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On shape and multiplicity of solutions for a singularly perturbed Neumann problem

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Abstract. We investigate the effect of the topology of the boundary $\partial\Omega$ and of the graph topology of the coefficient Q on the number of solutions of the nonlinear Neumann problem (1_d).

1. Introduction. In this paper we investigate the existence of solutions for the Neumann problem

$$(1_d)\qquad \begin{cases} -d^2\Delta u + u = Q(y)u^p, & y \in \Omega,\\ u > 0 \quad \text{for } y \in \Omega, & \dfrac{\partial u(y)}{\partial n} = 0 \quad \text{for } y \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^3-boundary $\partial\Omega$, $d > 0$ is a parameter, $1 < p < (N+2)/(N-2)$, $N \geq 3$, and n is the unit outer normal to $\partial\Omega$. It is assumed that the coefficient Q is positive, Hölder continuous with exponent $\alpha \in (0,1)$ on $\overline{\Omega}$ and $Q \not\equiv \text{Const}$ on $\partial\Omega$.

Problem (1_d) stems from the studies of pattern formation in biology. In particular, it can be viewed as the steady state problem for a chemotactic aggregation model proposed by Keller and Segel [6]. Problem (1_d) also appears in the study of activator-inhibitor systems in biological pattern formation theory due to Gierer and Meinhardt [5]. In the case of $Q(x) \equiv 1$ on Ω problem (1_d) has an extensive literature [6], [8]–[12], [14], [15]. In [11] and [12] Ni and Takagi proved that for every $d > 0$ sufficiently small, problem (1_d) has a nonconstant least energy solution u_d. The solutions u_d with d small exhibit concentration phenomena. Namely, each solution u_d attains its unique maximum at P_d on the boundary and $P_d \to P_\circ$, where $P_\circ$ is located

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on the "most curved" part of $\partial\Omega$, that is, the mean curvature attains its maximum at this point. Wei [15] constructed a solution u_d on some energy level which has only one local maximum point $x_d \in \partial\Omega$ and $x_d \to x_\circ$, where $x_\circ$ is a critical point of $H(x)$. He also established a partial converse showing that each nondegenerate critical point $x_\circ$ of $H(x)$ generates a solution u_d having only one maximum point x_d, with $x_d \to x_\circ$. Li [8] showed that the nondegeneracy assumption can be replaced by C^1-stability.

The effect of the topology of $\partial\Omega$ on the existence of multiple solutions was studied by Wang [14]. He proved that for small $d > 0$ problem (1_d), with $Q(y) \equiv 1$, has at least $\mathrm{cat}(\partial\Omega)$ distinct single-peak solutions, where $\mathrm{cat}(\partial\Omega)$ is the Lusternik–Schnirelman category of $\partial\Omega$.

The main purpose of this work is to investigate the effect of the graph topology of the coefficient Q and the topology of the boundary $\partial\Omega$ on the existence of multiple solutions. It is assumed that the coefficient Q attains its maximum on the boundary $\partial\Omega$. We show the existence of the least energy solutions u_d, $d > 0$. These solutions achieve their maxima at $x_d \in \partial\Omega$ for d sufficiently small and $x_d \to x_\circ$, with $Q(x_\circ) = \max_{x\in\overline{\Omega}} Q(x)$. However, if $Q(x) = \mathrm{Const}$ on $\partial\Omega$, then the influence of the mean curvature is stronger. In this case solutions concentrate at points on $\partial\Omega$ where $H(x)$ attains its maximum. In Section 4 we extend the result of Wang [14] to the problem (1_d) and show that at levels close to the least energy level problem (1_d) has at least $\mathrm{cat}(\partial\Omega)$ solutions.

Section 5 is devoted to the construction of multi-peak solutions. We aim to show that local minima of the restriction of Q to $\partial\Omega$ generate multi-peak solutions. Our approach is a modification of the construction of multi-peak solutions from [4]. In Section 6 we study the effect of the graph topology of the coefficient Q on the existence of multiple solutions. We express the multiplicity of solutions in terms of the Lusternik–Schnirelman category of the set where Q attains its maximum on the boundary ∂Q. As in [14] it can be shown that solutions obtained in Section 4 are one-peak solutions, that is, they have at most one local maximum. However, we were unable to locate their concentration points. On the other hand, solutions constructed in Section 6, which are one-peak solutions, have maxima concentrating at points of $\partial\Omega$ where Q attains its maximum.

2. Least energy solutions. Solutions of problem (1_d) will be found as critical points of the variational functional $I_d : H^1(\Omega) \to \mathbb{R}$ defined by

$$I_d(u) = \frac{1}{2}\int_{\Omega} (d^2|\nabla u|^2 + u^2)\,dx - \frac{1}{p+1}\int_{\Omega} Q(x)|u|^{p+1}\,dx,$$

where $H^1(\Omega)$ is a Sobolev space equipped with the norm

$$\|u\|^2 = \int_\Omega (|\nabla u(x)|^2 + u(x)^2)\,dx.$$

It is easy to check that the functional I_d has a mountain pass structure:

(i) there exist constants $\varrho > 0$ and $\beta > 0$ such that $I_d(u) \geq \beta$ for $\|u\| = \varrho$;

(ii) there exists $\phi \in H^1(\Omega)$, with $\|\phi\| > \varrho$, such that $I_d(\phi) < 0$;

(iii) I_d satisfies the Palais–Smale condition: if $\{u_n\} \subset H^1(\Omega)$ is such that $I_d(u_n)$ is bounded and $I_d'(u_n) \to 0$ in $H^{-1}(\Omega)$ then $\{u_n\}$ is relatively compact in $H^1(\Omega)$.

Let $\Gamma = \{\gamma \in C([0,1], H^1(\Omega)) : \gamma(0) = 0,\ \gamma(1) = \phi\}$. By the mountain pass principle [1] for each $d > 0$ there exists $u_d \in H^1(\Omega)$ such that

$$I_d(u_d) = c_d = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_d(\gamma(t)).$$

Using the Hopf maximum principle we can assume that $u_d > 0$ on Ω (see [9, p. 9]). Repeating the argument from [9, pp. 12, 13] one can show that $c_d = O(d^N)$.

Let $P \in \partial\Omega$. In order to examine the behaviour of u_d near the boundary $\partial\Omega$ we introduce the diffeomorphism which straightens the boundary portion near the point P (see [10], [11]). We may assume that P is the origin and the inner normal to $\partial\Omega$ at P is pointing in the direction of the positive x_N-axis. Then there exists a smooth function $\psi_P(x')$, $x' = (x_1, \ldots, x_{N-1})$, defined for $|x'|$ small such that (a) $\psi_P(0) = 0$, $\nabla\psi_P(0) = 0$, and (b) $\partial\Omega \cap \mathcal{N} = \{(x', x_N) : x_N = \psi_P(x')\}$ and $\Omega \cap \mathcal{N} = \{(x', x_N) : x_N > \psi_P(x')\}$, where $\mathcal{N}$ is a neighbourhood of P.

For $y \in \mathbb{R}^N$ near 0, we define a mapping $x = \Phi_P(y) = (\Phi_{P,1}(y), \ldots, \Phi_{P,N}(y))$ by

$$\Phi_{P,j}(y) = \begin{cases} y_j - y_N \dfrac{\psi_P(y')}{\partial x_j}, & j = 1, \ldots, N-1, \\ y_N + \psi_P(y'), & j = N. \end{cases}$$

Since $\nabla\psi_P(0) = 0$, we have $\nabla\Phi_P(0) = I$. Therefore Φ_P has an inverse mapping $y = \Phi_P^{-1}(x)$ for $|x| < \delta'$, with $\delta' > 0$ small. We set $\Psi_P(x) = \Phi_P^{-1}(x)$. We notice that

$$|\Phi_P(y)| \leq C|y| \tag{2}$$

for some constant $C > 0$ and small $|y|$. Following the ideas from [12] and [14] we define a comparison function for I_d. Towards this end let w_M be the ground state solution [7] of

$$\begin{cases} -\Delta u + u = Mu^p & \text{in } \mathbb{R}^N, \\ u > 0 & \text{on } \mathbb{R}^N. \end{cases} \tag{3}$$

By rescaling we have $w_M = M^{-1/(p-1)} w_1$, where $w_1 > 0$ is spherically symmetric: $w_1(x) = w_1(|x|)$, $dw_1/dr < 0$ for $|x| > 0$. It is known that w_1 and its first order derivatives exponentially decay at infinity, that is,

$$w_1(x), |\nabla w_1(x)| \le C e^{-\mu|x|} \quad \text{on } \mathbb{R}^N, \tag{4}$$

for some constants $C > 0$ and $\mu > 0$. For $\varrho > 0$ we set

$$\zeta_\varrho(t) = \begin{cases} 1 & \text{for } 0 \le t \le \varrho, \\ 2 - \varrho^{-1} t & \text{for } \varrho < t \le 2\varrho, \\ 0 & \text{for } t \ge 2\varrho, \end{cases}$$

and

$$w_*^M(z) = \zeta_{k/d}(|z|) w_M(z),$$

where $k > 0$ is chosen so that the domain of definition of Φ_P contains the ball $B_{3k} = B(0, 3k)$. Further, let $D_j = \Phi_P(B_{jk}^+)$ for $j = 1, 2$, with $B_r^+ = B_r \cap \mathbb{R}_+^N$. We observe that $D_1 \subset D_2 \subset \Omega$. We define a comparison function by

$$\phi_d^M(x) = \begin{cases} w_*^M(\Psi_P(x)/d) & \text{in } D_2, \\ 0 & \text{elsewhere.} \end{cases} \tag{5}$$

In what follows we shall use for a fixed $P \in \overline{\Omega}$ the following functional:

$$I_{Q(P)}(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + u^2)\, dx - \frac{Q(P)}{p+1} \int_\Omega |u|^{p+1}\, dx.$$

3. Behaviour of the least energy solutions for small $d > 0$. We begin by estimating I_d at ϕ_d.

LEMMA 1. *Suppose $P \in \partial\Omega$. Let $\phi_d = \phi_d^{Q(P)}$ be defined by* (5). *Then*

$$M[\phi_d] = \sup_{t \ge 0} I_d(t\phi_d) = \frac{d^N}{2} I_{Q(P)}(w_{Q(P)}) A_d,$$

where $A_d > 0$ and $A_d \to 1$ as $d \to 0$.

Proof. We commence by observing that there exists $t_\circ(d) > 0$ (unique) such that

$$M[\phi_d] = I_d(t_\circ(d)\phi_d).$$

To simplify we assume that $P =$ the origin of the coordinates. Since

$$\frac{d}{dt} I_d(t\phi_d)\Big|_{t = t_\circ(d)} = 0,$$

we have

$$t_\circ(d) = \left(\frac{\int_\Omega (d^2 |\nabla \phi_d|^2 + \phi^2)\, dx}{\int_\Omega Q(x) \phi_d^{p+1}\, dx} \right)^{1/(p-1)}.$$

Hence

$$I_d(t_\circ(d)\phi_d) = \frac{p-1}{2(p+1)}\left(\frac{\int_\Omega (d^2|\nabla\phi_d|^2+\phi_d^2)\,dx}{\int_\Omega Q(x)\phi_d^{p+1}\,dx}\right)^{2/(p-1)} \int\limits_\Omega (d^2|\nabla\phi_d|^2+\phi_d^2)\,dx. \tag{6}$$

It follows from (3.14) and (3.15) in [12] that

$$d^2\int\limits_\Omega |\nabla\phi_d|^2\,dx = d^N\Big(\int\limits_{\mathbb{R}^N_+} (w'_{Q(0)})^2\,dx + O(d)\Big), \tag{7}$$

$$\int\limits_\Omega \phi_d^{p+1}\,dx = d^N\Big(\int\limits_{\mathbb{R}^N_+} w_{Q(0)}^{p+1}\,dx + O(d)\Big), \tag{8}$$

$$\int\limits_\Omega \phi_d^2\,dx = d^N\Big(\int\limits_{\mathbb{R}^N_+} w_{Q(0)}^2\,dx + O(d)\Big). \tag{9}$$

Using (8) we now write

$$\begin{aligned}\int\limits_\Omega Q(x)\phi_d^{p+1}\,dx &= Q(0)\int\limits_\Omega \phi_d^{p+1}\,dx + \int\limits_\Omega (Q(x)-Q(0))\phi_d^{p+1}\,dx \\ &= Q(0)d^N\Big(\int\limits_{\mathbb{R}^N_+} w_{Q(0)}^{p+1}\,dx + O(d)\Big) + \int\limits_\Omega (Q(x)-Q(0))\phi_d^{p+1}\,dx.\end{aligned}$$

The second integral on the right side can be estimated using (A.3) of Lemma A.1 in [12]:

$$\begin{aligned}&\int\limits_\Omega |Q(x)-Q(0)|\phi_d^{p+1}\,dx \\ &\quad\le L\int\limits_{D_2} |x|^\alpha w_*^{Q(0)}\left(\frac{\Psi(x)}{d}\right)^{p+1} dx \\ &\quad= L\int\limits_{B_{2k}^+} |\Phi(y)|^\alpha w_*^{Q(0)}\left(\frac{y}{d}\right)^{p+1} \det|D\Phi(y)|\,dy \\ &\quad\le LCd^{N+\alpha}\int\limits_{B_{2R}} |y|^\alpha w_*^{Q(0)}(y)^{p+1}(1-\overline{\alpha}dy_N + O(d^2|y|^2))\,dy,\end{aligned}$$

where $\overline{\alpha} = \varDelta\Psi(0)$ and C is a constant from (2) and L is a Hölder constant for Q. Using the fact that $w_1(z) \le Ce^{-\mu|z|}$ on $\mathbb{R}^N$, we derive from the last two estimates that

$$\int\limits_\Omega Q(x)\phi_d^{p+1}\,dx = d^N\Big(Q(0)\int\limits_{\mathbb{R}^N_+} w_{Q(0)}^{p+1}\,dx + O(d^\alpha)\Big). \tag{10}$$

The result follows from (6), (7), (9) and (10) with

$$A_d = \left(\frac{\int_{\mathbb{R}^N_+} (|\nabla w_{Q(0)}|^2 + w_{Q(0)}^2)\,dx + O(d)}{\int_{\mathbb{R}^N_+} Q(0) w_{Q(0)}^{p+1}\,dx + O(d^\alpha)} \right)^{2/(p-1)}.$$

LEMMA 2. *If u_d attains its local maximum at $x_d \in \overline{\Omega}$, then*

$$u_d(x_d) \ge M^{-1/(p-1)},$$

where $M = \max_{x\in\overline{\Omega}} Q(x)$. Also, there exists a constant $\eta > 0$ independent of x_d and d such that $u_d(x) \ge \eta$ for $x \in B(x_d, d) \cap \Omega$ if d is sufficiently small.

Proof. Suppose that $u_d(x_d) < M^{-1/(p-1)}$. If $x_d \in \Omega$, then

$$d^2 \Delta u_d = u_d - Q(x)u_d^p = u_d(1 - Q(x)u_d^{p-1}) \ge u_d(1 - Mu_d^{p-1}) > 0$$

in a small ball with centre at x_d. However, this contradicts the inequality $d^2\Delta u_d(x_d) \le 0$. Hence, $x_d \in \partial\Omega$ and $u_d(x) < u_d(x_d)$ for $x \in \Omega$ close to x_d. According to the Hopf boundary point lemma we have $\partial u_d(x_d)/\partial n > 0$, which does not match the boundary condition. The second assertion follows from the Harnack inequality (see Lemma 4.3 in [9] and p. 830 in [12]).

We are now in a position to locate the maximum points of u_d.

THEOREM 1. *Suppose that* $\max_{x\in\partial\Omega} Q(x) = \max_{x\in\overline{\Omega}} Q(x)$. *Let $u_d(P_d) = \max_{x\in\overline{\Omega}} u_d(x)$. Then $P_d \in \partial\Omega$ for small $d > 0$ and $P_d \to \overline{P}$ with $Q(\overline{P}) = \max_{x\in\partial\Omega} Q(x)$.*

Proof. We follow some ideas from [12]. The proof will be divided in several steps.

STEP I. There exists a constant $C > 0$ such that $\operatorname{dist}(P_d, \partial\Omega) \le Cd$. In the contrary case we can find a decreasing sequence $d_j \to 0$ such that

$$\varrho_j = \frac{\operatorname{dist}(P_j, \partial\Omega)}{d_j} \to \infty$$

as $j \to \infty$, where $P_j = P_{d_j}$. We define a function v_j on B_{ϱ_j} by

$$v_j(z) = u_{d_j}(P_j + d_j z) \qquad \text{for } z \in B_{\varrho_j}.$$

We may assume that $P_j \to \overline{P}$. Using the Schauder estimates we can show as in [12] that $v_j \to w_{Q(\overline{P})}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$, where $w_{Q(\overline{P})}$ is the ground state solution of equation (3) with $M = Q(\overline{P})$. Since $M_1 \le Q(x) \le M_2$ on $\overline{\Omega}$ for some constants $0 < M_1 < M_2$ and $w_{Q(\overline{P})}(z) = Q(\overline{P})^{-1/(p-1)} w_1(z)$, we see that

$$w_{Q(\overline{P})}(z), |\nabla w_{Q(\overline{P})}(z)| \le C_\circ e^{-\mu|z|} \qquad \text{on } \mathbb{R}^N,$$

for some constants $\mu > 0$ and $C_\circ > 0$ independent of P. For the mountain pass level we have the following estimate from below for each $R > 0$:

$$
\begin{aligned}
(11)\qquad c_{d_j} &\geq \int\limits_{|x-P_j|<d_jR} \frac{p-1}{2(p+1)} Q(x) u_{d_j}(x)^{p+1}\,dx \\
&= d_j^N \int\limits_{|z|<R} \frac{p-1}{2(p+1)} Q(P_j+d_jz) v_j(z)^{p+1}\,dz \\
&= d_j^N \int\limits_{|z|<R} \frac{p-1}{2(p+1)} Q(\overline{P}) w_{Q(\overline{P})}(z)^{p+1}\,dz \\
&\quad + d_j^N \int\limits_{|z|<R} \frac{p-1}{2(p+1)} Q(P_j+d_jz) v_j(z)^{p+1}\,dz \\
&\quad - d_j^N \int\limits_{|z|<R} \frac{p-1}{2(p+1)} Q(\overline{P}) w_{Q(\overline{P})}(z)^{p+1}\,dz \\
&= d_j^N \int\limits_{|z|<R} \frac{p-1}{2(p+1)} Q(\overline{P}) w_{Q(\overline{P})}(z)^{p+1}\,dz \\
&\quad + d_j^N \int\limits_{|z|<R} \frac{p-1}{2(p+1)} Q(P_j+d_jz)(v_j(z)^{p+1} - w_{Q(\overline{P})}(z)^{p+1})\,dz \\
&\quad + d_j^N \int\limits_{|z|<R} \frac{p-1}{2(p+1)} (Q(P_j+d_jz) - Q(\overline{P})) w_{Q(\overline{P})}(z)^{p+1}\,dz.
\end{aligned}
$$

Let us denote the last two integrals on the right side of this inequality by J_1 and J_2, respectively. Let $R > 0$ be a fixed and large number and set $d_R = C_\circ e^{-\mu R/2}$. We choose $j_\circ = j_\circ(R)$ sufficiently large so that $\|v_j - w_{Q(\overline{P})}\|_{C^2(\overline{B(0,R)})} \leq d_R$ for $j \geq j_\circ$. Thus

$$
(12)\qquad |J_1| \leq C_1 d_j^N R^N d_R
$$

for some constant $C_1 > 0$ and all $j \geq j_\circ$. Since $Q(P_j + d_jz) \to Q(\overline{P})$ uniformly on $\overline{B(0,2R)}$, we have

$$
(13)\qquad |J_2| \leq C_2 d_j^N o(1)
$$

for some constant $C_2 > 0$ and $j \geq j_\circ$. Inserting estimates (12) and (13) into (11) we get

$$
c_{d_j} \geq d_j^N \Bigg(\int\limits_{B(0,R)} \frac{p-1}{2(p+1)} Q(\overline{P}) w_{Q(\overline{P})}^{p+1}\,dx - C_1 R^N d_R + o(1) \Bigg).
$$

Next, we observe that

$$\int\limits_{B(0,R)} \frac{p-1}{2(p+1)} Q(\overline{P}) w_{Q(\overline{P})}(z)^{p+1}\,dx$$
$$= I_{Q(\overline{P})}(w_{Q(\overline{P})}) - \int\limits_{|z|>R} \frac{p-1}{2(p+1)} Q(\overline{P}) w_{Q(\overline{P})}(z)^{p+1}\,dz$$
$$\geq I_{Q(\overline{P})}(w_{Q(\overline{P})}) - C_3 e^{-\mu R}$$

for some constant $C_3 > 0$. Combining the last two estimates we derive the following estimate from below for the level c_{d_j}:

$$c_{d_j} \geq d_j^N (I_{Q(\overline{P})}(w_{Q(\overline{P})}) - Ce^{-\mu R} + o(1))$$

for some constants $\mu > 0$ and $C > 0$. On the other hand, by Lemma 1 we have

$$c_{d_j} \leq M[\phi_{d_j}] \leq \frac{d_j^N}{2} I_{Q(P)}(w_{Q(P)}) A_{d_j}$$

for each $P \in \partial\Omega$. Combining the last two estimates and letting first $j \to \infty$ and then $R \to \infty$ we obtain

$$\frac{p-1}{2(p+1)} \int\limits_{\mathbb{R}^N} Q(\overline{P}) w_{Q(\overline{P})}^{p+1}\,dx \leq \frac{p-1}{4(p+1)} \int\limits_{\mathbb{R}^N} Q(P) w_{Q(P)}^{p+1}\,dx$$

for each $P \in \partial\Omega$. Since $w_{Q(P)} = Q(P)^{-1/(p-1)} w_1$ for each P, we see that this inequality is equivalent to

$$Q(\overline{P}) \geq 2^{(p-1)/2} Q(P)$$

for each $P \in \partial\Omega$. This contradicts the fact that Q achieves its maximum on the boundary.

Step II. We show that $P_d \in \partial\Omega$ for $d > 0$ small. Arguing indirectly we may assume that $P_j = P_{d_j} \in \Omega$ for some decreasing sequence $d_j \to 0$. By Step I, $P_j \to P \in \partial\Omega$ and we assume that $P = 0$. Let $y = \Psi(x)$ be a diffeomorphism which straightens the boundary $\partial\Omega$ near P. We assume that the closed ball $\overline{B}_{2k}$ is contained in the domain of definition of $\Phi = \Psi^{-1}$ for some $k > 0$ and let $Q_j = \Psi(P_j) \in \overline{B}_k^+$ for all j. We set $v_j(y) = u_j(\Phi(y))$ for $y \in \overline{B}_{2k}^+$ and we extend v_j to $\overline{B}_{2k}$ by the reflection

$$\overline{v}_k(y) = \begin{cases} v_k(y) & \text{if } y \in B_{2k}^+, \\ v_k(y', -y_N) & \text{if } y \in B_{2k}^-, \end{cases}$$

where $B_{2k}^- = D_{2k} \cap \{y \in \mathbb{R}^N : y_N < 0\}$. Finally, we set $w_j(z) = \overline{v}_j(Q_j + d_j z)$ for $z \in \overline{B}_{k/d_j}$. Let $Q_j = (q_j', \alpha_j d_j)$, where $q_j' \in \mathbb{R}^{N-1}$ and $\alpha_j > 0$. As in [12] using the Schauder estimates one can show that $w_j \to w_{Q(P)}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$, where $w_{Q(P)}$ is a ground state solution of (3) with $M = Q(P)$. Here we have used the fact that $Q_j \to 0$. Since the ground state has only one maximum we deduce from Lemma 4.2 in [12], repeating the argument from p. 837 in

[12], that w_j attains only one maximum in a ball $B(0,R)$, where $R>0$ is chosen so that $R>\alpha_j$ for all j. If $\alpha_j>0$, then it follows from the definition of $\overline{v}_j$ that $Q_j^*=(q_j',-\alpha_j d_j)$ is also another local maximum point of w_j in $B(0,R)$, which is impossible.

Step III. We show that u_d has at most one local maximum. In the contrary case there exists a decreasing sequence d_j such that u_{d_j} has two local maxima at P_j and P_j'. By the previous part of the proof $P_j, P_j' \in \partial\Omega$. We may assume that $|P_j-P_j'|/d_j \to \infty$ as $j\to\infty$, since in the contrary case the scaled function from Step II has two local maxima in the ball $B(0,R)$. Assume $P_j \to \overline{P} \in \partial\Omega$. We now introduce the diffeomorphism $y=\Psi(x)$ which straightens a part of the boundary $\partial\Omega$ around P_j and define, as in Step II, v_j, $\overline{v}_j$ and w_j. Using the Schauder estimates we show that, up to a subsequence, $w_j \to w_{Q(\overline{P})}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $w_{Q(\overline{P})}$ is the ground state solution of (3) with $M=Q(\overline{P})$. Since $u_j=u_{d_j}$ satisfies (3) we see that

$$\begin{aligned} c_{d_j} &= \int_\Omega \frac{p-1}{2(p+1)} Q(x) u_j(x)^{p+1}\,dx \\ &= \int_{D_1} \frac{p-1}{2(p+1)} Q(x) u_j(x)^{p+1}\,dx + \int_{\Omega-D_1} \frac{p-1}{2(p+1)} Q(x) u_j(x)^{p+1}\,dx \\ &= I_1 + I_2, \end{aligned}$$

where $D_1=\Phi(B_{Rd_j})$. As in Step I (see also p. 832 in [12]) we check that

$$I_1 \ge d_j^N\bigg[\int_{B_k^+} \frac{p-1}{2(p+1)} Q(\overline{P}) w_{Q(\overline{P})}(z)^{p+1}\,dz - C_1 R^N d_R - C_2 d_j\bigg] \tag{14}$$

for some constants $C_1>0$ and $C_2>0$. It follows from Lemma 2 that u_j is bounded away from zero on $B_{d_j}(P_j')\cap\Omega$, uniformly in j. Consequently,

$$I_2 = \int_{\Omega-D_1} \frac{p-1}{2(p+1)} Q(x) u_j(x)^{p+1}\,dx \ge \eta \tag{15}$$

for some constant $\eta>0$ independent of j. Estimates (14) and (15) give the following estimate of c_{d_j} from below:

$$c_{d_j} \ge d_j^N\bigg(\frac{1}{2} I_{Q(\overline{P})}(w_{Q(\overline{P})}) + \eta - Ce^{-\mu R} - C_2 d_j\bigg). \tag{16}$$

Here we have used the exponential decay of $w_{Q(\overline{P})}$. On the other hand, by Lemma 1 we have

$$c_{d_j} \le M[\phi_{d_j}] \le d_j^N\bigg(\frac{1}{2} I_{Q(\overline{P})}(w_{Q(\overline{P})}) + O(d^\alpha)\bigg).$$

The last estimate contradicts (16) if d_j is sufficiently small and R sufficiently large.

Step IV. In the final step we show that if $u_d(P_d) = \max_{x\in\overline{\Omega}} u_d(x)$, then $P_d \to P^*$, where $Q(P^*) = \max_{x\in\partial\Omega} Q(x)$, $P^* \in \partial\Omega$. Obviously, by the previous steps $P_d \in \partial\Omega$. Suppose that $P_d \to P^* \in \partial\Omega$. As in the paper [12] (see pp. 837–838 there) we can establish the following estimate of c_d from below:

$$c_d \ge d^N\left(\frac{1}{2}I_{Q(P^*)}(w_{Q(P^*)}) - Ce^{-\mu R} + o(1)\right),$$

where $C > 0$ and $\mu > 0$ are constants independent of R and $o(1) \to 0$ as $d \to 0$. Using this and the estimate from Lemma 1 we get

$$\begin{aligned}\frac{p-1}{4(p+1)}Q(P^*)^{-1/(p-1)} &\int\limits_{\mathbb{R}^N} w_1(z)^{p+1}\,dz - Ce^{-\mu R} + o(1)\\ &\le \frac{p-1}{4(p+1)}Q(P)^{-1/(p-1)}\int\limits_{\mathbb{R}^N} w_1(z)^{p+1}\,dz + O(d^\alpha)\end{aligned}$$

for every $P \in \partial\Omega$. Letting $d \to 0$ and then $R \to \infty$ we obtain

$$Q(P^*)^{-1/(p-1)} \le Q(P)^{-1/(p-1)}$$

for each $P \in \partial\Omega$ and the claim follows.

Inspection of the proof of Step 1 of Theorem 1 shows that if $Q(x) \le hQ(y)$ for all $x \in \Omega$ and y in $\partial\Omega$ for some constant $1 < h < 2^{(p-1)/2}$, then the points P_d where u_d achieve their maxima concentrate at a boundary point of Ω. However, it is not clear whether $P_d \in \partial\Omega$ for small d under this assumption on Q.

If $Q(x) = M$ on $\partial\Omega$ and $Q(x) \le M$ for all $x \in \Omega$, where $M > 0$ is a constant, then according to Theorem 1, u_d concentrates on the boundary. The question is how to locate a point of $\partial\Omega$ at which the concentration occurs. We show that, under an additional assumption on the behaviour of $Q(x)$ near the boundary, the concentration occurs at a point where the mean curvature of $\partial\Omega$ attains its maximum. We need the following asymptotic formula for c_d.

Proposition 1. *Suppose that $Q(x) \le M$ on $\overline{\Omega}$ and $Q(x) = M$ on $\partial\Omega$ for some constant $M > 0$, and moreover,*

$$|Q(x) - M| = O(\mathrm{dist}(x,\partial\Omega)^k) \tag{17}$$

for x close to $\partial\Omega$ and some constant $k > 1$. Then

$$c_d = d^N\{I_M(w_M) - (N-1)\gamma M^{-1/(p-2)}H(P_d) + o(d)\} \tag{18}$$

as $d \to 0$, where $u_d(P_d) = \max_{x\in\Omega} u_d(x)$, $H(P)$ denotes the mean curvature of $\partial\Omega$ at P and

$$\gamma = \frac{1}{N+1}\int\limits_{\mathbb{R}^N_+} w'(|z|)^2 z_N\,dz > 0.$$

(*The quantity $o(d)$ in* (18) *is independent of P_d.*)

The proof of Proposition 1 is identical to that of Proposition 2.1 in [11]. In the Appendix we provide a sketch of the proof.

Similarly, we have

$$M[\phi_d] \le d^N \left\{ \frac{1}{2} I_M(w_M) - (N-1)\gamma H(P) M^{-2/(p-1)} + o(d) \right\} \tag{19}$$

for all $P \in \partial\Omega$.

THEOREM 2. *Suppose that the assumptions of Proposition* 1 *hold and let* $u_d(P_d) = \max_{x\in\Omega} u_d(x)$. *Then* $P_d \in \partial\Omega$ *for small* d *and* $P_d \to \overline{P}$, *with* $H(\overline{P}) = \max_{P\in\partial\Omega} H(P)$.

Proof. Since $c_d \le M[\phi_d]$, it follows from (18) and (19) that $H(P_d) \ge H(P) + o(d)$ for d sufficiently small and all $P \in \partial\Omega$. Letting $d \to 0$ we get $H(\overline{P}) \ge H(P)$ for all $P \in \partial\Omega$.

4. Multiple single peak solutions. In this section we calculate a lower bound on the number of single peak solutions to problem (1_d) in terms of the category of $\partial\Omega$. Explicitly, we show there are at least $\mathrm{cat}(\partial\Omega)$ distinct nonconstant solutions provided d is sufficiently small.

Assume throughout this section that $Q \in C^2(\overline{\Omega})$ and $0 < M_1 = \min_\Omega Q(x) < M_2 = \max_{\overline{\Omega}} Q(x) = \max_{\partial\Omega} Q(x)$.

It is convenient for our purposes to consider a different functional than $I_d(u)$ as used in the previous section. We look for minima of the functional

$$E_d(u) = \int_\Omega (d^2|\nabla u|^2 + u^2)\,dx$$

constrained to the manifold

$$V_1^Q(\Omega) = \Big\{ u \in H^1(\Omega) : \int_\Omega Q(x)|u|^{p+1}\,dx = 1 \Big\}.$$

One may check that if u is a critical point of E_d on $V_1^Q(\Omega)$ then $v = [E_d(u)]^{1/(p-1)} u$ is a solution of (1_d). Furthermore, we have the following relation between I_d and E_d:

$$I_d(v) = \frac{p-1}{2(p+1)} [E_d(u)]^{(p+1)/(p-1)}.$$

Therefore, an absolute minimum of E_d corresponds to the least energy of I_d.

Let

$$c_d = \min_{u\in V_1^Q(\Omega)} E_d(u),$$

which is easily seen to be achieved on $V_1^Q(\Omega)$. Also, a critical point of E_d with absolute minimum critical value corresponds to a critical point of I_d and therefore a least energy solution of (1_d).

There exists a one-to-one correspondence between the solutions of (1_d) of arbitrary sign, and the solutions of the rescaled problem

$$(1_{1/d}) \qquad \begin{cases} -\Delta u + u = Q_d(x)u^p, & x \in \Omega_{1/d}, \\ \dfrac{\partial u(x)}{\partial n} = 0, & x \in \partial\Omega_{1/d}, \end{cases}$$

where $\Omega_{1/d} = \{x \in \mathbb{R}^N : dx \in \Omega\}$ and $Q_d(x) = Q(dx)$. This correspondence is given by

$$\sigma(u)(x) = d^{N/(p+1)}u(dx).$$

The functional associated with problem $(1_{1/d})$ is

$$\widetilde{E}_{1/d}(u) = \int\limits_{\Omega_{1/d}} (|\nabla u|^2 + u^2)\,dx$$

for $u \in V_1^Q(\Omega_{1/d}) = \{u \in H^1(\Omega_{1/d}) : \int_{\Omega_{1/d}} Q_d(x)|u|^{p+1}\,dx = 1\}$.

By direct computation as shown in [14], Lemma 1.1, we have the following

Lemma 3. *For any* $u \in V_1^Q(\Omega)$,

$$\widetilde{E}_{1/d}[\sigma(u)] = d^{-N(p-1)/(p+1)}E_d(u)$$

and therefore

$$\min_{V_1^Q(\Omega_{1/d})} \widetilde{E}_{1/d} = d^{-N(p-1)/(p+1)} \min_{V_1^Q(\Omega)} E_d(u).$$

The following notation is needed: for $\alpha > 0$ and $r \ge 1$ define

$$V_\alpha(\Omega_r) = \Big\{u \in H^1(\Omega_r) : \int\limits_{\Omega_r} |u|^{p+1}\,dx = \alpha\Big\},$$

$$V_\alpha^Q(\Omega_r) = \Big\{u \in H^1(\Omega_r) : \int\limits_{\Omega_r} Q_{1/r}(x)|u|^{p+1}\,dx = \alpha\Big\},$$

$$m(r,\alpha) = \min_{u\in V_\alpha(\Omega_r)} \widetilde{E}_r(u),$$

$$m_Q(r,\alpha) = \min_{u\in V_\alpha^Q(\Omega_r)} \widetilde{E}_r(u),$$

$$m(+,\alpha) = \min\Big\{\int\limits_{\mathbb{R}^N_+} (|\nabla u|^2 + u^2)\,dx : u \in H^1(\mathbb{R}^N_+),\ \int\limits_{\mathbb{R}^N_+} |u|^{p+1}\,dx = \alpha\Big\},$$

$$m(\infty,\alpha) = \min\Big\{\int\limits_{\mathbb{R}^N} (|\nabla u|^2 + u^2)\,dx : u \in H^1(\mathbb{R}^N),\ \int\limits_{\mathbb{R}^N} |u|^{p+1}\,dx = \alpha\Big\}.$$

Let w_1 be the ground state solution of (3) with $M = 1$ and let $\widetilde{w} = w_1/\|w_1\|_{L^{p+1}}$. The following result is Lemma 1.2 of [14].

Lemma 4. *For $r \geq 1$ and $\alpha > 0$:*

(i) $m(\infty, 1) = \int_{\mathbb{R}^N} (|\nabla \widetilde{w}|^2 + |\widetilde{w}|^2)\, dx$;

(ii) $m(r, \alpha) = \alpha^{2/(p+1)} m(r, 1)$ *where r may be $+$ or ∞ as well*;

(iii) $m(\infty, 2) = 2m(+, 1)$.

Now we define a comparison function in order to find an asymptotic estimate of c_d. Let $\varrho > 0$ be fixed throughout this section (different from ϱ in Section 2) such that the neighbourhood

$$N_\varrho(\partial\Omega) = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \partial\Omega) < \varrho\}$$

is homotopic to $\partial\Omega$. Define

$$\eta(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1, \\ 0 & \text{for } t \geq 2 \end{cases}$$

and $|\eta'(t)| \leq 2$. For $P \in \partial\Omega$ let $w_{Q(P)} \equiv w_P$ be the ground state solution of (3) with $M = Q(P)$ and set

$$\psi_d(P)(x) = \eta\left(\frac{|x - P|}{\varrho}\right) w_P\left(\frac{x - P}{d}\right)$$

for $x \in \overline{\Omega}$. The comparison function is defined as

$$\phi_d(P)(x) = \frac{\psi_d(P)(x)}{Q^{1/(p+1)}(x) \|\psi_d(P)\|_{L^{p+1}(\Omega)}}.$$

Clearly, $\phi_d(P) \in V_1^Q(\Omega)$. For $u \in V_1^Q(\Omega)$ we define a "barycenter" function as

$$\beta(u) = \int_\Omega Q(x)|u|^{p+1} x\, dx.$$

It is clear that $\beta(u) \in \overline{\operatorname{conv}}(\Omega)$ for $u \in V_1^Q(\Omega)$. Furthermore, using the arguments of Proposition 2 below, one may show

$$\beta(\phi_d(P)) = P + o(1) \quad \text{ as } d \to 0.$$

Proposition 2. *For $P \in \partial\Omega$,*

$$E_d(\phi_d(P)) = d^{N(p-1)/(p+1)} \left(m\left(+, \frac{1}{Q(P)}\right) + o(1) \right)$$

as $d \to 0$.

Proof. Upon substitution and expansion we obtain

$$\begin{aligned} E_d(\phi_d(P)) &= \frac{1}{\|\psi_d(P)\|_{L^{p+1}}^2} \int_\Omega \left(d^2 \left| \nabla \left(\frac{\psi_d}{Q^{1/(p+1)}} \right) \right|^2 + \left| \frac{\psi_d}{Q^{1/(p+1)}} \right|^2 \right) dx \\ &= \frac{1}{\|\psi_d\|_{L^{p+1}}^2} (I_1 + I_2 + I_3), \end{aligned}$$

where

$$I_1 = \int\limits_{\Omega} \frac{\eta^2(|x-P|/\varrho)}{Q^{2/(p+1)}}\left(\left|\nabla w_P\left(\frac{x-P}{d}\right)\right|^2 + w_P^2\left(\frac{x-P}{d}\right)\right)dx,$$

$$\begin{aligned} I_2 = d\int\limits_{\Omega}\Big[&\frac{2}{\varrho Q^{2/(p+1)}}\eta w_P \nabla\eta\cdot\nabla w_P \\ &+ 2\eta^2 w_P \frac{1}{Q^{1/(p+1)}}\nabla w_P\cdot\nabla\left(\frac{1}{Q^{1/(p+1)}}\right)\Big]dx, \end{aligned}$$

$$\begin{aligned} I_3 = d^2\int\limits_{\Omega}\Big[&\frac{1}{\varrho^2 Q^{2/(p+1)}}|\nabla\eta|^2 w_P^2 + \eta^2 w_P^2\left|\nabla\frac{1}{Q^{1/(p+1)}}\right|^2 \\ &+ \frac{2}{\varrho Q^{1/(p+1)}}\eta w_P^2\nabla\eta\cdot\nabla\left(\frac{1}{Q^{1/(p+1)}}\right)\Big]dx \end{aligned}$$

where we have dropped the arguments of η and w_P in I_2 and I_3 for clarity. We examine each integral in turn.

Using the substitution $z = (x-P)/d$ we obtain

$$I_1 = d^N \int\limits_{(\Omega-P)_{1/d}} \frac{\eta^2(|z|d/\varrho)}{Q(dz+P)^{2/(p+1)}}(|\nabla w_P(z)|^2 + w_P(z)^2)\,dz.$$

As w and $|\nabla w|$ decay exponentially and Q is bounded, for any $\varepsilon > 0$ there exists $R > 0$ such that

$$\int\limits_{(\Omega-P)_{1/d}\cap(|z|\geq R)} \frac{\eta^2}{Q^{2/(p+1)}}(|\nabla w_P|^2 + w_P^2)\,dz \leq \frac{\varepsilon}{2}.$$

For this R, $(\Omega - P)_{1/d} \cap (|z| \leq R) \to B_R^+(0)$ in measure as $d \to 0$. By the mean value theorem we find that

$$Q(dz+P)^{2/(p+1)} = Q(P)^{2/(p+1)} + O(d).$$

Choosing $d < \varrho/R$ sufficiently small we get

$$\left|\int\limits_{(\Omega-P)_{1/d}\cap(|z|\leq R)} \frac{|\nabla w_P|^2 + w_P^2}{Q(P+dz)^{2/(p+1)}}\,dz - \int\limits_{B_R^+} \frac{|\nabla w_P|^2 + w_P^2}{Q(P)^{2/(p+1)}}\,dx\right| \leq \frac{\varepsilon}{2}$$

and therefore

$$I_1 = d^N\left(\int\limits_{\mathbb{R}_+^N} \frac{|\nabla w_P|^2 + w_P^2}{Q(P)^{2/(p+1)}}\,dx + o(1)\right).$$

Using the same substitution as above we estimate I_2 to be

$$\begin{aligned}
I_2 = d^N \int\limits_{(\Omega-P)_{1/d}} d\bigg[&\frac{2\eta w_P}{\varrho Q(P+dz)^{2/(p+1)}}\nabla\eta\cdot\nabla w_p \\
&+ \frac{2\eta^2 w_P}{Q(P+dz)^{2/(p+1)}}\nabla w_P\cdot\nabla\bigg(\frac{1}{Q(P+dz)^{1/(p+1)}}\bigg)\bigg]dz \\
\le d^N\bigg[&\int\limits_{(\Omega-P)_{1/d}\cap(|z|\le\varrho/d)} d\frac{2w_P}{Q(P+dz)^{1/(p+1)}}\nabla w_P\cdot\nabla\bigg(\frac{1}{Q^{1/(p+1)}}\bigg)dz \\
&+ \int\limits_{(\Omega-P)_{1/d}\cap(\varrho/d\le|z|\le 2\varrho/d)} d\bigg[\frac{4w_P}{\varrho Q(P+dz)^{2/(p+1)}}|\nabla w_P| \\
&+ \frac{2w_p}{Q(P+dz)^{1/(p+1)}}\nabla w_P\cdot\nabla\bigg(\frac{1}{Q^{1/(p+1)}}\bigg)\bigg]dz\bigg].
\end{aligned}$$

Since w_P and $|\nabla w_P|$ decay exponentially and the terms involving Q are bounded independently of d, we have

$$I_2 \le d^N(o(1)).$$

Precisely the same reasoning shows that

$$I_3 \le d^N(o(1))$$

and so

$$\begin{aligned}
(20)\qquad \int\limits_{\Omega}\bigg(d^2\bigg|\nabla\bigg(\frac{\psi_d}{Q^{1/(p+1)}}\bigg)\bigg|^2 + \bigg|\frac{\psi_d}{Q^{1/(p+1)}}\bigg|^2\bigg)dx& \\
= d^N\bigg(\int\limits_{\mathbb{R}^N_+}\frac{|\nabla w_P|^2+w_P^2}{Q(P)^{2/(p+1)}}\,dx + o(1)\bigg)&.
\end{aligned}$$

Now we estimate $\|\psi_d\|^2_{L^{p+1}}$:

$$\begin{aligned}
\|\psi_d\|^2_{L^{p+1}} &= \bigg[\int\limits_{\Omega}\eta^{p+1}\bigg(\frac{|x-P|}{\varrho}\bigg)w_P^{p+1}\bigg(\frac{x-P}{d}\bigg)dx\bigg]^{2/(p+1)} \\
&= d^{2N/(p+1)}\bigg[\int\limits_{(\Omega-P)_{1/d}}\eta^{p+1}\bigg(\frac{|z|d}{\varrho}\bigg)w_P^{p+1}(z)\,dz\bigg]^{2/(p+1)} \\
&= d^{2N/(p+1)}\bigg[\int\limits_{(\Omega-P)_{1/d}\cap(|z|\le\varrho/d)} w_P^{p+1}(z)\,dz \\
&\qquad + \int\limits_{(\Omega-P)_{1/d}\cap(\varrho/d\le|z|\le 2\varrho/d)}\eta^{p+1}\bigg(\frac{|x-P|}{\varrho}\bigg)w_P^{p+1}(z)\,dz\bigg]^{2/(p+1)}
\end{aligned}$$

$$= d^{2N/(p+1)} \Big[\int_{\mathbb{R}^N_+} w_P^{p+1}(z)\, dz + o(1) \Big]^{2/(p+1)}$$

hence

$$\|\psi_d\|^2_{L^{p+1}} = d^{2N/(p+1)} \Big\{ \Big[\int_{\mathbb{R}^N_+} w_P^{p+1}(z)\, dz \Big]^{2/(p+1)} + o(1) \Big\}. \tag{21}$$

From equations (20) and (21) we find

$$\begin{aligned} E_d(\phi_d(P)) &= \frac{d^N \Big(\int_{\mathbb{R}^N_+} \frac{|\nabla w_P|^2 + w_P^2}{Q(P)^{2/(p+1)}}\, dx + o(1) \Big)}{d^{2N/(p+1)} ([\int_{\mathbb{R}^N_+} w_P^{p+1}(x)\, dx]^{2/(p+1)} + o(1))} \\ &= \frac{d^{N(p-1)/(p+1)}}{Q(P)^{2/(p+1)}} \Big(\frac{\int_{\mathbb{R}^N_+} |\nabla w_P|^2 + w_P^2}{(\int_{\mathbb{R}^N_+} w_P^{p+1})^{2/(p+1)}} + o(1) \Big) \\ &= \frac{d^{N(p-1)/(p+1)}}{Q(P)^{2/(p+1)}} (m(+,1) + o(1)) \\ &= d^{N(p-1)/(p+1)} \Big(m\Big(+, \frac{1}{Q(P)}\Big) + o(1) \Big) \qquad \text{as } d \to 0. \end{aligned}$$

Proposition 3. *For* $c_d = \min_{V_1^Q(\Omega)} E_d(u)$, *we have*

$$c_d = d^{N(p-1)/(p+1)} \Big(m\Big(+, \frac{1}{M_2}\Big) + o(1) \Big).$$

Proof. As $\phi_d(P) \in V_1^Q(\Omega)$ for any $P \in \partial\Omega$, choosing $P \in \partial\Omega$ with $Q(P) = M_2$ we find that

$$c_d \le \frac{d^{N(p-1)/(p+1)}}{M_2^{2/(p+1)}} (m(+,1) + o(1)).$$

Suppose this inequality is strict, so that

$$\liminf_{d \to 0} d^{-N(p-1)/(p+1)} c_d < \frac{m(+,1)}{M_2^{2/(p+1)}}.$$

Then there is a subsequence $d_n \to 0$ and $u_n \in V_1^Q(\Omega)$ such that

$$c_{d_n} \equiv c_n = E_{d_n}(u_n)$$

and

$$\lim_{n \to \infty} d_n^{-N(p-1)/(p+1)} E_{d_n}(u_n) = A < m\Big(+, \frac{1}{M_2}\Big).$$

By rescaling $v_n = \sigma(u_n) = d_n^{N/(p+1)} u_n(d_n x)$ we have

$$\lim_{n \to \infty} d_n^{-N(p-1)/(p+1)} E_{d_n}(u_n) = \lim_{n \to \infty} \widetilde{E}_{1/d_n}(v_n) = A < m\Big(+, \frac{1}{M_2}\Big).$$

It is obvious that $A = \lim_n m_Q(1/d_n, 1)$ as v_n solves the rescaled problem $(1_{1/d_n})$. Defining

$$\mu_n = \chi_n(x)Q_n(x)|v_n|^{p+1},$$

where χ_n is the characteristic function of $\Omega_{1/d_n} \equiv \Omega_n$ and $Q_n(x) = Q(d_n x)$, we get

$$\int_{\mathbb{R}^N} \mu_n = 1$$

as $v_n \in V_1^Q(\Omega_n)$. Therefore we may apply the Concentration-Compactness principle.

Following the arguments of [14, Lemma 2.1] and replacing the function v_n there with $Q_n^{1/(p+1)} v_n$ we find that Vanishing of the sequence μ_n does not occur.

If Dichotomy occurs, then there exists $\lambda \in (0,1)$ such that for

$$\Theta_n(t) = \sup_{y\in\mathbb{R}^N} \int_{B_t(y)} \chi_n Q_n |v_n|^{p+1}\,dx \quad \text{and} \quad \Theta(t) = \lim_{n\to\infty} \Theta_n(t),$$

we have $\lim_{t\to\infty} \Theta(t) = \lambda$. For any $\varepsilon > 0$ there exists R_0 such that $\Theta(R_0) \geq \lambda - \varepsilon/6$ and there exist $\{y_n\} \subset \mathbb{R}^N$ and $n_0 \geq 0$ such that for $n \geq n_0$,

$$\Theta_n(R_0) - \varepsilon/6 \leq \int_{B_{R_0}(y_n)} \chi_n Q_n |v_n|^{p+1}\,dx$$

and

$$|\Theta(R_0) - \Theta_n(R_0)| \leq \varepsilon/6.$$

Therefore

$$\lambda - \frac{\varepsilon}{6} \leq \Theta(R_0) \leq \Theta_n(R_0) + \frac{\varepsilon}{6} \leq \int_{B_{R_0}(y_n)} \chi_n Q_n |v_n|^{p+1}\,dx + \frac{\varepsilon}{3}$$

and so

$$\int_{B_{R_0}(y_n)} \chi_n Q_n |v_n|^{p+1}\,dx \geq \lambda - \frac{\varepsilon}{2}$$

for all $n \geq n_0$. We may also choose $R_n \to \infty$ such that

$$\Theta_n(2R_n) \leq \lambda + \varepsilon/2.$$

With η as defined above, let $\xi = 1 - \eta$ and

$$v_n^1(x) = \chi_n(x)\eta\left(\frac{|x-y_n|}{R_0}\right)v_n(x), \qquad v_n^2(x) = \chi_n(x)\xi\left(\frac{|x-y_n|}{R_n}\right)v_n(x).$$

It is standard to show that

$$\left|1 - \int_{\mathbb{R}^N} Q_n(|v_n^1|^{p+1} + |v_n^2|^{p+1})\,dx\right| \leq \varepsilon$$

for all $n \geq n_0$. Thus for all $n \geq n_0$,

$$\Big|\int_{\mathbb{R}^N} Q_n|v_n^1|^{p+1}\,dx - \lambda\Big| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \Big|\int_{\mathbb{R}^N} Q_n|v_n^1|^{p+1}\,dx - (1-\lambda)\Big| \leq \frac{\varepsilon}{2}.$$

Hence

$$\int_{\mathbb{R}^N} Q_n|v_n^1|^{p+1}\,dx = \lambda + \varepsilon_n^1 \qquad \text{with } |\varepsilon_n^1| \leq \varepsilon,$$
$$\int_{\mathbb{R}^N} Q_n|v_n^2|^{p+1}\,dx = 1-\lambda - \varepsilon_n^2 \qquad \text{with } |\varepsilon_n^2| \leq \varepsilon.$$

By choosing R_0 sufficiently large and keeping ε fixed we get

$$\int_{\Omega_n}(|\nabla v_n|^2+|v_n|^2)\,dx - \int_{\Omega_n}(|\nabla v_n^1|^2+|v_n^1|^2)\,dx - \int_{\Omega_n}(|\nabla v_n^2|^2+|v_n^2|^2)\,dx \geq -2\varepsilon.$$

Thus

$$\begin{aligned}
(22)\qquad A &= \lim_{n\to\infty}\int_{\Omega_n}(|\nabla v_n|^2+|v_n|^2)\,dx \\
&\geq \lim_{n\to\infty}\Big(\int_{\Omega_n}(|\nabla v_n^1|^2+|v_n^1|^2)\,dx + \int_{\Omega_n}(|\nabla v_n^2|^2+|v_n^2|^2)\,dx\Big) - 2\varepsilon \\
&\geq \lim_{n\to\infty}(m_Q(1/d_n,\lambda+\varepsilon_n^1) + m_Q(1/d_n, 1-\lambda+\varepsilon_n^2)) - 2\varepsilon \\
&\geq \lim_{n\to\infty}((\lambda-\varepsilon)^{2/(p+1)}m_Q(1/d_n,1) \\
&\qquad + (1-\lambda-\varepsilon)^{2/(p+1)}m_Q(1/d_n,1)) - 2\varepsilon \\
&= (\lambda-\varepsilon)^{2/(p+1)}A + (1-\lambda-\varepsilon)^{2/(p+1)}A - 2\varepsilon.
\end{aligned}$$

Now $A > 0$ from the following reasoning: as the embedding $H^1(\Omega_n) \to L^{p+1}(\Omega_n)$ is compact, there is a constant depending only on the cone condition of Ω_n such that $\|v_n\|_{L^{p+1}}^{p+1} \leq \widetilde{c}_n\widetilde{E}_{1/d_n}(v_n)$. As the cone condition is independent of n, we have $\widetilde{c}_n \equiv \widetilde{c}$. As $v_n \in V_1^Q(\Omega_n)$ it is trivial to see that $\|v_n\|_{L^{p+1}}^{p+1} \geq 1/M_2 > 0$. Thus $\widetilde{E}_{1/d_n}(v_n) \geq \widetilde{c}/M_2$ and $A = \lim_n \widetilde{E}_{1/d_n}(v_n) > 0$.

So dividing the final equation in (22) by A gives

$$1 \geq (\lambda-\varepsilon)^{2/(p+1)} + (1-\lambda-\varepsilon)^{2/(p+1)} - 2\varepsilon$$

and letting $\varepsilon \to 0$ we get the contradiction

$$1 \geq \lambda^{2/(p+1)} + (1-\lambda)^{2/(p+1)} > 1.$$

Therefore Dichotomy does not occur.

With Vanishing and Dichotomy ruled out the sequence μ_n must be tight, so there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that for any $\varepsilon > 0$ there exists $R > 0$ such that

$$(23)\qquad \int_{B_R(y_n)} \chi_n Q_n|v_n|^{p+1}\,dx \geq 1-\varepsilon.$$

Assertion 1. *There exists $C > 0$ such that* $\operatorname{dist}(y_n, \partial\Omega_n) \le C$.

If not, then $\operatorname{dist}(y_n, \partial\Omega_n) \to \infty$. From (23), $y_n \in \Omega_n$. For $\varepsilon > 0$ fixed, let $R > 0$ be such that (23) holds. Choose n large such that $B_{2R}(y_n) \subset \Omega_n$ and define

$$w_n(x) = \eta\left(\frac{|x - y_n|}{R}\right) v_n(x),$$

which clearly belongs to $H^1(\Omega_n)$. Choosing R larger if necessary, we have

$$\int_{\Omega_n} (|\nabla v_n|^2 + |v_n|^2)\, dx - \int_{\Omega_n} (|\nabla w_n|^2 + |w_n|^2)\, dx \ge -2\varepsilon.$$

Let

$$\lambda_n \equiv \int_{\mathbb{R}^N} \chi_n Q_n |w_n|^{p+1}\, dx \ge 1 - \varepsilon$$

and define

$$\overline{Q}_n = \frac{\lambda_n}{\int_{\mathbb{R}^N} |w_n|^{p+1}\, dx}.$$

Then $\overline{Q}_n \in [M_1, M_2]$ and so

$$\begin{aligned}
\int_{\Omega_n} (|\nabla v_n|^2 + |v_n|^2)\, dx &\ge \int_{\mathbb{R}^N} (|\nabla w_n|^2 + |w_n|^2)\, dx - 2\varepsilon \\
&\ge m\left(\infty, \frac{\lambda_n}{\overline{Q}_n}\right) - 2\varepsilon = \frac{\lambda_n^{2/(p+1)}}{\overline{Q}_n^{2/(p+1)}} m(\infty, 1) - 2\varepsilon \\
&\ge \frac{(1-\varepsilon)^{2/(p+1)}}{\overline{Q}_n^{2/(p+1)}} 2^{(p-1)/(p+1)} m(+, 1) - 2\varepsilon \\
&\ge \frac{(1-\varepsilon)^{2/(p+1)}}{M_2^{2/(p+1)}} 2^{(p-1)/(p+1)} m(+, 1) - 2\varepsilon \\
&= (1-\varepsilon)^{2/(p+1)} 2^{(p-1)/(p+1)} m\left(+, \frac{1}{M_2}\right) - 2\varepsilon.
\end{aligned}$$

Letting $n \to \infty$ and $\varepsilon \to 0$ gives

$$m\left(+, \frac{1}{M_2}\right) > A \ge 2^{(p-1)/(p+1)} m\left(+, \frac{1}{M_2}\right),$$

a contradiction. Therefore there exists $C > 0$ such that

$$\operatorname{dist}(y_n, \partial\Omega_n) \le C.$$

Choose $\overline{t}_n \in \partial\Omega_n$ with $\operatorname{dist}(y_n, \overline{t}_n) \le C$ and let $t_n = d_n \overline{t}_n \in \partial\Omega$. Let T_n be a unitary matrix such that $\widetilde{\Omega}_n = T_n(\Omega_n - \overline{t}_n)$ has y^N as an inner normal direction to $\partial\widetilde{\Omega}_n$ at 0.

We need the following result of [14] (Assertion 3):

Assertion 2. *For any fixed $R_1 > 0$, as $n \to \infty$,*

$$T_n(\Omega_n - \overline{t}_n) \cap B_{R_1}(0)$$

converges to $B^+_{R_1}(0)$ in the following sense: There exists $K_1 > 0$ and for each $\delta > 0$ there exists n_δ such that

$$\{x \in B^+_{R_1}(0) : x^N \geq \delta\} \subset T_n(\Omega_n - \overline{t}_n) \qquad \text{for all } n \geq n_\delta,$$

and

$$L^N\{x \in T_n(\Omega_n - \overline{t}_n) \cap B_{R_1}(0) : x^N \leq \delta\} \leq K_1\delta$$

where $L^N\{\ldots\}$ is N-dimensional Lebesgue measure.

The next result is Assertion 4 of [14]:

Assertion 3. *$\|v_n\|_{L^\infty(\Omega_n)}$ is uniformly bounded independently of n.*

Now we complete the proof of Proposition 3. As μ_n is tight, for each $\varepsilon > 0$ there exists $R > 0$ such that (23) holds. Since $\operatorname{dist}(y_n, \partial\Omega_n) \leq C$, setting $R_1 = R + C$ gives

$$\int\limits_{B_{R_1}(\overline{t}_n)} \chi_n Q_n |v_n|^{p+1}\, dx \geq 1 - \varepsilon.$$

By Assertions 2 and 3, for the given ε choose $\delta_1 > 0$ such that

$$K_1\delta_1 M_2 \|v_n\|^{p+1}_{L^\infty(\Omega_n)} < \varepsilon.$$

Then there exists n_{δ_1} such that

$$\begin{aligned}
&\int\limits_{B_{R_1}(\overline{t}_n)} \chi_n Q_n |v_n|^{p+1}\, dx \\
&\quad = \int\limits_{\{x \in B^+_{R_1}(0) : x^N \geq \delta_1\}} Q_n(T_n^{-1}x + \overline{t}_n)|v_n(T_n^{-1}x + \overline{t}_n)|^{p+1}\, dx \\
&\qquad + \int\limits_{\{x \in B_{R_1}(0) : x^N < \delta_1\} \cap T_n(\Omega_n + \overline{t}_n)} Q_n(T_n^{-1}x + \overline{t}_n)|v_n(T_n^{-1}x + \overline{t}_n)|^{p+1}\, dx \\
&\quad \geq 1 - \varepsilon \qquad \text{for all } n \geq n_{\delta_1}.
\end{aligned}$$

As $K_1\delta_1 M_2 \|v_n\|^{p+1}_{L^\infty(\Omega_n)} < \varepsilon$ the second integral is less than ε by the second part of Assertion 2. Thus

$$\int\limits_{\{x \in B^+_{R_1}(0) : x^N \geq \delta_1\}} Q_n(T_n^{-1}x + \overline{t}_n)|v_n(T_n^{-1}x + \overline{t}_n)|^{p+1}\, dx \geq 1 - 2\varepsilon$$

for all $n \geq n_{\delta_1}$. Define $x_{\delta_1} = (0, \ldots, 0, \delta_1)$ and

$$\widetilde{v}_n(x) = \eta(|x|/R_1) v_n(T_n^{-1}(x + x_{\delta_1}) + \overline{t}_n)$$

for $x \in \mathbb{R}^N_+$, which is well defined by the first part of Assertion 2 for n sufficiently large. Choose R_1 so large to ensure that

$$\int\limits_{\Omega_n} (|\nabla v_n|^2 + v_n^2)\,dx - \int\limits_{\mathbb{R}^N_+} (|\nabla \widetilde{v}_n|^2 + \widetilde{v}_n^2)\,dx \geq -2\varepsilon$$

and let

$$\lambda_n = \int\limits_{\{x\in B^+_{R_1}(0):x^N\geq\delta_1\}} Q_n(T_n^{-1}x+\bar{t}_n)|v_n(T_n^{-1}x+\bar{t}_n)|^{p+1}\,dx \geq 1-2\varepsilon.$$

Defining

$$\overline{Q}_n = \frac{\lambda_n}{\int_{\mathbb{R}^N_+} |\widetilde{v}_n|^{p+1}\,dx}$$

we get

$$\begin{aligned}
A &= \lim_{n\to\infty} \int\limits_{\Omega_n} (|\nabla v_n|^2 + v_n^2)\,dx \geq \lim_n \int\limits_{\mathbb{R}^N} (|\nabla \widetilde{v}_n|^2 + \widetilde{v}_n^2)\,dx - 2\varepsilon \\
&\geq \lim_n m\biggl(+, \frac{\lambda_n}{\overline{Q}_n}\biggr) - 2\varepsilon = \lim_n \frac{\lambda_n^{2/(p+1)}}{\overline{Q}_n^{2/(p+1)}} m(+,1) - 2\varepsilon \\
&\geq (1-2\varepsilon)^{2/(p+1)} \frac{1}{M_2^{2/(p+1)}} m(+,1) - 2\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ gives the contradiction

$$m\biggl(+, \frac{1}{M_2}\biggr) > A \geq m\biggl(+, \frac{1}{M_2}\biggr),$$

which completes the proof.

An obvious modification of the arguments used in the above proof reveals the following.

Lemma 5. *Let $d_n \to 0$ as $n\to\infty$ and assume a sequence $v_n \in V_1^Q(\Omega_{1/d_n})$ satisfies*

$$\int\limits_{\Omega_{1/d_n}} (|\nabla v_n|^2 + v_n^2)\,dx \to m\biggl(+, \frac{1}{M_2}\biggr)$$

as $n \to \infty$. Then there exists a subsequence v_n, points $y_n \in \mathbb{R}^N$ and a constant C independent of n such that for n sufficiently large,

$$\forall\varepsilon\ \exists R>0 \qquad \lim_{n\to\infty} \int\limits_{B_R(y_n)\cap\Omega_{1/d_n}} Q_n|v_n|^{p+1}\,dx \geq 1-\varepsilon$$

and

$$\operatorname{dist}(y_n, \partial\Omega_n) \leq C.$$

Now we are in a position to prove

PROPOSITION 4. *For $\varrho > 0$ such that $N_\varrho(\partial\Omega)$ is homotopic to $\partial\Omega$, there exist $\varepsilon_1 > 0$ and $d_1 > 0$ such that for any $d \in (0, d_1]$,*

$$\beta(u) \in N_\varrho(\partial\Omega)$$

for all $u \in E_d^{c_d + \varepsilon_1 d^{N(p-1)/(p+1)}}$.

Proof. Suppose not. Then for each $\varepsilon > 0$ and $d > 0$ there exists $u \in E_d^{c_d + \varepsilon d^{N(p-1)/(p+1)}}$ with $\beta(u) \notin N_\varrho(\partial\Omega)$. We may choose subsequences $d_n \to 0$, $\varepsilon_n \to 0$ and functions

$$u_n \in E_{d_n}^{c_{d_n} + \varepsilon_n d_n^{N(p-1)/(p+1)}}$$

with $\beta(u_n) \notin N_\varrho(\partial\Omega)$. Using Proposition 3 we have

$$\lim_{n\to\infty} d_n^{-N(p-1)/p+1} E_{d_n}(u_n) = m\biggl(+, \frac{1}{M_2}\biggr).$$

Rescaling, we find $v_n(x) = d_n^{N/(p+1)} u_n(d_n x) \in V_1^Q(\Omega_n)$ and

$$\lim_{n\to\infty} \int\limits_{\Omega_n} (|\nabla v_n|^2 + v_n^2)\, dx = m\biggl(+, \frac{1}{M_2}\biggr).$$

By Lemma 5, there is a subsequence v_n, points $y_n \in \mathbb{R}^N$ and $C > 0$ such that for any $\varepsilon > 0$ there exists $R > 0$ such that

$$\lim_{n\to\infty} \int\limits_{B_R(y_n)\cap\Omega_n} Q_n|v_n|^{p+1}\, dx \ge 1 - \varepsilon$$

and

$$\mathrm{dist}(y_n, \partial\Omega_n) \le C.$$

We may choose $t_n \in \partial\Omega$ such that $\mathrm{dist}(y_n, t_n/d_n) \le C$ and $t_n \to t \in \partial\Omega$. Without loss of generality we may assume that $\beta(u_n) \to 0 \in \mathbb{R}^N$.

By direct calculation,

$$\int\limits_{\Omega_n} Q_n|v_n|^{p+1} x\, dx = \frac{\beta(u_n)^1}{d_n}.$$

As $\beta(u_n) \notin N_\varrho(\partial\Omega)$, we have $t \ne 0$, so without loss of generality, $t = (t^1, \ldots, t^N)$ with $t^1 > 0$.

For $\varepsilon > 0$ arbitrary, let $R_1 > 0$ such that

$$\lim_{n\to\infty} \int\limits_{B_{R_1}(t_n/d_n)\cap\Omega_n} Q_n|v_n|^{p+1}\, dx \ge 1 - \varepsilon.$$

Let $s = \min\{y^1 : (y^1, \ldots, y^N) \in \partial\Omega\}$. For n large we then have

$$
\begin{aligned}
\frac{\beta(u_n)^1}{d_n} &= \int\limits_{\Omega_n} Q_n|v_n|^{p+1}x^1\,dx \\
&= \int\limits_{\Omega_n\cap B_{R_1}(t_n/d_n)} Q_n|v_n|^{p+1}x^1\,dx + \int\limits_{\Omega_n-B_{R_1}(t_n/d_n)} Q_n|v_n|^{p+1}x^1\,dx \\
&\ge \left(\frac{t_n^1}{d_n}-R_1\right)\int\limits_{\Omega_n\cap B_{R_1}(t_n/d_n)} Q_n|v_n|^{p+1}\,dx \\
&\quad -\frac{|s|}{d_n}\int\limits_{\Omega_n-B_{R_1}(t_n/d_n)} Q_n|v_n|^{p+1}\,dx \\
&\ge \left(\frac{t_n^1}{d_n}-R_1\right)(1-\varepsilon)-\frac{|s|}{d_n}\varepsilon
\end{aligned}
$$

so

$$\beta(u_n)^1 \ge (t_n^1 - d_nR_1)(1-\varepsilon) - |s|\varepsilon.$$

Letting $n \to \infty$ and $\varepsilon \to 0$ gives the contradiction $0 \ge t^1 > 0$.

We recall that the Lusternik–Schnirelman category of a subset $A \subset \partial\Omega$, denoted by $\mathrm{cat}_{\partial\Omega}(A)$, equals k if A can be covered by k closed contractible sets in $\partial\Omega$, but not $k-1$ such sets. We set $\mathrm{cat}_{\partial\Omega}(\partial\Omega) = \mathrm{cat}(\partial\Omega)$.

Lemmas 6 and 7 below are proved exactly as in [14].

Lemma 6. *Let ε_1 be given as in Proposition* 3. *For any $\varepsilon \in (0,\varepsilon_1)$ there exists $d_\varepsilon > 0$ such that*

$$\mathrm{cat}(E_d^{c_d+\varepsilon_d}) \ge 2\,\mathrm{cat}(\partial\Omega)$$

where $\varepsilon_d = d^{N(p-1)/(p+1)}\varepsilon$.

Lemma 7. *Let u be a critical point of $E_d(u)$ with*

$$E_d(u) \le 2^{(p-1)/(p+1)}c_d.$$

Then u does not change sign.

Now we prove the main result of this section.

Theorem 3. *If d is sufficiently small, then* (1_d) *has at least* $\mathrm{cat}(\partial\Omega)$ *distinct solutions.*

Proof. From Proposition 3,

$$c_d = d^{N(p-1)/(p+1)}\left(m\left(+,\frac{1}{M_2}\right)+o(1)\right)$$

as $d \to 0$. For ε_1 as in Proposition 3, choose $\varepsilon_0 \in (0,\varepsilon_1]$ such that

$$\varepsilon_0 < (2^{(p-1)/(p+1)}-1)m\left(+,\frac{1}{M_2}\right).$$

Then there exists d_0 such that for all $d \in (0, d_0)$,

$$c_d + d^{N(p-1)/(p+1)} c_d < 2^{(p-1)/(p+1)} c_d.$$

For this $\varepsilon_0 \in (0, \varepsilon_1]$, there exists, by Lemma 6, a $d_0' > 0$ such that

$$\operatorname{cat}(E_d^{c_d+\varepsilon_d}) \geq 2 \operatorname{cat}(\partial\Omega)$$

for all $d \in (0, d_0')$, where $\varepsilon_d = d^{N(p-1)/(p+1)} \varepsilon_0$.

Applying the minimax method here yields at least $2 \operatorname{cat}(\partial\Omega)$ critical points of E_d. From Lemma 7 none of the solutions change sign and so there are at least $\operatorname{cat}(\partial\Omega)$ solutions of (1_d).

5. Multi-peak solutions. This section is devoted to the construction of multi-peak solutions. We aim to show that local minima of the restriction of Q to $\partial\Omega$ generate multi-peak solutions. Our approach is a modification of the construction from [4].

For a fixed integer $k \geq 1$ we set $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$, $x = (x^1, \ldots, x^k) \in \mathbb{R}^{kN}$, where $x^i \in \mathbb{R}^N$, $i = 1, \ldots, k$. For $R > 0$ we define

$$D_{d,R} = \left\{ x : x^i \in \partial\Omega,\ i = 1, \ldots, k,\ \frac{|x^i - x^j|}{d} \geq R,\ i \neq j \right\}.$$

We denote by U the ground state solution of (3) with $M = 1$. We use the notation $U_{d,z}(y) = U((y - z)/d)$. It is convenient for our purposes to define a scalar product in $H^1(\Omega)$ by

$$\langle u, v \rangle_d = \int_\Omega (d^2 \nabla u \nabla v + uv)\, dx$$

and let $\|u\|_d = \langle u, u \rangle_d^{1/2}$. For every $x = (x^1, \ldots, x^k) \in \partial\Omega \times \ldots \times \partial\Omega$, we set

$$E_{d,x,k} = \left\{ v \in H^1(\Omega) : \langle U_{d,x^i}, v \rangle_d = \left\langle \frac{\partial U_{d,x^i}}{\partial \tau_{i,j}}, v \right\rangle_d = 0,\right.$$
$$\left. i = 1, \ldots, k,\ j = 1, \ldots, N-1 \right\},$$

where $\{\tau_{i,1}, \ldots, \tau_{i,N-1}\}$ form an orthogonal basis of the tangent space to $\partial\Omega$ at x^i. Let $Q_\circ = \min_{x \in \partial\Omega} Q(x)$ and set

$$M_{d,\delta,R} = \{(\alpha, x, v) : |\alpha_i - Q_\circ^{-1/(p-1)}| \leq \delta,\ i = 1, \ldots, k,$$
$$x \in D_{d,R},\ v \in E_{d,x,k},\ \|v\|_d \leq \delta d^{N/2}\}$$

for some constant $\delta > 0$. We define a functional $J_d : M_{d,\delta,R} \to \mathbb{R}$ by

$$J_d(\alpha, x, v) = I_d\Big(\sum_{i=1}^k \alpha_i U_{d,x^i} + v\Big).$$

The proof of the following lemma is similar to that of Proposition 7 in [2].

LEMMA 8. *There exist constants $d_\circ > 0$, $\delta > 0$ and $R > 0$ such that for every $d \in (0, d_\circ]$, every $u \in H^1(\Omega)$ satisfying*

$$\Big\| u - \sum_{i=1}^{k} U_{d,x^i} \Big\|_d \le \delta d^{N/2}$$

for some $x = (x^1, \ldots, x^k) \in D_{d,R}$ admits a unique decomposition

$$u = \sum_{i=1}^{k} \alpha_{d,i} U_{d,x_d^i} + v_d,$$

where $(\alpha_d, x_d, v_d) \in M_{d,\delta,k}$.

The next result is a consequence of Lemma 8 and shows that in order to find critical points of I_d it is sufficient to find critical points of the functional J_d.

PROPOSITION 5. *There exist $d_\circ > 0$, $\delta > 0$ and $R > 0$ such that for every $d \in (0, d_\circ]$, a point $(\alpha, x, v) \in M_{d,\delta,R}$ is a critical point of J_d if and only if*

$$u = \sum_{i=1}^{k} \alpha_i U_{d,x^i} + v$$

is a critical point of I_d.

It is known [13] that critical points of I_d of the form $u = \sum_{i=1}^{k} \alpha_i U_{d,x^i} + v$ are positive. According to Proposition 4 to find a critical point $(\alpha, x, v) \in M_{d,\delta,R}$ of J_d it is enough to find $(\alpha, x, v) \in M_{d,\delta,R}$ and constants A_l, B_{li}, $l = 1, \ldots, k$, $i = 1, \ldots, N-1$, such that

$$\frac{\partial J_d(\alpha, x, v)}{\partial \tau_{l,i}} = \sum_{j=1}^{N-1} B_{lj} \Big\langle \frac{\partial^2 U_{d,x^l}}{\partial \tau_{l,i} \partial \tau_{l,j}}, v \Big\rangle_d, \quad i = 1, \ldots, N-1,\ l = 1, \ldots, k, \tag{24}$$

$$\frac{\partial J_d(\alpha, x, v)}{\partial \alpha_l} = 0, \qquad l = 1, \ldots, k, \tag{25}$$

$$\frac{\partial J_d(\alpha, x, v)}{\partial v} = \sum_{i=1}^{k} A_i U_{d,x^i} + \sum_{l=1}^{k} \sum_{j=1}^{N-1} B_{lj} \frac{\partial U_{d,x^l}}{\partial \tau_{l,j}}. \tag{26}$$

In the first step we solve equations (25) and (26) for each fixed $x \in D_{d,R}$. Then we solve (24).

PROPOSITION 6. *There exist $d_\circ > 0$, $\delta > 0$ and $R > 0$ such that for each $d \in (0, \delta_\circ]$ there exists a C^1-map $(\alpha_d(x), v_d(x)) : D_{d,R} \to \mathbb{R}^k \times E_{d,x,k}$ satisfying*

$$\frac{\partial J_d(\alpha_d(x), x, v_d(x))}{\partial \alpha_l} = 0, \qquad l = 1, \ldots, k, \tag{27}$$

$$\left\langle \frac{\partial J_d(\alpha_d(x), x, v_d(x))}{\partial v}, w \right\rangle_d = 0 \tag{28}$$

for each $w \in E_{d,x,k}$, *and moreover*,

$$|\alpha_l - Q_\circ^{-1/(p-1)}| = O\Big(d + \sum_{i \neq j} \exp\Big(-\frac{1+\sigma}{2}\frac{|x^i - x^j|}{d}\Big)\Big), \qquad l = 1, \ldots, k,$$

$$\|v_d\|_d = O\Big(d^{N/2}\Big(d + \sum_{i \neq j} \exp\Big(-\frac{1+\sigma}{2}\frac{|x^i - x^j|}{d}\Big)\Big)\Big)$$

for some constant $\sigma > 0$.

Results of this nature are known and we refer to [4], [13]. Let (α_d, v_d) be the mapping from Proposition 6. Then there exists $x_d \in D_{d,R}$ such that

$$J_d(\alpha_d(x_d), x_d, v_d(x_d)) = \sup\{J_d(\alpha_d(x), x, v_d(x)) : x \in D_{d,R}\}.$$

To proceed further we need some estimates of J_d. Let $H(x)$ be the mean curvature of $\partial\Omega$.

LEMMA 9. *Let* $x \in \partial\Omega$ *and set*

$$K(x) = \frac{1}{2}H(x) \int\limits_{\mathbb{R}^{N-1}} U^{p+1}(y', 0)|y'|^2\, dy.$$

Then

$$\int\limits_{\Omega} Q(y) U_{d,x}^{p+1}(y)\, dy = d^N Q(x)(A - dK(x) + O(d^2)) + O(d^{N+\alpha}),$$

where $A = \frac{1}{2}\int_{\mathbb{R}^N} U^{p+1}(y)\, dy$ *and* α *is a Hölder exponent for* Q.

Proof. For simplicity we assume that $x = 0$. We choose the coordinate system so that

$$\Omega \cap B_\tau(0) = \{y_N > f(y')\}, \qquad \partial\Omega \cap B_\tau(0) = \{y_N = f(y')\},$$

where $\tau > 0$ is a small constant and $f(y')$ satisfies

$$f(y') = \frac{1}{2}\sum_{i=1}^{N-1} \varrho_i y_i^2 + O(|y'|^3) \qquad \text{for } y' \in B_\tau^{N-1}(0) = \{y' : |y'| \le \tau\}.$$

Then $H(0) = (N-1)^{-1}\sum_{i=1}^{N-1} \varrho_i$. Letting $\Omega_{d,x} = \{y : dy + x \in \Omega\}$, $\Omega_d = \Omega_{d,0}$ and using the result from [4] (see also [8]) we get

$$\begin{aligned}\int\limits_{\Omega} Q(y) U_{d,0}^{p+1}(y)\, dy &= Q(0)\int\limits_{\Omega} U_{d,0}^{p+1}(y)\, dy + \int\limits_{\Omega} (Q(y) - Q(0)) U_{d,0}^{p+1}(y)\, dy \\ &= d^N Q(0)(A - dK(0) + O(d^2)) \\ &\quad + \int\limits_{\Omega} (Q(y) - Q(0)) U_{d,0}^{p+1}(y)\, dy.\end{aligned}$$

It follows from the Hölder condition for Q that

$$\Big|\int_{\Omega}(Q(y)-Q(0))U_{d,0}^{p+1}(y)\,dy\Big| \le C\int_{\Omega}|y|^{\alpha}U_{d,0}^{p+1}(y)\,dy = Cd^{N+\alpha}\int_{\Omega_d}|y|^{\alpha}U^{p+1}(y)\,dy \le Cd^{N+\alpha}\int_{\mathbb{R}^N}|y|^{\alpha}U^{p+1}(y)\,dy$$

for some constant $C>0$ and the result follows.

We also need the following asymptotic relation (see Lemma A.2 in [4]):

$$\int_{\Omega}(d^2|\nabla U_{d,x}|^2+U_{d,x}^2)\,dx = d^N(A-dK(x)+dM(x)+O(d^2)), \tag{29}$$

where $K(x)$ is as in Lemma 9 and

$$M(x)=\frac{1}{2}H(x)\int_{\mathbb{R}^{N-1}}U(y',0)\frac{\partial U(y',0)}{\partial r}|y'|\,dy.$$

PROPOSITION 7. *Let x_d be a point in $D_{d,R}$ where J_d attains its maximum. Then*

$$\frac{|x_d^i-x_d^j|}{d}\to\infty \qquad \text{as } d\to 0,\ i\ne j,$$

$x_d^i\to x^i\in\partial\Omega$ as $d\to 0$ and x^i satisfies $Q(x^i)=\min_{\partial\Omega}Q(x)$, $i=1,\dots,k$.

Proof. Expanding $J_d(\alpha,x,v)$ around $(\overline{Q},x,0)$, where $\overline{Q}=(Q_\circ^{-1/(p-1)},\dots,Q_\circ^{-1/(p-1)})$ and using Proposition 6 we get

$$J_d(\alpha_d(x),x,v_d(x)) = J_d(\overline{Q},x,0)+O\bigg(d^N\bigg(d^2+\sum_{i\ne j}\exp\bigg(-(1+\sigma)\frac{|x^i-x^j|}{d}\bigg)\bigg)\bigg).$$

Since x_d is a point in $D_{d,R}$ where the maximum is achieved we have

$$J_d(\alpha_d(x_d),x_d,v_d(x_d))\ge J_d(\alpha_d(z_d),z_d,v_d(z_d))$$

for every $z_d\in D_{d,R}$. Thus

$$J_d(\overline{Q},x_d,0)+O\bigg(d^N\bigg(d^2+\sum_{i\ne j}\exp\bigg(-(1+\sigma)\frac{|x_d^i-x_d^j|}{d}\bigg)\bigg)\bigg) \ge J_d(\overline{Q},z_d,0)+O\bigg(d^N\bigg(d^2+\sum_{i\ne j}\exp\bigg(-(1+\sigma)\frac{|z_d^i-z_d^j|}{d}\bigg)\bigg)\bigg). \tag{30}$$

Let e_i, $i=1,\dots,k$, be tangent vectors to $\partial\Omega$ at $x_\circ$, with $e_i\ne e_j$ for $i\ne j$, and $z_i(t)$ be a curve in $\partial\Omega$ satisfying $z_i(0)=x_\circ$, $z_i'(0)=e_i$, where $Q(x_\circ)=Q_\circ$. Let $z_d^i=z^i(d^{1/2})$, $i=1,\dots,k$. Then

$$\frac{|z_d^i - z_d^j|}{d} = \frac{|e_i - e_j| + o(1)}{d^{1/2}} \to \infty \qquad \text{as } d \to 0.$$

Therefore $z_d \in D_{d,R}$ if $d > 0$ is sufficiently small. It follows from Lemma 9 and (29) that

$$\begin{aligned}
(31)\quad & J_d(\overline{Q}, z_d, 0) \\
&= \frac{1}{2}\Big\| \sum_{i=1}^k Q_\circ^{-1/(p-1)} U_{d,z_d^i} \Big\|_d^2 - \frac{1}{p+1} \int_\Omega Q(y) \Big| \sum_{i=1}^k Q_\circ^{-1/(p-1)} U_{d,z_d^i} \Big|^{p+1} dy \\
&= \frac{1}{2} \sum_{i=1}^k Q_\circ^{-2/(p-1)} \|U_{d,z_d^i}\|_d^2 \\
&\quad - \frac{1}{p+1} \sum_{i=1}^k Q_\circ^{-(p+1)/(p-1)} \int_\Omega Q(y) U_{d,z_d^i}^{p+1}\, dy + O(d^N e^{-c_\circ/d^{1/2}}) \\
&= \frac{d^N}{2} \Big(\sum_{i=1}^k Q_\circ^{-2/(p-1)} (A - dK(z_d^i) + dM(z_d^i) + O(d^2)) \Big) \\
&\quad - \frac{d^N}{p+1} \sum_{i=1}^k Q(z_d^i) Q_\circ^{-(p+1)/(p-1)} (A - dK(z_d^i) + O(d^2)) + O(d^{N+\alpha}) \\
&= \frac{kd^N}{2} (Q_\circ^{-2/(p-1)} (A - dK(x_\circ) + dM(x_\circ) + O(d^{3/2}))) \\
&\quad - \frac{kd^N}{p+1} Q_\circ^{-2/p-1} (A - dK(x_\circ) + O(d^{3/2})) + O(d^{N+\alpha}) \\
&= k\left(\frac{1}{2} - \frac{1}{p+1} \right) d^N A Q_\circ^{-2/(p-1)} + O(d^{N+\alpha})
\end{aligned}$$

for some constant $c_\circ > 0$. For $J_d(\overline{Q}, x_d, 0)$ we have the estimate

$$\begin{aligned}
J_d(\overline{Q}, x_d, 0) &= \frac{1}{2}\Big\| \sum_{i=1}^k Q_\circ^{-1/(p-1)} U_{d,x_d^i} \Big\|_d^2 \\
&\quad - \frac{1}{p+1} \int_\Omega Q(y) \Big| \sum_{i=1}^k Q_\circ^{-1/(p-1)} U_{d,x_d^i} \Big|^{p+1} dy \\
&= \sum_{i=1}^k \Big(\frac{1}{2} Q_\circ^{-2/(p-1)} \|U_{d,x_d^i}\|_d^2 \\
&\quad - \frac{1}{p+1} Q_\circ^{-(p+1)/(p-1)} \int_\Omega Q(y) U_{d,x_d^i}^{p+1}\, dy \Big)
\end{aligned}$$

$$+\frac{1}{2}\sum_{i\neq j} Q_\circ^{-2/(p-1)}\langle U_{d,x_d^i}, U_{d,x_d^j}\rangle_d$$
$$-\frac{1}{p+1}\Big[\int_\Omega Q(y)\Big|\sum_{i=1}^k Q_\circ^{-1/(p-1)}U_{d,x_d^i}\Big|^{p+1}dy$$
$$-\sum_{i=1}^k\int_\Omega Q_\circ^{-(p+1)/(p-1)}Q(y)U_{d,x_d^i}^{p+1}\,dy\Big].$$

We denote the last two terms on the right side by P_1 and $-\frac{1}{p+1}P_2$, respectively. For P_1 we have the estimate (see [4])

$$\begin{aligned}P_1 &= \frac{1}{2}\sum_{i\neq j} Q_\circ^{-2/(p-1)}\langle U_{d,x_d^i}, U_{d,x_d^j}\rangle_d\\ &= \sum_{i<j} Q_\circ^{-2/(p-1)}\int_\Omega U_{d,x_d^i}^p U_{d,x_d^j}\,dy\\ &\quad + O\Big(d^{N+1}\sum_{i\neq j}\exp\Big(-\frac{(1-\theta)|x_d^i-x_d^j|}{d}\Big)\Big).\end{aligned}$$

To estimate P_2 we make repeated use of the inequality

$$\big||a+b|^p-a^p-b^p-pa^{p-1}b-pab^{p-1}\big| \le \begin{cases} Ca^{p/2}b^{p/2} & \text{if } 2<p\le 3,\\ C(a^{p-2}b^2+a^2b^{p-2}) & \text{if } p>3,\end{cases}$$

and we get

$$\begin{aligned}P_2 &= \int_\Omega Q(y)\Big|\sum_{i=1}^k Q_\circ^{-1/(p-1)}U_{d,x_d^i}\Big|^{p+1}dy-\sum_{i=1}^k\int_\Omega Q_\circ^{-(p+1)/(p-1)}Q(y)U_{d,x_d^i}^{p+1}\,dy\\ &= Q_\circ^{-(p+1)/(p-1)}\int_\Omega Q(y)\Big(\sum_{i=2}^k U_{d,x_d^i}\Big)^{p+1}dy\\ &\quad - Q_\circ^{-(p+1)/(p-1)}\sum_{i=2}^k\int_\Omega Q(y)U_{d,x_d^i}^{p+1}\,dy\\ &\quad + (p+1)Q_\circ^{-(p+1)/(p-1)}\int_\Omega Q(y)\Big(\sum_{i=2}^k U_{d,x_d^i}\Big)^p U_{d,x_d^1}\,dy\\ &\quad + (p+1)Q_\circ^{-(p+1)/(p-1)}\int_\Omega Q(y)U_{d,x_d^1}^p\sum_{i=2}^k U_{d,x_d^i}\,dy\\ &\quad + O\Big(d^N\sum_{i\neq j}\exp\Big(-\frac{(1+\sigma)|x_d^i-x_d^j|}{d}\Big)\Big)\end{aligned}$$

$$= (p+1)Q_\circ^{-(p+1)/(p-1)} \int_\Omega Q(y) \sum_{j=1}^{k-1} \Big(\sum_{i=j+1}^{k} U_{d,x_d^i}\Big)^p U_{d,x_d^j}\,dy$$
$$+ (p+1)Q_\circ^{-(p+1)/(p-1)} \int_\Omega Q(y) \sum_{i<j} U_{d,x_d^i}^p U_{d,x_d^j}\,dy$$
$$+ O\bigg(d^N \sum_{i\neq j} \exp\bigg(-\frac{(1+\sigma)|x_d^i - x_d^j|}{d}\bigg)\bigg).$$

Since

$$Q_\circ^{-2/(p-1)} \sum_{i<j} \int_\Omega U_{d,x_d^i}^p U_{d,x_d^j}\,dy - Q_\circ^{-(p+1)/(p-1)} \int_\Omega Q(y) \sum_{i<j} U_{d,x_d^i}^p U_{d,x_d^j}\,dy$$
$$= Q_\circ^{-2/(p-1)} \int_\Omega \bigg(1 - \frac{Q(y)}{Q_\circ}\bigg) \sum_{i<j} U_{d,x_d^i}^p U_{d,x_d^j}\,dy < 0,$$

we derive the following estimate:

$$\text{(32)} \quad J_d(\overline{Q}, x_d, 0)$$
$$\le d^N \bigg[\frac{k}{2} A \overline{Q}_\circ^{-2/(p-2)} - \frac{1}{p+1} A \sum_{i=1}^{k} Q_\circ^{-(p+1)/(p-1)} Q(x_d^i)$$
$$- \bigg(\frac{1}{2} - \frac{1}{p+1}\bigg) d \sum_{i=1}^{k} Q_\circ^{-2/(p-1)} K(x_d^i) + O(d^2)\bigg] + O(d^{N+\alpha})$$
$$- Q_\circ^{-(p+1)/(p-1)} \int_\Omega Q(y) \sum_{j=1}^{k-1} \Big(\sum_{i=j+1}^{k} U_{d,x_d^i}\Big)^p U_{d,x_d^j}\,dy$$
$$+ O\bigg(d^N \sum_{i\neq j} \exp\bigg(-\frac{(1+\sigma)|x_d^i - x_d^j|}{d}\bigg)\bigg).$$

Inserting estimates (31) and (32) into (30) we get

$$\text{(33)} \quad -\frac{k}{p+1} A Q_\circ^{-2/(p-1)}$$
$$\le -\frac{1}{p+1} A Q_\circ^{-(p+1)/(p-1)} \sum_{i=1}^{k} Q(x_d^i)$$
$$- d^{-N} Q_\circ^{-(p+1)/(p-1)} \int_\Omega Q(y) \sum_{j=1}^{k-1} \Big(\sum_{i=j+1}^{k} U_{d,x_d^i}\Big)^p U_{d,x_d^j}\,dy$$
$$+ O\bigg(\sum_{i\neq j} \exp\bigg(\frac{(1+\sigma)|x_d^i - x_d^j|}{d}\bigg)\bigg) + O(d^\alpha).$$

Since

$$d^{-N}Q_\circ^{-(p+1)/(p-1)}\int_\Omega Q(y)\sum_{j=1}^{k-1}\Big(\sum_{i=j+1}^{k}U_{d,x_d^i}\Big)^pU_{d,x_d^j}\,dy \\ \geq c_\circ\sum_{i\neq j}\exp\bigg(-\frac{(1+\sigma/2)|x_d^i-x_d^j|}{d}\bigg)$$

for some $c_\circ>0$ and for large $R>0$ we deduce from (33) that

$$\sum_{i=1}^{k}Q(x_d^i)+O(d^\alpha)+c_\circ\sum_{i\neq j}\exp\bigg(-\Big(1+\frac{\sigma}{2}\Big)\frac{|x_d^i-x_j^d|}{d}\bigg) \\ +O\bigg(\sum_{i\neq j}\exp\bigg(-(1+\sigma)\frac{|x_d^i-x_d^j|}{d}\bigg)\bigg)\leq kQ_\circ.$$

Taking R sufficiently large we deduce that

$$\sum_{i=1}^{k}Q(x_d^i)+O(d^\alpha)+c_1\sum_{i\neq j}\exp\bigg(-\Big(1+\frac{\sigma}{2}\Big)\frac{|x_i^i-x_d^j|}{d}\bigg)\leq kQ_\circ. \tag{34}$$

Since $Q(x_d^i)\geq Q_\circ$, $i=1,\ldots,k$, we see that

$$\lim_{d\to0}\frac{|x_d^i-x_d^j|}{d}=\infty$$

and necessarily $x_d^i\to x_i$ as $d\to0$. Letting $d\to0$ in (34) we get

$$\sum_{i=1}^{k}Q(x_i)\leq kQ(x_\circ)=k\min_{x\in\partial\Omega}Q(x)$$

and the result follows.

From Propositions 6 and 7 we deduce the following existence result:

THEOREM 4. *For each positive integer k, there exists a $d_\circ=d_\circ(k)$ such that for each $d\in(0,d_\circ]$, problem (1_d) has a solution of the form*

$$u_d=\sum_{i=1}^{k}\alpha_d^iU_{d,x_d^i}+v_d,$$

where

$$\alpha_d^i\to Q_i^{-1/(p-1)},\qquad i=1,\ldots,k,$$
$$\frac{|x_d^i-x_d^j|}{d}\to\infty\quad\textit{for } i\neq j,\qquad x_d^i\to x^i$$

as $d\to0$ and x_d^i satisfies $Q(x^i)=\min_{\partial\Omega}Q(x)$, $k=1,\ldots,l$. In particular, if $Q|_{\partial\Omega}$ has only one global minimum point at $x_\circ$, then $x_d^i\to x_\circ$ as $d\to0$, $i=1,\ldots,k$.

6. Effect of the graph topology of the coefficient Q. In this section we examine the effect of the graph topology of the coefficient Q on the number of one-peak solutions. We follow some ideas from the paper [3], where the effect of the graph topology of Q was studied for the Dirichlet problem. Let $Q_{\mathrm{m}} = \max_{x\in\partial\Omega} Q(x)$ and set

$$\mathcal{M} = \{x \in \partial\Omega : Q(x) = Q_{\mathrm{m}}\} \quad \text{and} \quad \mathcal{M}_\varrho = \{x \in \partial\Omega : \mathrm{dist}(x, \mathcal{M}) < \varrho\},$$

where $\varrho > 0$ is a small number. The sets $E_{d,x,k}$, with $k = 1$, introduced in Section 5, are now denoted by $E_{d,x}$, that is,

$$E_{d,x} = \left\{ v \in H^1(\Omega) : \langle U_{d,x}, v\rangle_d = \left\langle \frac{\partial U_{d,x}}{\partial \tau_i}, v \right\rangle_d = 0,\ i = 1, \ldots, N-1 \right\},$$

where $\{\tau_i\}$, $i = 1, \ldots, N-1$, is an orthogonal basis of the tangent space to $\partial\Omega$ at x. For each $\delta > 0$ small enough we define

$$\mathcal{M}_{d,\varrho} = \{(\alpha, x, v) : |\alpha - Q_{\mathrm{m}}^{-1/(p-1)}| \le \delta,\ x \in \mathcal{M}_\varrho,\ v \in E_{d,x},\ \|v\|_d \le \delta d^{N/2}\}.$$

For $(\alpha, x, v) \in \mathcal{M}_{d,\varrho}$ we set

$$J_d(\alpha, x, v) = I_d(\alpha U_{d,x} + v).$$

As in Section 5 we have the following result:

PROPOSITION 8. *There exist $d_\circ > 0$, $\delta > 0$ such that for every $d \in (0, d_\circ)$ a point $(\alpha, x, v) \in \mathcal{M}_{d,\varrho}$ is a critical point of the functional $J_d(\alpha, x, v)$ if and only if $\alpha U_{d,x} + v$ is a critical point of I_d.*

Consequently, to find a critical point $(\alpha, x, v) \in \mathcal{M}_{d,\varrho}$ we need to solve the following problem: find constants A, B_i, $i = 1, \ldots, N-1$, and $(\alpha, x, v) \in \mathcal{M}_{d,\varrho}$ such that

$$\frac{\partial J_d}{\partial \tau_i} = \sum_{j=1}^{N-1} B_j \left\langle \frac{\partial^2 U_{d,x}}{\partial \tau_i \partial \tau_j}, v \right\rangle_d, \qquad i = 1, \ldots, N-1, \tag{35}$$

$$\frac{\partial J_d}{\partial \alpha} = 0, \tag{36}$$

$$\frac{\partial J_d}{\partial v} = A U_{d,x} + \sum_{j=1}^{N-1} B_j \frac{\partial U_{d,x}}{\partial \tau_j}. \tag{37}$$

We need a result analogous to Proposition 6.

PROPOSITION 9. *There exist $d_\circ > 0$, $\delta > 0$ such that for every $d \in (0, d_\circ)$, there exists a C^1-mapping $(\alpha_d(x), v_d(x)) : \mathcal{M}_\varrho \to \mathbb{R} \times E_{d,x}$ satisfying*

$$\frac{\partial J_d(\alpha_d(x), x, v_d(x))}{\partial \alpha} = 0, \qquad \left\langle \frac{\partial J_d(\alpha_d(x), x, v_d(x))}{\partial v}, w \right\rangle_d = 0$$

for every $w \in E_{d,x}$ *and*

$$|\alpha_d(x) - Q_{\mathrm{m}}^{-1/(p-1)}| = O(d^{N/2+\alpha}), \qquad \|v_d\|_d = O(d^{N/2+1}).$$

Proof. Let $J^*(x,w) = J(\alpha,x,v) = I_d(\alpha U_{d,x} + v)$, where $w = (\beta, v) = (\alpha - Q_{\mathrm{m}}^{-1/(p-d)}, v)$. As in [4] (see also [13]) we expand $J^*(x,w)$ at $w=0$:

$$J^*(w,x) = J^*(0,x) + \langle f_{d,x}, w\rangle_d + \tfrac{1}{2}\langle Q_{d,x}w, w\rangle + R_{d,x}(w),$$

where $f_{d,x} \in \mathbb{R} \times E_{d,x}$ is given by

$$\begin{aligned}\langle f_{d,x}, w\rangle_d &= \Big[Q_m^{-1/(p-1)}\|U_{d,x}\|_d^2 - Q_{\mathrm{m}}^{-p/(p-1)}\int\limits_{\Omega} Q(y)U_{d,x}^{p+1}\,dy\Big]\beta \\ &\quad - Q_{\mathrm{m}}^{-p/(p-1)}\int\limits_{\Omega} Q(y)U_{d,x}^{p} v\,dy \\ &= J_1 \cdot \beta + J_2(v),\end{aligned}$$

$Q_{d,x}$ is a linear map from $\mathbb{R} \times E_{d,x}$ to $\mathbb{R} \times E_{d,x}$ and $R_{d,x}$ is the higher order term satisfying

$$R_{d,x}^{(i)}(w) = O(\|w\|_d^{\min(p-i,3-i)}), \qquad i = 0,1,2.$$

Repeating the argument from [4] we show that $Q_{d,x}$ is invertible and $\|Q_{d,x}^{-1}\| \le C$, where C is independent of d and x. Obviously, equations (36) and (37) are equivalent to

$$f_{d,x} + Q_{d,x}w + R'_{d,x}(w) = 0.$$

From the implicit function theorem this equation has a solution $w_d \in \mathbb{R} \times E_{d,x}$ and w_d satisfies

$$\|w_d\|_d \le C\|f_{d,x}\|.$$

We now estimate $\|f_{d,x}\|$. For the term J_1 of β we have (we eventually drop the d,x subscript)

$$\begin{aligned}J_1 &= Q_{\mathrm{m}}^{-1/(p-1)}\|U_{d,x}\|_d^2 - Q_{\mathrm{m}}^{-p/(p-1)}\int\limits_{\Omega} Q(y)U_{d,x}^{p+1}\,dy \\ &= Q_{\mathrm{m}}^{-1/(p-1)}\Big[\int\limits_{\Omega}(d^2|\nabla U|^2 + U^2)\,dy - Q_{\mathrm{m}}^{-1}\int\limits_{\Omega} Q(y)U^{p+1}\,dy\Big] \\ &= Q_{\mathrm{m}}^{-1/(p-1)}\Big[\int\limits_{\Omega}\Big(U^2 - \frac{Q(y)}{Q_{\mathrm{m}}}U^{p+1}\Big)dy + \int\limits_{\Omega} d^2|\nabla U|^2\,dy\Big] \\ &= Q_{\mathrm{m}}^{-1/(p-1)}\Big[\int\limits_{\Omega}\Big(U^2 - \frac{Q(y)}{Q_{\mathrm{m}}}U^{p+1}\Big)dy \\ &\qquad + \int\limits_{\partial\Omega} d^2 U\frac{\partial U}{\partial \nu}\,d\sigma - \int\limits_{\Omega} d^2 U \Delta U\,dy\Big]\end{aligned}$$

$$= Q_{\mathrm{m}}^{-1/(p-1)}\bigg[\int_{\Omega}(-d^2\Delta U_{d,x} + U_{d,x} - U_{d,x}^p)\,dy$$

$$+ \int_{\Omega}\bigg(1 - \frac{Q(y)}{Q_{\mathrm{m}}}\bigg)U_{d,x}^{p+1}\,dy + \int_{\partial\Omega} d^2 U \frac{\partial U}{\partial\nu}\,d\sigma\bigg].$$

After scaling $z = (y-x)/d$ we find the first integral to be 0. The third integral is $O(d^{N+1})$. For the second integral we have the estimate

$$\int_{\Omega}\bigg(1 - \frac{Q(y)}{Q_{\mathrm{m}}}\bigg)U_{d,x}^{p+1}\,dy = d^N(A - dK(x) + O(d^2))$$

$$-d^N\frac{Q(x)}{Q_{\mathrm{m}}}(A - dK(x) + O(d^2)) + O(d^{N+\alpha})$$

$$= d^N A\bigg(1 - \frac{Q(x)}{Q_{\mathrm{m}}}\bigg) + O(d^{N+\alpha}) + o(d^{N+1}).$$

The estimates for α_d and $\|v_d\|_d$ easily follow.

Problem $(1)_d$ is reduced to finding $x \in \mathcal{M}_\varrho$ such that equation (34) is satisfied.

PROPOSITION 10. *Let (α_d, v_d) be the mapping from Proposition 9. Then*

$$J_d(\alpha_d(x), x, v_d(x)) = d^N\bigg[\frac{A}{2Q_{\mathrm{m}}^{2/(p-1)}}\bigg(1 - \frac{2}{p+1}\frac{Q(x)}{Q_{\mathrm{m}}}\bigg) + O(d)\bigg],$$

where A is the constant defined in Lemma 9.

Proof. It follows from Proposition 9 that

$$\int_{\Omega} d^2|\alpha_d\nabla U_{d,x} + \nabla v_d|^2\,dx + \int_{\Omega}(\alpha_d U_{d,x} + v_d)^2\,dx$$

$$= \alpha_d^2\|U_{d,x}\|_d^2 + \|v_d\|_d^2 + \langle U_{d,x}, v\rangle = \alpha_d^2\|U_{d,x}\|_d^2 + O(d^{N+2}).$$

Next we have

$$\int_{\Omega} Q(y)|\alpha_d U_{d,x} + v_d|^{p+1}\,dy$$

$$= \int_{\Omega} Q(y)|\alpha_d U_{d,x}|^{p+1}\,dy$$

$$+ (p+1)\int_{\Omega} Q(y)|\alpha_d U_{d,x} + \theta(y)v_d|^{p-1}(\alpha_d U_{d,x} + \theta(y)v_d)\,dy,$$

where $0 < \theta \le 1$. Hence

$$
\begin{aligned}
\int_\Omega & Q(y)|\alpha_d U_{d,x} + \theta(y) v_d|^p |v_d|\, dy \\
&\le 2^p \int_\Omega Q(y)|\alpha_d U_{d,x}|^p |v_d|\, dy + 2^p \int_\Omega Q(y)|v_d|^{p+1}\, dy \\
&\le 2^p \max_{\overline{\Omega}} Q \Big(\int_\Omega |\alpha_d U_{d,x}|^{p+1}\, dy \Big)^{p/(p+1)} \Big(\int_\Omega |v_d|^{p+1}\, dy \Big)^{1/(p+1)} \\
&\quad + 2^p \int_\Omega |v_d|^{p+1}\, dx \\
&= O(d^{Np/(p+1)+N/2+1}) + O(d^{(N/2+1)(p+1)}).
\end{aligned}
$$

Inserting the above estimates into $I_d(\alpha_d U_{d,x} + v_d)$ we get

$$
\begin{aligned}
I_d(\alpha_d U_{d,x} + v_d) &= \frac{\alpha_d^2}{2} \|U_{d,x}\|_2^2 - \frac{1}{p+1} \int_\Omega Q(y)|\alpha_d U_{d,x}|^{p+1}\, dy \\
&\quad + O(d^{N+2}) + (O(d^{N+\alpha}) + O(d^N)) O(d^{N/2+1}) \\
&= \frac{\alpha_d^2}{2} \|U_{d,x}\|_2^2 - \frac{1}{p+1} \int_\Omega Q(y)|\alpha_d U_{d,x}|^{p+1}\, dy + O(d^{N+1}).
\end{aligned}
$$

Applying Lemma 9 and the estimate (29) we obtain

$$
\begin{aligned}
I_d(\alpha_d U_{d,x} + v) &= \frac{\alpha_d^2}{2} [d^N (A - dK(x) + dM(x) + O(d^2))] \\
&\quad - \frac{\alpha_d^{p+1}}{p+1} [d^N Q(x)(A - dK(x) + O(d^2))] \\
&\quad + O(d^{N+\alpha}) + O(d^{N+1}) \\
&= \frac{\alpha_d^2}{2} d^N A \Big(1 - \frac{2\alpha_d^{p-1}}{p+1} Q(x) \Big) \\
&\quad + \frac{\alpha_d^2 d^N}{2} (d(M(x) - K(x)) + O(d^2)) \\
&\quad + \frac{\alpha_d^{p+1}}{p+1} d^N Q(x)(dK(x) + O(d^2)) + O(d^{N+\alpha}) + O(d^{N+1}) \\
&= \frac{\alpha_d^2}{2} d^N A \Big(1 - \frac{2\alpha_d^{p-1}}{p+1} Q(x) \Big) + O(d^{N+1}).
\end{aligned}
$$

The result follows by applying the estimate $|\alpha_d - Q_{\mathrm{m}}^{-1/(p-1)}| = O(d^{N/2+\alpha})$ from Proposition 9.

To proceed further we define a functional $F : \mathcal{M}_\varrho \to \mathbb{R}$ by $F(x) = J_d(\alpha_d(x), x, v_d(x))$, where (α_d, v_d) is the mapping from Proposition 9. Let

$$a_d = d^N\left[\frac{A}{2Q_{\mathrm{m}}^{2/(p-1)}}\left(1-\frac{2}{p+1}\right)+d^s\right],$$

where $0 < s < 1$. We show that there exists $d_1 > 0$ sufficiently small such that the flow defined by

$$\frac{dY(t)}{dt} = -\mathrm{grad}\ F(Y(t)),$$
$$Y_\circ = Y(0) \in F^{a_d} = \{x \in \mathcal{M}_\varrho : F(x) < a_d\}$$

does not leave $\mathcal{M}_\varrho$ for $0 < d \le d_1$.

LEMMA 10. *There exists $d_1 > 0$ such that for every $d \in (0, d_1)$ and every $x \in \partial\mathcal{M}_\varrho$ we have $F(x) > a_d$.*

Proof. Arguing by contradiction assume that there are sequences $d_n \to 0$ and $\{x_n\} \subset \partial M_\varrho$ such that $F(x_n) \le a_{d_n}$. We then have

$$\frac{A}{2Q_{\mathrm{m}}^{2/(p-1)}}\left(1-\frac{2}{p+1}\frac{Q(x_n)}{Q_{\mathrm{m}}}\right)+O(d_n) \le \frac{A}{2Q_{\mathrm{m}}^{2/(p-1)}}\left(1-\frac{2}{p+1}\right)+d_n^s.$$

We may assume that $x_n \to x_\circ \in \partial\mathcal{M}_\varrho$. Letting $n \to 0$ in the last inequality we deduce that $Q(x_\circ)/Q_{\mathrm{m}} \ge 1$, which contradicts the fact that $x_\circ \in \partial\mathcal{M}_\varrho$ and $Q(x_\circ) < Q_{\mathrm{m}}$.

LEMMA 11. *There exists $d_2 > 0$ small enough such that $\mathcal{M} \subset F^{a_d}$ for $0 < d \le d_2$.*

Proof. In the contrary case there exist sequences $d_n \to 0$ and $\{x_n\} \subset \mathcal{M}$ such that $F(x_n) > a_{d_n}$. From this we get

$$d_n^N\left[\frac{A}{2Q_{\mathrm{m}}^{2/(p-1)}}\left(1-\frac{2}{p+1}\frac{Q(x_n)}{Q_{\mathrm{m}}}\right)+O(d_n)\right]$$
$$> d_n^N\left[\frac{A}{2Q_{\mathrm{m}}^{2/(p-1)}}\left(1-\frac{2}{p+1}\right)+d_n^s\right],$$

which is impossible since $Q(x_n) = Q_{\mathrm{m}}$ and $0 < s < 1$.

THEOREM 4. *Suppose that $Q(x) \not\equiv$ Const on $\partial\Omega$. Then there exists a constant $\varrho > 0$ such that problem (1_d) has $\mathrm{cat}_{\mathcal{M}_\varrho}(\mathcal{M})$ solutions for $0 < d \le \min(d_1, d_2)$, of the following type*:

$$u_d = \alpha_d U_{d,x_d} + v_d$$

where as $d \to 0$, $\alpha_d \to Q_{\mathrm{m}}^{-1/(p-1)}$, $\|v_d\|_d \to 0$ and $x_d \to x_\circ \in \mathcal{M}$.

Proof. In view of Lemmas 10 and 11 we conclude that

$$\#\{x \in \mathcal{M}_\varrho : DF(x) = 0\} \ge \mathrm{cat}_{\mathcal{M}_\varrho}(F^{a_d}) \ge \mathrm{cat}_{\mathcal{M}_\varrho}(\mathcal{M})$$

and the result readily follows.

Appendix. We only describe the main steps of the proof of Proposition 1 and for more details we refer to the paper [11]. Let Φ_{P_d} be the mapping defined in Section 2 associated with $P_d \in \partial\Omega$, where u_d attains its maximum on $\overline{\Omega}$. In what follows we assume $P_d = P$ and write ψ, Φ, Φ_j, Ψ, Ψ_j instead of ψ_{P_d}, Φ_{P_d}, $\Phi_{P_d,j}$, Ψ_{P_d}, $\Psi_{P_d,j}$, respectively. We assume that Φ is defined on an open set containing the closed ball $\overline{B}_{3k} = \overline{B(0,3k)}$, where $k > 0$ is a small constant. We set

$$w_d(z) = \begin{cases} u_d(\Phi(dz)) & \text{for } z \in \overline{B}^+_{3k/d}, \\ u_d(\Phi(dz', -dz_N)) & \text{for } z \in \overline{B}^-_{3k/d}. \end{cases}$$

As in [12] we check that $w_d \to w_M$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N)$. We need a first order approximation in d of w_d. Towards this end we set

$$w_d(z) = w_M(z) + d\widetilde{w}_d(z). \tag{A$_1$}$$

The function w_d satisfies the equation

$$\sum_{i,j=1}^N a^*_{ij}(z)\frac{\partial w_d}{\partial z_i \partial z_j} + d\sum_{j=1}^N b^*_j(z)\frac{\partial w_d}{\partial z_j} - w_d + Q(\Phi(dz))w_d^p = 0 \tag{A$_2$}$$

in $\overline{B}_{3k/d}$, where the coefficients a^*_{ij} and b^*_j are expressed in terms of the derivatives of Ψ (see p. 835 in [12]), and moreover,

$$\frac{\partial w_d}{\partial z_N}(z', 0) = 0 \quad \text{for } |z'| < \frac{3k}{d}.$$

Let $\chi \in C^2(\mathbb{R})$ be a cut-off function satisfying $0 \le \chi(t) \le 1$ on $\mathbb{R}$, $\chi(t) = 0$ if $|t| \ge 3/2$ and $\chi(t) = 1$ if $|t| < 1$. We set $\chi_R(z) = \chi(|z|/R)$ for $z \in \mathbb{R}^N$ and $W_R(z) = \chi_R(z)w_d(z)$. We see that $W_R \in C^2(\mathbb{R}^N)$, $W_R(z) = w_d(z)$ on B_R and $W_R(z) = 0$ on $\mathbb{R}^N - B_{2R}$. From now on we assume that $R = k/d$. We let $d\phi_R(z) = W_R(z) - w_M(z)$, and we see that $\phi_R(z) = \widetilde{w}_d$ on $B_{k/d}$. The function ϕ_R satisfies the equation

$$L\phi_R + A_R\phi_R + g_R + h(\phi_R) = 0,$$

where

$$\begin{aligned} L &= \Delta - 1 - pMw_M^{p-1}, \\ g_R &= \frac{1}{d}A_R w_M - \frac{1}{d}(\Delta\chi_R + A_R\chi_R)w_d - \frac{1}{d}\sum_{i,j=1}^N (a^*_{ij} + a^*_{ji})\frac{\partial \chi_R}{\partial z_i}\frac{\partial w_d}{\partial z_j}, \\ h(\phi_R) &= \frac{1}{d}[Q(\Phi(dz))\chi_R w_d^p - Mw_M^p + pMw_M^{p-1}d\phi_R], \end{aligned}$$

and

$$A_R = d\chi_{2R}(z)\Big(2|z_N|\sum_{i,j=1}^{N-1}\psi_{d,ij}\frac{\partial^2}{\partial z_i\partial z_j} - \alpha_d \operatorname{sgn} z_N \frac{\partial}{\partial z_N}\Big)$$
$$+\chi_{3R}\Big(\sum_{i,j=1}^{N-1}\alpha_{ij}(z)\frac{\partial^2}{\partial z_i\partial z_j} + d\sum_{j=1}^{N}\beta_j(z)\frac{\partial}{\partial z_j}\Big),$$

where

$$\psi_{d,ij} = \frac{\partial^2\psi_d}{\partial z_i\partial z_j}(0) \quad \text{and} \quad \alpha_d = \Delta\psi_d(0).$$

Using Lemmas 4.2 and 4.3 from [11] we can formulate the existence result for the following problem:

$$(\mathrm{A}_3) \qquad \Delta\phi - \phi + Mw_M^{p-1}\phi + 2|z_N|\sum_{i,j=1}^{N-1}\psi_{d,ij}\frac{\partial^2 w_M}{\partial z_i\partial z_j} - \alpha_d(\operatorname{sgn} z_N)\frac{\partial w_M}{\partial z_N} = 0 \quad \text{in } \mathbb{R}^N,$$

$$(\mathrm{A}_4) \qquad \phi(z)\to 0 \quad \text{as } |z|\to\infty.$$

PROPOSITION A_1. *Problem* (A_3)–(A_4) *has a unique solution* $\phi\in C^2(\mathbb{R}^N)$ *satisfying*

$$\int_{\mathbb{R}^N}\phi\frac{\partial w_M}{\partial z_j}\,dz = 0 \qquad \textit{for } j=1,\ldots,N.$$

Furthermore, ϕ *decays exponentially at infinity*:

$$|\phi(z)|\le Ce^{-\mu|z|} \qquad \textit{for } z\in\mathbb{R}^N,$$

for some constants $C>0$ *and* $\mu>0$ *independent of* d. *The function* ϕ *is even in* z_N *and*

$$(\mathrm{A}_5) \qquad \lim_{d\to 0} d^{-1}\sup_{z\in B_{k/d}}|w_d(z) - (w_M(z)+d\phi(z))| = 0.$$

Sketch of proof. Since $w_d(z)-(w_M(z)+d\phi(z)) = d(\phi_R(z)-\phi(z))$ on B_R, for the proof of the last assertion (A_5) it is sufficient to show that $\sup|\phi_R(z)-\phi(z)|\to 0$ as $R\to\infty$. This is established by writing the following decomposition:

$$\phi_R(z) = \sum_{j=1}^{N} a_j(R)\phi_j(z) + \zeta_R(z),$$

where

$$\phi_j(z) = c_\circ\frac{\partial w_M}{\partial z_j}, \qquad c_\circ = \Big(\frac{N}{\int_{\mathbb{R}^N} w_M'(|z|)^2\,dz}\Big)^{1/2}$$

and showing that $a_j(R) \to 0$ and $\zeta_R(z) \to \phi$ as $R \to \infty$ (see Lemmas 4.7–4.9 in [11]).

Now we are in a position to establish the asymptotic formula (18) for c_d. Multiplying (A_3) by w_M and integrating over $\mathbb{R}^N$ we obtain

$$\int_{\mathbb{R}^N_+} (p-1) M w_M^p \phi \, dz = 2\gamma \alpha_d M^{-2/(p-1)} + \alpha_d M^{-2/(p-1)} \int_{\mathbb{R}^N_+} w_1 \frac{\partial w_1}{\partial z_N} \, dz \tag{A_6}$$

(for details see Lemma 3.1 in [11]). Since w_d satisfies (1) we have, using formula (A_3) from [12],

$$\begin{aligned}
c_d &= \int_{\Omega} \frac{p-1}{2(p+1)} Q(x) u_d^{p+1} \, dx \\
&= \int_{D_1} \frac{p-1}{2(p+1)} Q(x) u_d^{p+1} \, dx + \int_{\Omega - D_1} \frac{p-1}{2(p+1)} Q(x) u_d^{p+1} \, dx \\
&= \int_{D_1} \frac{p-1}{2(p+1)} Q(x) u_d^{p+1} \, dx + O(e^{-\mu/d}) \\
&= d^N \int_{B^+_{k/d}} \frac{p-1}{2(p+1)} w_d(z)^{p+1} M(1 - \alpha_d d z_n + O(d^2|z|^2)) \\
&\quad + \int_{D_1} \frac{p-1}{2(p+1)} (Q(x) - M) u_d^{p+1} \, dx + O(e^{-\mu/d}) \\
&= I_1 + I_2 + O(e^{-\mu/d}).
\end{aligned}$$

We estimate I_1 using Proposition A_1:

$$\begin{aligned}
I_1 &= \frac{p-1}{2(p+1)} d^N \int_{B^+_{k/d}} M w_d(z)^{p+1} (1 - d\alpha_d z_N) \, dz + d^{N+2} \\
&= \frac{p-1}{2(p+1)} d^N \int_{B^+_{k/d}} M (w_M + d(\phi + o(1)))^{p+1} (1 - d\alpha_d z_N) \, dz + O(d^{N+2}) \\
&= d^N \bigg[\frac{p-1}{2(p+1)} \int_{B^+_{k/d}} M w_M^{p+1} \, dx + \frac{d}{2}(p-1) \int_{B^+_{k/d}} M w_M^p \phi \, dx \\
&\quad - \alpha_d \frac{p-1}{2(p+1)} \int_{B^+_{k/d}} M w_M^{p+1} z_N \, dz + o(d) \bigg]
\end{aligned}$$

$$= d^N\left[\frac{p-1}{2(p+1)}\int_{\mathbb{R}^N_+} Mw_M^{p+1}\,dz + \frac{d}{2}(p-1)\int_{\mathbb{R}^N_+} Mw_M^p\phi\,dz\right.$$
$$\left.-\frac{p-1}{2(p+1)}\alpha_d d\int_{\mathbb{R}^N_+} Mw_M^{p+1}z_N\,dz + o(1)\right].$$

It then follows from (A_6) and an obvious modification of formula (3.13) in [12] that

$$I_1 = d^N\left\{\tfrac{1}{2}I_M(w_M) - \gamma d\alpha_d M^{-2/(p-1)}\right\} + o(1)d^N.$$

Finally, using (17) we check that $I_2 = o(1)d^N$ and the result follows.

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