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Permanence and global exponential stability of Nicholson-type delay systems

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Abstract. We present several results on permanence and global exponential stability of Nicholson-type delay systems, which correct and generalize some recent results of Berezansky, Idels and Troib [Nonlinear Anal. Real World Appl. 12 (2011), 436–445].

1. Introduction. Recently, to describe the models of Marine Protected Areas and B-cell Chronic Lymphocytic Leukemia dynamics that belong to the class of Nicholson-type delay differential systems, L. Berezansky, L. Idels and L. Troib [BIT] considered the delay systems

$$\begin{cases} x_1'(t) = -a_1x_1(t) + b_1x_2(t) + c_1x_1(t-\tau)e^{-x_1(t-\tau)}, \\ x_2'(t) = -a_2x_2(t) + b_2x_1(t) + c_2x_2(t-\tau)e^{-x_2(t-\tau)}, \end{cases} \tag{1}$$

with initial conditions

$$x_i(s) = \varphi_i(s), \qquad s \in [-\tau, 0],\ \varphi_i(0) > 0, \tag{2}$$

where $\varphi_i \in C([-\tau,0],[0,+\infty))$, a_i, b_i, c_i and τ are nonnegative constants, $i = 1, 2$.

In [BIT], L. Berezansky, L. Idels and L. Troib claim the following results:

THEOREM A (see Theorem 2.3 in [BIT]). *Suppose $c_1 > a_1 > 0$ and $c_2 > a_2 > 0$. Then the solution of system* (1)–(2) *is bounded from below by a positive constant, and moreover*

$$\liminf_{t\to+\infty} x_1(t) \geq \frac{c_1^2}{ea_1^2}e^{-\frac{c_1}{a_1 e}}, \qquad \liminf_{t\to+\infty} x_2(t) \geq \frac{c_2^2}{ea_2^2}e^{-\frac{c_2}{a_2 e}}.$$

THEOREM B (see Theorem 4.1 in [BIT]). *Suppose*

$$\max\{c_1, c_2\} < \min\{a_1 - b_1, a_2 - b_2\}. \tag{3}$$

Then the trivial solution of system (1)–(2) *is globally asymptotically stable.*

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Unfortunately, Theorem A is incorrect, as can be seen from the following example.

Example. Consider the system

$$\begin{cases} x_1'(t) = -ax_1(t) + cx_1(t-\tau)e^{-x_1(t-\tau)}, \\ x_2'(t) = -ax_2(t) + cx_2(t-\tau)e^{-x_2(t-\tau)}, \end{cases} \tag{4}$$

where $c > a > 0$ and $c/a \in (1,2)$. Obviously, (4) is a special case of (1) with $a_1 = a_2$, $c_1 = c_2$ and $b_1 = b_2 = 0$.

Consider the trivial solution $(x_1(t), x_2(t)) = (\ln \frac{c}{a}, \ln \frac{c}{a})$. Theorem A implies

$$\liminf_{t\to+\infty} x_1(t) = \liminf_{t\to+\infty} x_2(t) = \ln \frac{c}{a} \geq \frac{c^2}{ea^2} e^{-\frac{c}{ae}} > \frac{1}{e} e^{-\frac{2}{e}}.$$

Letting $c/a \to 1+$, we obtain

$$0 \geq \frac{1}{e} e^{-\frac{2}{e}},$$

which is a contradiction.

Since Theorem A is incorrect, the proof of Theorem 2.4 in [BIT] has to be amended; this is done in Section 2. Moreover, as shown in Section 3, the global asymptotical stability of Theorem B can be replaced by global exponential stability, and the condition (3) can be relaxed to $\rho(D) < 1$, where $\rho(D)$ denotes the spectral radius of

$$D = \begin{pmatrix} c_1/a_1 & b_1/a_1 \\ b_2/a_2 & c_2/a_2 \end{pmatrix}.$$

The main purpose of this paper is to employ a novel proof to establish some criteria to guarantee the permanence and global exponential stability of system (1)–(2), and our conditions are weak.

2. Permanence

Definition 2.1. System (1)–(2) is said to be *permanent* if there are positive constants m_i and M_i such that

$$m_i \leq \liminf_{t\to+\infty} x_i(t) \leq \limsup_{t\to+\infty} x_i(t) \leq M_i \quad \text{for all } i = 1, 2.$$

Theorem 2.1 (see Theorem 2.4 in [BIT]). *System (1)–(2) is permanent if*

$$a_1a_2 - b_1b_2 > 0, \quad c_1 > a_1 > 0 \quad \textit{and} \quad c_2 > a_2 > 0.$$

Proof. By Theorem 2.2 in [BIT], we need only prove that there exist positive constants m_1 and m_2 such that

$$\liminf_{t\to+\infty} x_1(t) \geq m_1, \quad \liminf_{t\to+\infty} x_2(t) \geq m_2. \tag{5}$$

From Theorem 2.1 in [BIT] and the first equation of (1), we have

$$x_1'(t) \geq -a_1x_1(t) + c_1x_1(t-\tau)e^{-x_1(t-\tau)}, \qquad x_1(t) > 0, \quad t \in [0,+\infty). \tag{6}$$

We next prove that there exists a positive constant m_1 such that

$$\liminf_{t\to+\infty} x_1(t) \geq m_1. \tag{7}$$

Suppose, for the sake of contradiction, $\liminf_{t\to+\infty} x_1(t) = 0$. For each $t \geq 0$, we define

$$\theta(t) = \max\{\xi : \xi \leq t,\, x_1(\xi) = \min_{0\leq s\leq t} x_1(s)\}.$$

Observe that $\theta(t) \to +\infty$ as $t \to +\infty$, and

$$\lim_{t\to+\infty} x_1(\theta(t)) = 0. \tag{8}$$

However, $x_1(\theta(t)) = \min_{0\leq s\leq t} x_1(s)$, and so $x_1'(\theta(t)) \leq 0$ whenever $\theta(t) > 0$. According to (6), we have

$$0 \geq x_1'(\theta(t)) \geq -a_1x_1(\theta(t)) + c_1x_1(\theta(t)-\tau)e^{-x_1(\theta(t)-\tau)},$$

and consequently

$$a_1x_1(\theta(t)) \geq c_1x_1(\theta(t)-\tau)e^{-x_1(\theta(t)-\tau)} \quad \text{whenever } \theta(t) > 0. \tag{9}$$

This together with (8) implies that

$$\lim_{t\to+\infty} x_1(\theta(t)-\tau) = 0. \tag{10}$$

Thus, we get

(11)

$$\frac{a_1}{c_1} \geq \frac{x_1(\theta(t)-\tau)e^{-x_1(\theta(t)-\tau)}}{x_1(\theta(t))} \geq \frac{x_1(\theta(t)-\tau)e^{-x_1(\theta(t)-\tau)}}{x_1(\theta(t)-\tau)} = e^{-x_1(\theta(t)-\tau)}$$

whenever $\theta(t) > \tau$.

Letting $t \to +\infty$, (8), (10) and (11) imply that

$$\frac{a_1}{c_1} \geq 1,$$

which contradicts the assumption that $c_1 > a_1 > 0$. Hence, (7) holds. The second inequality of (5) can be proven similarly. This completes the proof of Theorem 2.1. ∎

3. Global exponential stability. In this section, for a matrix $A = (a_{ij})_{n\times n}$, A^T denotes the transpose of A, A^{-1} denotes the inverse of A, and $\rho(A)$ denotes the spectral radius of A. For a matrix or vector A, the inequality $A \geq 0$ means that all entries of A are non-negative; $A > 0$ is defined similarly. For matrices or vectors A and B, $A \geq B$ (resp. $A > B$) means that $A - B \geq 0$ (resp. $A - B > 0$).

Definition 3.1. A real non-singular $n \times n$ matrix $K = (k_{ij})$ is said to be an *M-matrix* if $k_{ij} \le 0$ for all $i, j = 1, 2, \ldots, n, i \neq j$, and $K^{-1} \ge 0$.

Lemma 3.1 (see [BP, HJ, L]). *Let $K = (k_{ij})_{n\times n}$ with $k_{ij} \le 0$, $i, j = 1, \ldots, n$, $i \neq j$. Then the following statements are equivalent.*

(1) *K is an M-matrix.*
(2) *There exists a vector $\eta = (\eta_1, \ldots, \eta_n) > (0, \ldots, 0)$ such that $\eta K > 0$.*
(3) *There exists a vector $\xi = (\xi_1, \ldots, \xi_n)^T > (0, \ldots, 0)^T$ such that $K\xi > 0$.*

Lemma 3.2 (see [BP, HJ, L]). *Let $A \ge 0$ be an $n \times n$ matrix and $\rho(A) < 1$. Then $(E_n - A)^{-1} \ge 0$, where E_n denotes the identity matrix of size n.*

Theorem 3.1. *Suppose*

$$\rho(D) < 1, \qquad D = \begin{pmatrix} c_1/a_1 & b_1/a_1 \\ b_2/a_2 & c_2/a_2 \end{pmatrix}.$$

Then the trivial solution of system (1)–(2) *is globally exponentially stable.*

Proof. Since $\rho(D) < 1$, by Lemma 3.2, $E_2 - D$ is an M-matrix. Therefore, by Lemma 3.1, there exists a vector $\xi = (\xi_1, \xi_2)^T > 0$ such that $(E_2 - D)\xi > 0$. Then

$$-a_1\xi_1 + \xi_1 c_1 + \xi_2 b_1 < 0, -a_2\xi_2 + \xi_2 c_2 + \xi_1 b_2 < 0. \tag{12}$$

Hence, there exists a sufficiently small constant $\lambda > 0$ such that

$$(\lambda - a_1)\xi_1 + c_1\xi_1 e^{\lambda\tau} + \xi_2 b_1 < 0, \qquad (\lambda - a_2)\xi_2 + c_2\xi_2 e^{\lambda\tau} + b_2\xi_1 < 0. \tag{13}$$

We consider the Lyapunov functions

$$V_1(t) = x_1(t)e^{\lambda t}, \qquad V_2(t) = x_2(t)e^{\lambda t}. \tag{14}$$

Calculating the derivative of $V_i(t)$ along the solution $x(t) = (x_1(t), x_2(t))$ of system (1)–(2) with the initial value $\varphi = (\varphi_1, \varphi_2)$, from Theorem 2.1 in [BIT] and the two equations of (1), for $t \ge 0$, we have

$$\begin{aligned} V_1'(t) &= (\lambda - a_1)x_1(t)e^{\lambda t} + c_1 x_1(t-\tau)e^{-x_1(t-\tau)}e^{\lambda t} + b_1 x_2(t)e^{\lambda t} \\ &\le (\lambda - a_1)x_1(t)e^{\lambda t} + c_1 x_1(t-\tau)e^{\lambda t} + b_1 x_2(t)e^{\lambda t}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} V_2'(t) &= (\lambda - a_2)x_2(t)e^{\lambda t} + c_2 x_2(t-\tau)e^{-x_2(t-\tau)}e^{\lambda t} + b_2 x_1(t)e^{\lambda t} \\ &\le (\lambda - a_2)x_2(t)e^{\lambda t} + c_2 x_2(t-\tau)e^{\lambda t} + b_2 x_1(t)e^{\lambda t}. \end{aligned} \tag{16}$$

Let $m > 1$ be such that

$$m\xi_i > \sup_{-\tau \le s \le 0} \varphi_i(s) > 0, \qquad i = 1, 2.$$

It follows from (14) that

$$V_i(t) = x_i(t)e^{\lambda t} < m\xi_i \quad \text{for all } t \in [-\tau, 0],\ i = 1, 2.$$

We claim that

$$V_i(t) = x_i(t)e^{\lambda t} < m\xi_i \quad \text{for all } t > 0,\ i = 1, 2. \tag{17}$$

Otherwise, one of the following cases must occur.

CASE 1: There exists $t_1 > 0$ such that

$$V_1(t_1) = m\xi_1 \quad \text{and} \quad V_j(t) < m\xi_j \quad \text{for all } t \in [-\tau, t_1),\ j = 1, 2. \tag{18}$$

CASE 2: There exists $t_2 > 0$ such that

$$V_2(t_2) = m\xi_2 \quad \text{and} \quad V_j(t) < m\xi_j \quad \text{for all } t \in [-\tau, t_2),\ j = 1, 2. \tag{19}$$

If Case 1 holds, then calculating the derivative of $V_1(t) - m\xi_1$ and making use of (15), (18) yields

$$\begin{aligned} 0 &\le (V_1(t_1) - m\xi_1)' = V_1'(t_1) \\ &\le (\lambda - a_1)x_1(t_1)e^{\lambda t_1} + c_1x_1(t_1 - \tau)e^{\lambda t_1} + b_1x_2(t_1)e^{\lambda t_1} \\ &= (\lambda - a_1)x_1(t_1)e^{\lambda t_1} + c_1x_1(t_1 - \tau)e^{\lambda(t_1-\tau)}e^{\lambda\tau} + b_1x_2(t_1)e^{\lambda t_1} \\ &\le (\lambda - a_1)m\xi_1 + c_1m\xi_1e^{\lambda\tau} + b_1m\xi_2 \\ &= [(\lambda - a_1)\xi_1 + c_1\xi_1e^{\lambda\tau} + b_1\xi_2]m, \end{aligned} \tag{20}$$

which contradicts the fact that $(\lambda - a_1)\xi_1 + c_1\xi_1e^{\lambda\tau} + \xi_2b_1 < 0$. This implies that (17) holds.

If Case 2 holds, then calculating the derivative of $V_2(t) - m\xi_2$ and making use of (16), (19) yields

$$\begin{aligned} 0 &\le (V_2(t_2) - m\xi_2)' = V_2'(t_2) \\ &\le (\lambda - a_2)x_2(t_2)e^{\lambda t_2} + c_2x_2(t_2 - \tau)e^{\lambda t_2} + b_2x_1(t_2)e^{\lambda t_2} \\ &= (\lambda - a_2)x_2(t_2)e^{\lambda t_2} + c_2x_2(t_2 - \tau)e^{\lambda(t_2-\tau)}e^{\lambda\tau} + b_2x_1(t_2)e^{\lambda t_2} \\ &\le (\lambda - a_2)m\xi_2 + c_2m\xi_2e^{\lambda\tau} + b_2m\xi_1 \\ &= [(\lambda - a_2)\xi_2 + c_2\xi_2e^{\lambda\tau} + b_2\xi_1]m, \end{aligned} \tag{21}$$

which contradicts the fact that $(\lambda - a_2)\xi_2 + c_2\xi_2e^{\lambda\tau} + b_2\xi_1 < 0$. This implies that (17) holds.

Therefore, from (17), we obtain

$$x_i(t) < m\xi_ie^{-\lambda t} \quad \text{for all } t > 0,\ i = 1, 2. \tag{22}$$

It follows that $(x_1(t), x_2(t))$ converges exponentially to $(0, 0)$ as $t \to +\infty$. This ends the proof of Theorem 3.1.

REMARK 3.1. One can easily show that $\max\{c_1, c_2\} < \min\{a_1 - b_1, a_2 - b_2\}$ implies the row norm of the matrix D is less than 1. Therefore, $\rho(D) < 1$. Hence, Theorem 4.1 of [BIT] is a special case of our Theorem 3.1. Moreover, exponential convergence is an important dynamic behavior since it gives a rate of convergence. This implies that our results improve those in [BIT].

4. An example

EXAMPLE 4.1. Consider the Nicholson-type delay system

$$\begin{cases} x_1'(t) = -20x_1(t) + \frac{10}{9}x_2(t) + 10x_1(t-\tau)e^{-x_1(t-\tau)}, \\ x_2'(t) = -40x_2(t) + 80x_1(t) + 20x_2(t-\tau)e^{-x_2(t-\tau)}. \end{cases} \tag{23}$$

Obviously, $a_1 = 20$, $b_1 = 10/9$, $c_1 = 10$, $a_2 = 40$, $b_2 = 80$, $c_2 = 20$, and

$$D = \begin{pmatrix} c_1/a_1 & b_1/a_1 \\ b_2/a_2 & c_2/a_2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/18 \\ 2 & 1/2 \end{pmatrix}.$$

An easy computation shows that $\rho(D) = 5/6 < 1$. Thus, from Theorem 3.1, every solution $(x_1(t), x_2(t))$ of system (23) with initial conditions (2) converges exponentially to $(0,0)$ as $t \to +\infty$.

REMARK 4.1. System (23) is a very simple form of Nicholson-type delay system. One can observe that

$$\max\{c_1, c_2\} = 20 > -40 = \min\{a_1 - b_1, a_2 - b_2\}.$$

Therefore, no results in [BIT, Theorem 4.1] can be applied to (23). This implies that the results in Theorem 3.1 of this paper are essentially new.

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