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Multiplicity of positive solutions for a nonlinear fourth order equation

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Abstract. We study the existence and multiplicity of positive solutions of the nonlinear fourth order problem

$$\begin{cases} u^{(4)} = \lambda f(u) & \text{in } (0,1), \\ u(0) = a \geq 0, \quad u'(0) = a' \geq 0, \quad u(1) = b \geq 0, \quad u'(1) = -b' \leq 0. \end{cases}$$

The methods employed are upper and lower solutions and degree theory arguments.

1. Introduction. In this paper we are concerned with the existence and multiplicities of positive solutions of the nonlinear fourth order problem

$$\begin{cases} u^{(4)} = \lambda f(u) & \text{in } (0,1), \\ u(0) = a, \quad u'(0) = a', \quad u(1) = b, \quad u'(1) = -b'. \end{cases} \tag{1}$$

To emphasize the dependence on λ, we shall often denote the problem by $(1)_\lambda$, with u_λ denoting a corresponding solution. Throughout this paper we assume, without further mention, that λ is a positive parameter, a, a', b, b' are nonnegative constants, and f is continuous on $[0,\infty)$ with $f : (0,\infty) \to (0,\infty)$.

If we make the convention that $f(u) = f(0)$ when $u < 0$, then by the maximum principle for fourth order inequalities given below, all nontrivial solutions $u \in C^4[0,1]$ of $(1)_\lambda$ are positive, i.e., $u(x) > 0$ for $x \in (0,1)$.

Our main result is as follows:

Theorem 1.1. *Assume the following hypotheses*:

$$(\mathrm{H}_1) \qquad f_\infty \equiv \lim_{u \to +\infty} \frac{f(u)}{u} = \infty.$$

$$(\mathrm{H}_2) \qquad f_0 \equiv \lim_{u \to 0} \frac{f(u)}{u} = \infty.$$

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(H_3) *f is nondecreasing.*

Then there exists a positive number λ^* such that $(1)_\lambda$ has at least two positive solutions for $0 < \lambda < \lambda^*$, at least one positive solution for $\lambda = \lambda^*$ and no positive solutions for $\lambda > \lambda^*$.

In the case of a second order equation, analogous results have been proved under homogeneous boundary conditions with various hypotheses concerning the smoothness and growth properties of f. Among them, one may refer, with further references therein, to Dang, Schmitt and Shivaji [2], Ha and Lee [5] and Lin [7]. For systems of second order equations, see Dunninger and Wang [4].

The paper is organized as follows. In Section 2, some notation and preliminaries are introduced. Next, in Section 3, we use the method of upper and lower solutions to establish an existence result. In Section 4, the proof of the main theorem is given using degree theory arguments.

2. Preliminaries. Define

$$T(\lambda, u)(x) = \varphi(x) + \lambda \int_0^1 g(x, y) f(u(y))\, dy$$

where φ satisfies

$$\begin{cases} \varphi^{(4)} = 0 \quad \text{in } (0,1), \\ \varphi(0) = a, \quad \varphi'(0) = a', \quad \varphi(1) = b, \quad \varphi'(1) = -b', \end{cases}$$

and $g(x, y)$ is Green's function given explicitly by

$$g(x,y) = \begin{cases} x^2(1-y)^2(3y - x - 2xy)/6, & x \le y, \\ y^2(1-x)^2(3x - y - 2xy)/6, & x \ge y. \end{cases}$$

Then $(1)_\lambda$ is equivalent to the fixed point equation

$$T(\lambda, u) = u$$

in the usual Banach space $C[0,1]$ equipped with the sup norm. The operator $T : [0,\infty) \times C[0,1] \to C[0,1]$ is clearly completely continuous.

We shall need to make use of the following maximum principle [3]:

THEOREM 2.1. *Let $u \in C^4[0,1]$ satisfy the differential inequality*

$$u^{(4)} \ge 0 \quad \text{in } (0,1).$$

(i) *If $u(0) \ge 0$, $u'(0) \ge 0$, $u(1) \ge 0$, $u'(1) \le 0$, then either $u > 0$ or $u \equiv 0$ in $(0,1)$.*

(ii) *Let u be nonconstant and satisfy $u'(0) = u'(1) = 0$. If u assumes its minimum value at $x = 0$, then $u'''(0) < 0$, whereas if u assumes its minimum value at $x = 1$, then $u'''(1) > 0$.*

By reversing the above inequalities, we obtain analogous statements for the maximum value.

We shall need the following *a priori* estimate.

LEMMA 2.2. *Assume* (H_1). *For each* $\mu > 0$ *there exists a positive constant* M, *which depends on* μ, *such that* $\|u\|_3 < M$ *for all solutions* u *of* $(1)_\lambda$ *with* $\lambda \geq \mu$, λ *in a compact interval. Here* $\|u\|_3 = \sum_{i=0}^{3} \max_{x\in[0,1]} |u^i(x)|$.

Proof. Denote by λ_1 the smallest positive eigenvalue of

$$\begin{cases} \phi^{(4)} = \lambda\phi \quad \text{in } (0,1), \\ \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0, \end{cases}$$

with ϕ_1 denoting the corresponding eigenfunction satisfying $\phi_1(x) > 0$ for $x \in (0,1)$. In view of the above maximum principle, $\phi_1'''(0) < 0$ and $\phi_1'''(1) > 0$, from which we can readily deduce that $\phi_1''(0) > 0$ and $\phi_1''(1) > 0$.

By (H_1), there is a positive constant c_2 such that $f(u) \geq c_1 u - c_2$ for all $u \geq 0$, where $c_1\mu > \lambda_1$. Multiplying the equation in $(1)_\lambda$ by ϕ_1 and integrating, we have

$$\begin{aligned} \lambda_1 \int_0^1 \phi_1 u\,dx &= \int_0^1 \phi_1 u^{(4)}\,dx + [\phi_1' u'' - \phi_1'' u' + \phi_1''' u - \phi_1 u''']_0^1 \\ &\geq \mu \int_0^1 \phi_1 f(u)\,dx + b'\phi_1''(1) + a'\phi_1''(0) + b\phi_1'''(1) - a\phi_1'''(0) \\ &\geq c_1\mu \int_0^1 \phi_1 u\,dx - c_3 \end{aligned}$$

for some positive constant c_3. Consequently, it follows that

$$\int_0^1 \phi_1 u\,dx \leq \frac{c_3}{c_1\mu - \lambda_1}, \qquad \int_0^1 \phi_1 f(u)\,dx \leq c_4 \tag{2}$$

for some positive constant c_4.

Let $\varrho(y) = y^2(1-y)^2$. By a result obtained in [1], there is a positive constant k such that

$$0 \leq g(x,y) \leq k\varrho(y) \quad \text{for all } 0 \leq x, y \leq 1. \tag{3}$$

Furthermore, since $\phi_1''(0) > 0$, $\phi_1''(1) > 0$, $\varrho''(0) > 0$ and $\varrho''(1) > 0$, several applications of L'Hôpital's rule show that

$$\lim_{x\to 0^+} \frac{\varrho(x)}{\phi_1(x)} > 0, \qquad \lim_{x\to 1^-} \frac{\varrho(x)}{\phi_1(x)} > 0.$$

Hence, there are positive constants k_1 and k_2 such that

$$k_1\varrho(x) \leq \phi_1(x) \leq k_2\varrho(x) \quad \text{for all } x \in [0,1]. \tag{4}$$

Using (2), (3), (4) in the integral representation

$$u(x) = \varphi(x) + \lambda \int_0^1 g(x,y) f(u(y))\, dy \tag{5}$$

we deduce that the solutions u_λ of $(1)_\lambda$ are a priori bounded in $C[0,1]$ for $\lambda \geq \mu$, λ belonging to a compact interval. By using this a priori bound in (5), we see that the u_λ are also a priori bounded in $C^3[0,1]$. ∎

3. Existence. In order to prove the existence of positive solutions, we rely on the method of upper and lower solutions.

Consider the system

$$\begin{cases} u^{(4)} = F(u) & \text{in } (0,1), \\ u(0) = \gamma_0, \quad u'(0) = \gamma_0', \quad u(1) = \gamma_1, \quad u'(1) = \gamma_1', \end{cases} \tag{6}$$

where γ_i, γ_i' $(i = 0, 1)$ are given constants. A function $\overline{u} \in C^4[0,1]$ is called an *upper solution* of (6) if

$$\begin{cases} \overline{u}^{(4)} \geq F(\overline{u}) & \text{in } (0,1), \\ \overline{u}(0) \geq \gamma_0, \quad \overline{u}'(0) \geq \gamma_0', \quad \overline{u}(1) \geq \gamma_1, \quad \overline{u}'(1) \leq \gamma_1'. \end{cases}$$

Similarly, a function $\underline{u} \in C^4[0,1]$ is called a *lower solution* of (6) if it satisfies the reverse of the above inequalities.

We define

$$[\underline{u}, \overline{u}] = \{u \in C^4[0,1] : \underline{u}(x) \leq u(x) \leq \overline{u}(x),\ x \in [0,1]\}.$$

THEOREM 3.1. *Let $\overline{u}, \underline{u}$ be upper and lower solutions, respectively, of (6) such that $\underline{u}(x) \leq \overline{u}(x)$ for $x \in [0,1]$. Let $F \in C[\underline{u}, \overline{u}]$ be nondecreasing. Then (6) has a solution $u \in C^4[0,1]$ such that $\underline{u}(x) \leq u(x) \leq \overline{u}(x)$ for all $x \in [0,1]$.*

Proof. For a given function $u \in C[0,1]$ define

$$u^*(x) = \begin{cases} \overline{u}(x) & \text{if } u(x) \geq \overline{u}(x), \\ u(x) & \text{if } \underline{u}(x) \leq u(x) \leq \overline{u}(x), \\ \underline{u}(x) & \text{if } u(x) \leq \underline{u}(x), \end{cases}$$

and set $F^*(x,u) = F(u^*(x))$. Since the function F^* is bounded and continuous on $(-\infty, \infty)$, an application of the Schauder fixed point theorem yields the existence of a solution $u \in C^4[0,1]$ of $u^{(4)} = F^*(x,u)$ satisfying the boundary conditions in (6). To complete the proof it suffices to show that $u \in [\underline{u}, \overline{u}]$, for then u satisfies $u^{(4)} = F(u)$. To this end, we note that since $u^* \in [\underline{u}, \overline{u}]$ and F is nondecreasing, $z = \overline{u} - u$ satisfies

$$\begin{cases} z^{(4)} \geq F(\overline{u}) - F(u^*) \geq 0 & \text{in } (0,1), \\ z(0) \geq 0, \quad z'(0) \geq 0, \quad z(1) \geq 0, \quad z'(1) \leq 0. \end{cases}$$

Hence by the maximum principle, $u(x) \le \overline{u}(x)$ for $x \in [0,1]$. Similarly, we can show $\underline{u}(x) \le u(x)$ for $x \in [0,1]$ and the proof is complete. ∎

Let Γ denote the set of $\lambda > 0$ such that a positive solution of $(1)_\lambda$ exists, and let $\lambda^* = \sup \Gamma$. The main result of this section is as follows:

THEOREM 3.2. *Assume* (H_1)–(H_3). *Then* $(1)_\lambda$ *has a positive solution* u_λ *for all* $0 < \lambda \le \lambda^* < \infty$. *Moreover, for* $\lambda < \lambda^*$, *we have*

$$\begin{cases} 0 < u_\lambda(x) < u_{\lambda^*}(x), & x \in (0,1), \\ u'''_\lambda(0) > u'''_{\lambda^*}(0), & u'''_\lambda(1) < u'''_{\lambda^*}(1), \end{cases}$$

where u_{λ^*} *is a solution of* $(1)_{\lambda^*}$.

Proof. Let $e \in C^4[0,1]$ denote the unique positive solution of

$$\begin{cases} e^{(4)} = 1 \quad \text{in } (0,1), \\ e(0) = a, \quad e'(0) = a', \quad e(1) = b, \quad e'(1) = -b'. \end{cases}$$

Then $e^{(4)} = 1 \ge \lambda f(e)$ if λ is sufficiently small, and so e is a positive upper solution of $(1)_\lambda$. In order to determine an appropriate lower solution we distinguish the following two cases:

(i) If 0 is not a solution of $(1)_\lambda$, then we can choose 0 as a lower solution of $(1)_\lambda$.

(ii) If 0 is a solution of $(1)_\lambda$ (in particular, this implies $a = a' = b = b' = 0$), then we can choose $\varepsilon\phi_1$ as a positive lower solution of $(1)_\lambda$ for $\varepsilon > 0$ sufficiently small. In fact, since $f_0 = \infty$, we have

$$(\varepsilon\phi_1)^{(4)} - \lambda f(\varepsilon\phi_1) = \varepsilon\phi_1\left(\lambda_1 - \lambda\frac{f(\varepsilon\phi_1)}{\varepsilon\phi_1}\right) \le 0$$

for $\varepsilon > 0$ sufficiently small. Moreover, since $e''(0) > 0$, $e''(1) > 0$, $\phi_1''(0) > 0$ and $\phi_1''(1) > 0$, several applications of L'Hôpital's rule show that

$$\lim_{x\to 0^+} \frac{e(x)}{\phi_1(x)} > 0, \qquad \lim_{x\to 1^-} \frac{e(x)}{\phi_1(x)} > 0.$$

Hence we can choose ε even smaller, if necessary, so that $e(x) \ge \varepsilon\phi_1(x)$ for $x \in [0,1]$.

Thus in either case, Theorem 3.1 implies that $(1)_\lambda$ has a positive solution for λ sufficiently small, i.e., $\lambda^* > 0$.

Now let $u \in C^4[0,1]$ be a solution of $(1)_\lambda$. By (H_1) and (H_2), there is a constant $c > 0$ such that $f(u) \ge cu$ for all $u \ge 0$. Thus by a computation, as in the proof of Lemma 2.2, we have

$$(\lambda_1 - c\lambda)\int_0^1 \phi_1 u\,dx \ge 0$$

and by choosing λ sufficiently large we see that a positive solution of $(1)_\lambda$ cannot exist. Hence $\lambda^* < \infty$.

To see that $\lambda^* \in \Gamma$, let $\lambda_n \to \lambda^*$ where $\lambda_n \in \Gamma$. By Lemma 2.2, the corresponding solutions u_n of $(1)_{\lambda_n}$ are bounded in $C[0,1]$. Since the operator T is compact, a standard limiting process shows that $\lambda^* \in \Gamma$.

Let u_{λ^*} be a solution of $(1)_{\lambda^*}$ and let $\lambda < \lambda^*$. Then u_{λ^*} is a positive upper solution of $(1)_\lambda$, and consequently, $(1)_\lambda$ has a positive solution $u_\lambda \le u_{\lambda^*}$. Since f is nondecreasing, it easily follows from the maximum principle that $0 < u_\lambda(x) < u_{\lambda^*}(x)$ for $x \in (0,1)$, $u_\lambda'''(0) > u_{\lambda^*}'''(0)$ and $u_\lambda'''(1) < u_{\lambda^*}'''(1)$. ∎

4. Main theorem. Define

$$\overline{f}(x,u) = \begin{cases} f(u_{\lambda^*}(x)) & \text{if } u \ge u_{\lambda^*}(x), \\ f(u) & \text{if } 0 \le u \le u_{\lambda^*}(x), \\ f(0) & \text{if } u \le 0, \end{cases}$$

and let $\overline{T}(\lambda,u)$ be the integral operator analogous to $T(\lambda,u)$ defined by $\overline{f}$. Since $\overline{f}$ is bounded, all solutions u of $u = \overline{T}(\lambda,u)$ are a priori bounded in $C^3[0,1]$ for λ belonging to a compact interval. Without loss of generality we can assume $\|u\|_3 < M$, where $M > 0$ is given by Lemma 2.2.

Consider

$$\Omega = \{u \in C^3[0,1] : \|u\|_3 < M,\ u < u_{\lambda^*} \text{ in } (0,1), \\ u'''(0) > u_{\lambda^*}'''(0),\ u'''(1) < u_{\lambda^*}'''(1)\}.$$

Then Ω is bounded and open in $C^3[0,1]$.

LEMMA 4.1. *Let $0 < \lambda < \lambda^*$. If u is a solution of $u = \overline{T}(\lambda,u)$, then $u \in \Omega$.*

Proof. First note that

$$(u - u_{\lambda^*})^{(4)} \le \lambda(\overline{f}(x,u) - f(u_{\lambda^*})) \quad \text{in } (0,1).$$

At all points $x \in (0,1)$ where $u(x) \ge u_{\lambda^*}(x)$,

$$\overline{f}(x,u) - f(u_{\lambda^*}) = f(u_{\lambda^*}) - f(u_{\lambda^*}) = 0.$$

On the other hand, at all points $x \in (0,1)$ where $u(x) \le u_{\lambda^*}(x)$, the monotonicity of f implies

$$\overline{f}(x,u) - f(u_{\lambda^*}) = f(u) - f(u_{\lambda^*}) \le 0.$$

Hence we have

$$\begin{cases} (u - u_{\lambda^*})^{(4)} \le 0 \quad \text{in } (0,1), \\ (u - u_{\lambda^*})(0) = (u - u_{\lambda^*})'(0) = (u - u_{\lambda^*})(1) = (u - u_{\lambda^*})'(1) = 0, \end{cases}$$

which, by the maximum principle, yields $u \in \Omega$. ∎

Proof of Theorem 1.1. Let $0 < \lambda < \lambda^*$. By Theorem 3.2, there exists a solution u_λ of $(1)_\lambda$, and moreover, $u_\lambda \in \Omega$. Since $\overline{T}$ is bounded for λ in

compact intervals,

$$d(I - \overline{T}(\lambda, \cdot), B(u_\lambda, R), 0) = 1$$

if R is sufficiently large, where $B(u_\lambda, R)$ is the ball centered at u_λ with radius R in $C^3[0,1]$. By Lemma 4.1, $d(I - \overline{T}(\lambda, \cdot), \Omega, 0)$ is defined. Moreover, since $\overline{T}(\lambda, \cdot)$ has no fixed point in $B(u_\lambda, R)\backslash\Omega$, by the excision property of degree we have

$$d(I - T(\lambda, \cdot), \Omega, 0) = d(I - \overline{T}(\lambda, \cdot), \Omega, 0) = 1.$$

On the other hand, it follows from Lemma 2.2 that for $\mu > 0$ there exists an $M > 0$ such that for $\lambda \geq \mu$, λ in compact intervals,

$$d(I - T(\lambda, \cdot), B(0, M), 0) = \text{constant}$$

where $B(0, M)$ is the ball centered at 0 with radius M in $C^3[0,1]$. The latter degree must equal 0 since $(1)_\lambda$ has no solutions for $\lambda > \lambda^*$. Hence by the excision property of degree,

$$d(I - T(\lambda, \cdot), B(0, M) \setminus \overline{\Omega}, 0) = -1$$

and the existence of a second positive solution to $(1)_\lambda$ follows.

Remark. The above analysis can be readily extended to problems of the form

$$\begin{cases} u^{(4)} = \lambda f(u) & \text{in } (0,1), \\ u(0) = a \geq 0, \quad u''(0) = a' \leq 0, \quad u(1) = b \geq 0, \quad u''(1) = -b' \leq 0, \end{cases}$$

and thus provides an improvement and generalization of a related result in [6].

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