t,x)|,$$

$\mathcal{P}$ is continuous. For convenience we will use the notation

$$\mathcal{P}_t f = (\mathcal{P}f)(t,\cdot,\cdot).$$

Hence we can rewrite (9) as follows:

$$\frac{d}{dt}(\mathcal{P}_t r) = \mathcal{P}_t\bigg(Lb + LBu + \frac{dL}{dt}\,u\bigg). \tag{11}$$

Integrating (11) with respect to t from 0 to $\overline{t}$ we obtain

$$r(\overline{t},\overline{x}) = \mathcal{P}_0 r^0 + \int_0^{\bar{t}} \mathcal{P}_t\bigg(Lb + LB\,u + \frac{dL}{dt}\,u\bigg)\,dt. \tag{12}$$

We thus arrived at a system of n integral equations in place of the original system (1):

$$\begin{aligned} u(\overline{t},\overline{x}) = {} & L^{-1}(\overline{t},\overline{x})\mathcal{P}_0 r^0 + L^{-1}(\overline{t},\overline{x})\int_0^{\bar{t}} \mathcal{P}_t(Lb + LBu)\,dt \\ & + L^{-1}(\overline{t},\overline{x})\int_0^{\bar{t}} \mathcal{P}_t\bigg(\frac{dL}{dt}\,u\bigg)\,dt. \end{aligned} \tag{13}$$

To generalize the notion of solution we can treat the last integral as a Stieltjes integral with respect to t and rewrite (13) in the form

$$\begin{aligned} u(\overline{t},\overline{x}) = {} & L^{-1}(\overline{t},\overline{x})\mathcal{P}_0 r^0 + L^{-1}(\overline{t},\overline{x})\int_0^{\bar{t}} \mathcal{P}_t(Lb + LBu)\,dt \\ & + L^{-1}(\overline{t},\overline{x})\int_0^{\bar{t}} \mathcal{P}_t(dL\cdot u). \end{aligned} \tag{14}$$

If the function u is continuous then for the existence of the Stieltjes integral in (14) it is sufficient that the entries of the matrix L have bounded variation along the characteristic curves for $t \in [0,\overline{t}]$ (respectively L_{sj} along $x_s(t;\overline{t},\overline{x})$, $s,j = 1,\dots,n$).

By a continuous generalized solution of the Cauchy problem (1)–(2) we understand a function satisfying the integral system (14).

3. Existence theorem. Let us formulate our main result:

THEOREM 1. *Let the entries of the matrices $L(t,x)$ and $L^{-1}(t,x)$ be continuous functions on $[0,T]\times[\alpha_0,\beta_0]$. Suppose $L(t,x)$ has bounded variation along each characteristic curve $x=x_k(t;\overline{t},\overline{x})$ contained in $[0,T]\times[\alpha_0,\beta_0]$, i.e. $L_{kj}(t,x_k(t;\overline{t},\overline{x}))$, $k,j=1,\dots,n$, has bounded variation as a function of t. Let in addition this variation be a continuous function of $\overline{t},\overline{x}$. Assume that the entries of the matrices $D(t,x)$, $B(t,x)$, $b(t,x)$ are continuous on $[0,T]\times[\alpha_0,\beta_0]$. Let the entries of $D(t,x)$ satisfy the Lipschitz condition with respect to x and let the initial data $u^0(x)$ be continuous on $[\alpha,\beta]$. Then there exists a unique function $u(t,x)$ of class $C^0(G)$ which satisfies* (1)–(2).

To prove the existence we will use the Banach fixed point theorem. We first define (for $T^*\in(0,\widetilde{T}]$) the set

$$G_{T^*}=G\cap([0,T^*]\times[\alpha_0,\beta_0]). \tag{15}$$

For the proof we consider the linear operator $\mathcal{Q}$, which transforms the vector function $u\in C^0(G_{T^*})$ into the vector function $U\in C^0(G_{T^*})$, $U=\mathcal{Q}(u)$, where according to (14),

$$\begin{aligned}\mathcal{Q}(u)=U(\overline{t},\overline{x})=L^{-1}(\overline{t},\overline{x})\mathcal{P}_0r^0+L^{-1}(\overline{t},\overline{x})\int_0^{\bar t}\mathcal{P}_t(Lb+LBu)\,dt\\ +L^{-1}(\overline{t},\overline{x})\int_0^{\bar t}\mathcal{P}_t(dL\cdot u).\end{aligned} \tag{16}$$

We will show that for sufficiently small T^* the mapping $\mathcal{Q}$ is a contraction.

We shall need the following

LEMMA 1. *Let $f,g:[0,\widetilde{T}]\times G\to\mathbb{R}$. Assume that the functions $f(\tau,t,x)$ and $g(\tau,t,x)$ are continuous on $[0,\widetilde{T}]\times G$. Moreover let g be of bounded variation with respect to the variable τ for any fixed t and x, and its variation be a continuous function of t,x* [1]. *Then*

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[1] Continuity of $h(x,y)$ and bounded variation with respect to y do not guarantee that the total variation is a continuous function of x. An example is the function

$$h:[-1,1]\times[-\pi,\pi]\to\mathbb{R},\quad h(x,y)=\begin{cases}x\sin\frac{y}{x}, & x>0,\\ 0, & x\le 0.\end{cases}$$

It is continuous on $[-1,1]\times[-\pi,\pi]$ and has bounded variation with respect to y (for any fixed x). Moreover its total variation is

$$V_{-\pi}^{\pi}(h(x,\cdot))=\begin{cases}0, & x\in[-1,0],\\ 2x\left[\frac{2}{x}\right]+2x\left|\sin\frac{\pi}{x}-\sin\left(\frac{\pi}{2}\left[\frac{2}{x}\right]\right)\right|, & x\in(0,1].\end{cases}$$

This is not a continuous function of x because $\lim_{x\to0^+}V_{-\pi}^{\pi}(h(x,\cdot))=4$, whereas $\lim_{x\to0^-}V_{-\pi}^{\pi}(h(x,\cdot))=0$.

$$J(t,x)=\int_0^t f(\tau;t,x)\,dg(\tau;t,x) \tag{17}$$

is a continuous function with respect to both variables.

REMARK. In the integral (17), t and x are treated as parameters.

Proof. Let $V_{t_1}^{t_2}(g(\tau;t,x))$ denote the variation of the function g for $\tau \in [t_1,t_2]$. We have the estimate

$$\begin{aligned}
|J(t,x)-J(t_0,x_0)| &= \Big|\int_0^t f(\tau;t,x)\,dg(\tau;t,x)-\int_0^{t_0} f(\tau;t_0,x_0)\,dg(\tau;t_0,x_0)\Big| \\
&\le \Big|\int_0^t f(\tau;t,x)\,dg(\tau;t,x)-\int_0^{t} f(\tau;t_0,x_0)\,dg(\tau;t_0,x_0)\Big| \\
&\quad+\Big|\int_0^t f(\tau;t_0,x_0)\,dg(\tau;t_0,x_0)-\int_0^{t_0} f(\tau;t_0,x_0)\,dg(\tau;t_0,x_0)\Big| \\
&\le \Big|\int_0^t [f(\tau;t,x)-f(\tau;t_0,x_0)]\,dg(\tau;t,x)\Big| \\
&\quad+\Big|\int_0^t f(\tau;t_0,x_0)\,d[g(\tau;t,x)-g(\tau;t_0,x_0)]\Big| \\
&\quad+\Big|\int_{t_0}^t f(\tau;t_0,x_0)\,dg(\tau;t_0,x_0)\Big| \\
&\le \max_{\tau\in[0,T]}|f(\tau;t,x)-f(\tau;t_0,x_0)|\cdot V_0^T(g(\tau;t,x)) \\
&\quad+\max_{\tau\in[0,T]}|f(\tau;t_0,x_0)|\cdot V_0^T(g(\tau;t,x)-g(\tau;t_0,x_0)) \\
&\quad+\max_{\tau\in[0,T]}|f(\tau;t_0,x_0)|\cdot V_{t_0}^t(g(\tau;t_0,x_0)).
\end{aligned}$$

As f is continuous, we have

$$\lim_{\substack{t\to t_0\\ x\to x_0}}\max_{\tau\in[0,T]}|f(\tau;t,x)-f(\tau;t_0,x_0)|=0.$$

The function g is continuous and has bounded variation with respect to τ for any fixed t_0 and x_0, therefore [3] we have

$$\lim_{t\to t_0} V_{t_0}^t(g(\tau;t_0,x_0))=0.$$

According to the assumptions, the variation is a continuous function of t,x. Hence

$$\lim_{\substack{t\to t_0\\ x\to x_0}} V_0^T(g(\tau;t,x)-g(\tau;t_0,x_0))=0.$$

From the above remarks it follows that

$$\lim_{\substack{t\to t_0\\ x\to x_0}} J(t,x) = J(t_0,x_0),$$

which proves the lemma. ■

By Lemma 1, $\mathcal{Q}$ maps $C^0(G_{T^*})$ into itself. We claim that it is possible to choose the time T^* so that the linear mapping $\mathcal{Q}$ will be a contraction. Indeed, let u and $\overline{u}$ be vector functions from the space $C^0(G_{T^*})$ with norm

$$\|u\| = \max_{(\bar{t},\overline{x})\in G_{T^*}} \max_{k=1,\dots,n} |u_k(\overline{t},\overline{x})|.$$

We obtain

$$\begin{aligned}
&\|\mathcal{Q}(u)-\mathcal{Q}(\overline{u})\| = \|U-\overline{U}\|\\
&\quad\le \Big\|L^{-1}\int_0^{\bar{t}} \mathcal{P}_t(LB(u-\overline{u}))\,dt\Big\| + \Big\|L^{-1}\int_0^{\bar{t}} \mathcal{P}_t(dL(u-\overline{u}))\Big\|\\
&\quad\le \|u-\overline{u}\|\,\|L^{-1}\|(T^*\|LB\| + n\max_{(\bar{t},\overline{x})\in G_{T^*}}\max_{s,j=1,\dots,n} V_0^{T^*}(L_{sj}(t,x_s(t;\overline{t},\overline{x})))).
\end{aligned}$$

It follows that

$$\begin{aligned}
(18)\quad &\|U-\overline{U}\|\\
&\le \|L^{-1}\|(T^*\|LB\| + n\max_{(\bar{t},\overline{x})\in G_{T^*}}\max_{s,j=1,\dots,n} V_0^{T^*}(L_{sj}(t,x_s(t;\overline{t},\overline{x}))))\|u-\overline{u}\|.
\end{aligned}$$

Here $V_0^{T^*}(L_{sj}(t,x_s(t;\overline{t},\overline{x})))$ stands for the variation of the function L_{sj} with respect to t. The entries of the matrix L and L^{-1} are continuous on $[0,T^*]\times[\alpha_0,\beta_0]$. Hence $\|L^{-1}\|<\infty$, $\|L\,B\|<\infty$.

According to (18), $\mathcal{Q}$ is a contraction mapping if T^* satisfies

$$(19)\quad \|L^{-1}\|(T^*\|LB\| + n\max_{(\bar{t},\overline{x})\in G_{T^*}}\max_{s,j=1,\dots,n} V_0^{T^*}(L_{sj}(t,x_s(t;\overline{t},\overline{x})))) < 1.$$

Our task is now to show that (19) holds for some $T^*>0$.

The first term in brackets contains T^* and can be made arbitrarily small for small T^*:

$$(20)\qquad T^*\|L^{-1}\|\,\|L\,B\|\to 0 \quad \text{as } T^*\to 0.$$

Similarly [3] for all $s,j=1,\dots,n$ we have

$$(21)\qquad V_0^{T^*}(L_{sj}(t,x_s(t;\overline{t},\overline{x})))\to 0 \quad \text{as } T^*\to 0$$

Since $\mathcal{Q}$ is a contraction, being a linear mapping in a Banach space it is also continuous. From the fact that G_{T^*} is a compact set we deduce that $T^*>0$ can be chosen in such a way that (21) holds for all $(\overline{t},\overline{x})\in G_{T^*}$ uniformly.

We conclude from (20) and (21) that there exists T^*, $0<T^*\le\widetilde{T}$, such that (19) is satisfied. By the Banach principle there is a unique fixed point of the mapping $\mathcal{Q}$. We have proved the existence of a local in time solution

of (14) on G_{T^*}. Since the norms of the entries of the matrices L, B, L^{-1} do not depend on t, we can extend the solution onto the whole set G. Indeed, taking now $t = T^*$ as the initial time and $u(T^*, x)$ as the new initial condition we come to the problem defined on the set

$$G_2 = \{(t,x) \in [T^*, \widetilde{T}] \times [\alpha_0, \beta_0] : X(t) \le x \le Y(t)\}.$$

We continue in this fashion obtaining a solution on the set G, which completes the proof.

REMARKS. To conclude, let us note that a (local in time) existence theorem similar to Theorem 1 can be proved for a semilinear system, i.e. when the RHS is a nonlinear continuous function Lipschitzian in u.

In applications it often happens that the coefficients of the system depend only on x:

$$\frac{\partial u}{\partial t} + A(x)\,\frac{\partial u}{\partial x} = b(x), \tag{22}$$

$$u(0,x) = u^0(x), \qquad x \in [\alpha, \beta]. \tag{23}$$

We claim that Theorem 1 is still true in this case if instead of the continuity of the vector function $b(x)$ we only assume that it is a derivative along the characteristic directions of some continuous function $f(x)$, i.e.

$$b(x) = \left(\frac{\partial}{\partial t} + D(x)\,\frac{\partial}{\partial x}\right) f(x) = D(x)\,\frac{\partial}{\partial x} f(x). \tag{24}$$

Using the integral formulation (14) for the Cauchy problem (22)–(23) we have

$$\begin{aligned} u(\bar{t},\bar{x}) = L^{-1}(\bar{t},\bar{x}) \cdot \mathcal{P}_0 r^0 + L^{-1}(\bar{t},\bar{x}) \int_0^{\bar{t}} \mathcal{P}_t(L \cdot df) \\ + L^{-1}(\bar{t},\bar{x}) \int_0^{\bar{t}} \mathcal{P}_t(dL \cdot u). \end{aligned}$$

Integrating by parts we obtain

$$\int_0^{\bar{t}} \mathcal{P}_t(L \cdot df) = -\int_0^{\bar{t}} \mathcal{P}_t(dL \cdot f) + \mathcal{P}_{\bar{t}}(Lf) - \mathcal{P}_0(Lf).$$

By the above, let us define a solution of the Cauchy problem (22)–(23) to be a C^0 function satisfying the following integral system:

$$\begin{aligned} u(\bar{t},\bar{x}) = -L^{-1}(\bar{t},\bar{x}) \int_0^{\bar{t}} \mathcal{P}_t(dL \cdot f) + L^{-1}(\bar{t},\bar{x}) \int_0^{\bar{t}} \mathcal{P}_t(dL \cdot u) \\ + L^{-1}(\bar{t},\bar{x}) \cdot \mathcal{P}_0 r^0 + L^{-1}(\bar{t},\bar{x}) \cdot \mathcal{P}_{\bar{t}}(Lf) - L^{-1}(\bar{t},\bar{x}) \cdot \mathcal{P}_0(Lf) \end{aligned}$$

The integral $\int_0^{\bar{t}} \mathcal{P}_t(dL \cdot f)$ exists and, by Lemma 1, is a continuous function of $(\overline{t}, \overline{x})$.

If the matrix $D(x)$ is nonsingular and $b(x) \in \mathcal{L}^1([\alpha_0, \beta_0])$ then a vector function $f(x)$ as in (24) always exists and it is given by

$$f(x) = \int\limits_{\alpha_0}^{x} D^{-1}(y) b(y)\, dy.$$

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