Permanence and global exponential stability of Nicholson-type delay systems

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Abstract. We present several results on permanence and global exponential stability of Nicholson-type delay systems, which correct and generalize some recent results of Berezansky, Idels and Troib [Nonlinear Anal. Real World Appl. 12 (2011), 436–445].

1. Introduction. Recently, to describe the models of Marine Protected Areas and B-cell Chronic Lymphocytic Leukemia dynamics that belong to the class of Nicholson-type delay differential systems, L. Berezansky, L. Idels and L. Troib [BIT] considered the delay systems

\[
\begin{align*}
  x'_1(t) & = -a_1 x_1(t) + b_1 x_2(t) + c_1 x_1(t - \tau) e^{-x_1(t-\tau)}, \\
  x'_2(t) & = -a_2 x_2(t) + b_2 x_1(t) + c_2 x_2(t - \tau) e^{-x_2(t-\tau)},
\end{align*}
\]

with initial conditions

\[
  x_i(s) = \varphi_i(s), \quad s \in [-\tau, 0], \varphi_i(0) > 0,
\]

where \( \varphi_i \in C([-\tau, 0], [0, +\infty)) \), \( a_i, b_i, c_i \) and \( \tau \) are nonnegative constants, \( i = 1, 2 \).

In [BIT], L. Berezansky, L. Idels and L. Troib claim the following results:

**Theorem A** (see Theorem 2.3 in [BIT]). Suppose \( c_1 > a_1 > 0 \) and \( c_2 > a_2 > 0 \). Then the solution of system (1)–(2) is bounded from below by a positive constant, and moreover

\[
\liminf_{t \to +\infty} x_1(t) \geq \frac{c_1^2}{ea_1^2} e^{-\frac{c_1}{a_1^2} \tau}, \quad \liminf_{t \to +\infty} x_2(t) \geq \frac{c_2^2}{da_2^2} e^{-\frac{c_2}{a_2^2} \tau}.
\]

**Theorem B** (see Theorem 4.1 in [BIT]). Suppose

\[
\max\{c_1, c_2\} < \min\{a_1 - b_1, a_2 - b_2\}.
\]

Then the trivial solution of system (1)–(2) is globally asymptotically stable.

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Unfortunately, Theorem A is incorrect, as can be seen from the following example.

**Example.** Consider the system

\[
\begin{align*}
x_1'(t) &= -ax_1(t) + cx_1(t - \tau)e^{-x_1(t-\tau)}, \\
x_2'(t) &= -ax_2(t) + cx_2(t - \tau)e^{-x_2(t-\tau)},
\end{align*}
\]

where \(c > a > 0\) and \(c/a \in (1, 2)\). Obviously, (4) is a special case of (1) with \(a_1 = a_2, c_1 = c_2\) and \(b_1 = b_2 = 0\).

Consider the trivial solution \((x_1(t), x_2(t)) = (\ln \frac{c}{a}, \ln \frac{c}{a})\). Theorem A implies

\[
\liminf_{t \to +\infty} x_1(t) = \liminf_{t \to +\infty} x_2(t) = \ln \frac{c}{a} \geq \frac{e^2}{e \cdot a^2} - \frac{e}{a^2} > e^{-\frac{2}{e}}.
\]

Letting \(c/a \to 1^+\), we obtain

\[
0 \geq e^{-\frac{2}{e}},
\]

which is a contradiction.

Since Theorem A is incorrect, the proof of Theorem 2.4 in [BIT] has to be amended; this is done in Section 2. Moreover, as shown in Section 3, the global asymptotical stability of Theorem B can be replaced by global exponential stability, and the condition (3) can be relaxed to \(\rho(D) < 1\), where \(\rho(D)\) denotes the spectral radius of

\[
D = \begin{pmatrix}
c_1/a_1 & b_1/a_1 \\
b_2/a_2 & c_2/a_2
\end{pmatrix}.
\]

The main purpose of this paper is to employ a novel proof to establish some criteria to guarantee the permanence and global exponential stability of system (1)–(2), and our conditions are weak.

### 2. Permanence

**Definition 2.1.** System (1)–(2) is said to be **permanent** if there are positive constants \(m_i\) and \(M_i\) such that

\[
m_i \leq \liminf_{t \to +\infty} x_i(t) \leq \limsup_{t \to +\infty} x_i(t) \leq M_i \quad \text{for all } i = 1, 2.
\]

**Theorem 2.1** (see Theorem 2.4 in [BIT]). **System (1)–(2) is permanent if**

\[
a_1a_2 - b_1b_2 > 0, \quad c_1 > a_1 > 0 \quad \text{and} \quad c_2 > a_2 > 0.
\]

**Proof.** By Theorem 2.2 in [BIT], we need only prove that there exist positive constants \(m_1\) and \(m_2\) such that

\[
\liminf_{t \to +\infty} x_1(t) \geq m_1, \quad \liminf_{t \to +\infty} x_2(t) \geq m_2.
\]
From Theorem 2.1 in [BIT] and the first equation of (1), we have
\begin{equation}
(6) \quad x'_1(t) \geq -a_1x_1(t) + c_1x_1(t-\tau)e^{-x_1(t-\tau)}, \quad x_1(t) > 0, \quad t \in [0, +\infty).
\end{equation}

We next prove that there exists a positive constant \( m_1 \) such that
\begin{equation}
(7) \quad \liminf_{t \to +\infty} x_1(t) \geq m_1.
\end{equation}

Suppose, for the sake of contradiction, \( \liminf_{t \to +\infty} x_1(t) = 0 \). For each \( t \geq 0 \), we define
\begin{equation}
\theta(t) = \max\{\xi : \xi \leq t, \ x_1(\xi) = \min_{0 \leq s \leq t} x_1(s)\}.
\end{equation}

Observe that \( \theta(t) \to +\infty \) as \( t \to +\infty \), and
\begin{equation}
(8) \quad \lim_{t \to +\infty} x_1(\theta(t)) = 0.
\end{equation}

However, \( x_1(\theta(t)) = \min_{0 \leq s \leq t} x_1(s) \), and so \( x'_1(\theta(t)) \leq 0 \) whenever \( \theta(t) > 0 \).

According to (6), we have
\begin{equation}
0 \geq x'_1(\theta(t)) \geq -a_1x_1(\theta(t)) + c_1x_1(\theta(t) - \tau)e^{-x_1(\theta(t) - \tau)},
\end{equation}

and consequently
\begin{equation}
(9) \quad a_1x_1(\theta(t)) \geq c_1x_1(\theta(t) - \tau)e^{-x_1(\theta(t) - \tau)} \quad \text{whenever } \theta(t) > 0.
\end{equation}

This together with (8) implies that
\begin{equation}
(10) \quad \lim_{t \to +\infty} x_1(\theta(t) - \tau) = 0.
\end{equation}

Thus, we get
\begin{equation}
(11) \quad \frac{a_1}{c_1} \geq \frac{x_1(\theta(t) - \tau)e^{-x_1(\theta(t) - \tau)}}{x_1(\theta(t))} \geq \frac{x_1(\theta(t) - \tau)e^{-x_1(\theta(t) - \tau)}}{x_1(\theta(t) - \tau)} = e^{-x_1(\theta(t) - \tau)}
\end{equation}

whenever \( \theta(t) > \tau \).

Letting \( t \to +\infty \), (8), (10) and (11) imply that
\begin{equation}
\frac{a_1}{c_1} \geq 1,
\end{equation}

which contradicts the assumption that \( c_1 > a_1 > 0 \). Hence, (7) holds. The second inequality of (5) can be proven similarly. This completes the proof of Theorem 2.1. \( \blacksquare \)

3. Global exponential stability. In this section, for a matrix \( A = (a_{ij})_{n \times n} \), \( A^T \) denotes the transpose of \( A \), \( A^{-1} \) denotes the inverse of \( A \), and \( \rho(A) \) denotes the spectral radius of \( A \). For a matrix or vector \( A \), the inequality \( A \geq 0 \) means that all entries of \( A \) are non-negative; \( A > 0 \) is defined similarly. For matrices or vectors \( A \) and \( B \), \( A \geq B \) (resp. \( A > B \)) means that \( A - B \geq 0 \) (resp. \( A - B > 0 \)).
DEFINITION 3.1. A real non-singular $n \times n$ matrix $K = (k_{ij})$ is said to be an $M$-matrix if $k_{ij} \leq 0$ for all $i, j = 1, 2, \ldots, n, i \neq j$, and $K^{-1} \geq 0$.

LEMMA 3.1 (see [BP, HJ, L]). Let $K = (k_{ij})_{n \times n}$ with $k_{ij} \leq 0, i, j = 1, \ldots, n, i \neq j$. Then the following statements are equivalent.

1. $K$ is an $M$-matrix.
2. There exists a vector $\eta = (\eta_1, \ldots, \eta_n) > (0, \ldots, 0)$ such that $\eta K > 0$.
3. There exists a vector $\xi = (\xi_1, \ldots, \xi_n)^T > (0, \ldots, 0)^T$ such that $K \xi > 0$.

LEMMA 3.2 (see [BP, HJ, L]). Let $A \geq 0$ be an $n \times n$ matrix and $\rho(A) < 1$. Then $(E_n - A)^{-1} \geq 0$, where $E_n$ denotes the identity matrix of size $n$.

THEOREM 3.1. Suppose

$$\rho(D) < 1, \quad D = \begin{pmatrix} \frac{c_1}{a_1} & \frac{b_1}{a_1} \\ \frac{b_2}{a_2} & \frac{c_2}{a_2} \end{pmatrix}.$$  

Then the trivial solution of system (1)–(2) is globally exponentially stable.

Proof. Since $\rho(D) < 1$, by Lemma 3.2, $E_2 - D$ is an $M$-matrix. Therefore, by Lemma 3.1, there exists a vector $\xi = (\xi_1, \xi_2)^T > 0$ such that $(E_2 - D) \xi > 0$.

Then

$$-a_1 \xi_1 + \xi_1 c_1 + \xi_2 b_1 < 0, -a_2 \xi_2 + \xi_2 c_2 + \xi_1 b_2 < 0.$$  

Hence, there exists a sufficiently small constant $\lambda > 0$ such that

$$\lambda - a_1) \xi_1 + c_1 \xi_1 e^{\lambda \tau} + \xi_2 b_1 < 0, \quad (\lambda - a_2) \xi_2 + c_2 \xi_2 e^{\lambda \tau} + b_2 \xi_1 < 0.$$  

We consider the Lyapunov functions

$$V_1(t) = x_1(t) e^{\lambda t}, \quad V_2(t) = x_2(t) e^{\lambda t}.$$  

Calculating the derivative of $V_i(t)$ along the solution $x(t) = (x_1(t), x_2(t))$ of system (1)–(2) with the initial value $\varphi = (\varphi_1, \varphi_2)$, from Theorem 2.1 in [BTT] and the two equations of (1), for $t \geq 0$, we have

$$V_1'(t) = (\lambda - a_1) x_1(t) e^{\lambda t} + c_1 x_1(t - \tau) e^{-(\lambda - a_1)(t - \tau)} e^{\lambda t} + b_1 x_2(t) e^{\lambda t} \leq (\lambda - a_1) x_1(t) e^{\lambda t} + c_1 x_1(t - \tau) e^{\lambda t} + b_1 x_2(t) e^{\lambda t},$$

and

$$V_2'(t) = (\lambda - a_2) x_2(t) e^{\lambda t} + c_2 x_2(t - \tau) e^{-(\lambda - a_2)(t - \tau)} e^{\lambda t} + b_2 x_1(t) e^{\lambda t} \leq (\lambda - a_2) x_2(t) e^{\lambda t} + c_2 x_2(t - \tau) e^{\lambda t} + b_2 x_1(t) e^{\lambda t}.$$  

Let $m > 1$ be such that

$$m \xi_i > \sup_{-\tau \leq s \leq 0} \varphi_i(s) > 0, \quad i = 1, 2.$$  

It follows from (14) that

$$V_i(t) = x_i(t) e^{\lambda t} < m \xi_i \quad \text{for all } t \in [-\tau, 0], \ i = 1, 2.$$
We claim that
\[ V_i(t) = x_i(t)e^{\lambda t} < m\xi_i \quad \text{for all } t > 0, \, i = 1, 2. \]
Otherwise, one of the following cases must occur.

\textbf{Case 1:} There exists \( t_1 > 0 \) such that
\[ V_1(t_1) = m\xi_1 \quad \text{and} \quad V_j(t) < m\xi_j \quad \text{for all } t \in [-\tau, t_1), \, j = 1, 2. \]

\textbf{Case 2:} There exists \( t_2 > 0 \) such that
\[ V_2(t_2) = m\xi_2 \quad \text{and} \quad V_j(t) < m\xi_j \quad \text{for all } t \in [-\tau, t_2), \, j = 1, 2. \]

If Case 1 holds, then calculating the derivative of \( V_1(t) - m\xi_1 \) and making use of (15), (18) yields
\[
0 \leq (V_1(t_1) - m\xi_1)' = V_1'(t_1) \\
\leq (\lambda - a_1)x_1(t_1)e^{\lambda t_1} + c_1x_1(t_1 - \tau)e^{\lambda t_1} + b_1x_2(t_1)e^{\lambda t_1} \\
= (\lambda - a_1)x_1(t_1)e^{\lambda t_1} + c_1x_1(t_1 - \tau)e^{\lambda (t_1 - \tau)}e^{\lambda \tau} + b_1x_2(t_1)e^{\lambda t_1} \\
\leq (\lambda - a_1)m\xi_1 + c_1m\xi_1e^{\lambda \tau} + b_1m\xi_2 \\
= [\lambda - a_1]\xi_1 + c_1\xi_1e^{\lambda \tau} + b_1\xi_2]m,
\]
which contradicts the fact that \( (\lambda - a_1)\xi_1 + c_1\xi_1e^{\lambda \tau} + \xi_2b_1 < 0 \). This implies that (17) holds.

If Case 2 holds, then calculating the derivative of \( V_2(t) - m\xi_2 \) and making use of (16), (19) yields
\[
0 \leq (V_2(t_2) - m\xi_2)' = V_2'(t_2) \\
\leq (\lambda - a_2)x_2(t_2)e^{\lambda t_2} + c_2x_2(t_2 - \tau)e^{\lambda t_2} + b_2x_1(t_2)e^{\lambda t_2} \\
= (\lambda - a_2)x_2(t_2)e^{\lambda t_2} + c_2x_2(t_2 - \tau)e^{\lambda (t_2 - \tau)}e^{\lambda \tau} + b_2x_1(t_2)e^{\lambda t_2} \\
\leq (\lambda - a_2)m\xi_2 + c_2m\xi_2e^{\lambda \tau} + b_2m\xi_1 \\
= [(\lambda - a_2)\xi_2 + c_2\xi_2e^{\lambda \tau} + b_2\xi_1]m,
\]
which contradicts the fact that \( (\lambda - a_2)\xi_2 + c_2\xi_2e^{\lambda \tau} + b_2\xi_1 < 0 \). This implies that (17) holds.

Therefore, from (17), we obtain
\[ x_i(t) < m\xi_i e^{-\lambda t} \quad \text{for all } t > 0, \, i = 1, 2. \]
It follows that \((x_1(t), x_2(t))\) converges exponentially to \((0, 0)\) as \( t \to +\infty \). This ends the proof of Theorem 3.1.

\textbf{Remark 3.1.} One can easily show that \( \max\{c_1, c_2\} < \min\{a_1 - b_1, a_2 - b_2\} \)
implies the row norm of the matrix \( D \) is less than 1. Therefore, \( \rho(D) < 1 \). Hence, Theorem 4.1 of [BIT] is a special case of our Theorem 3.1. Moreover, exponential convergence is an important dynamic behavior since it gives a rate of convergence. This implies that our results improve those in [BIT].
4. An example

Example 4.1. Consider the Nicholson-type delay system

\begin{align*}
    x'_1(t) &= -20x_1(t) + \frac{10}{9}x_2(t) + 10x_1(t - \tau)e^{-x_1(t-\tau)}, \\
    x'_2(t) &= -40x_2(t) + 80x_1(t) + 20x_2(t - \tau)e^{-x_2(t-\tau)}.
\end{align*}

(23)

Obviously, \( a_1 = 20, \ b_1 = 10/9, \ c_1 = 10, \ a_2 = 40, \ b_2 = 80, \ c_2 = 20, \) and

\[
    D = \begin{pmatrix}
        c_1/a_1 & b_1/a_1 \\
        b_2/a_2 & c_2/a_2
    \end{pmatrix} = \begin{pmatrix}
        1/2 & 1/18 \\
        2 & 1/2
    \end{pmatrix}.
\]

An easy computation shows that \( \rho(D) = 5/6 < 1. \) Thus, from Theorem 3.1, every solution \((x_1(t), x_2(t))\) of system (23) with initial conditions (2) converges exponentially to \((0, 0)\) as \( t \to +\infty. \)

Remark 4.1. System (23) is a very simple form of Nicholson-type delay system. One can observe that

\[
    \max\{c_1, c_2\} = 20 > -40 = \min\{a_1 - b_1, a_2 - b_2\}.
\]

Therefore, no results in [BIT, Theorem 4.1] can be applied to (23). This implies that the results in Theorem 3.1 of this paper are essentially new.

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