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Remarks on strongly Wright-convex functions

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Abstract. Some properties of strongly Wright-convex functions are presented. In particular it is shown that a function $f : D \to \mathbb{R}$, where D is an open convex subset of an inner product space X, is strongly Wright-convex with modulus c if and only if it can be represented in the form $f(x) = g(x) + a(x) + c\|x\|^2$, $x \in D$, where $g : D \to \mathbb{R}$ is a convex function and $a : X \to \mathbb{R}$ is an additive function. A characterization of inner product spaces by strongly Wright-convex functions is also given.

1. Introduction. Let $(X, \|\cdot\|)$ be a normed space, D a convex subset of X and let $c > 0$. A function $f : D \to \mathbb{R}$ is called:

- *strongly convex with modulus c* if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2 \tag{1.1}$$

  for all $x, y \in D$ and $t \in [0,1]$;
- *strongly Wright-convex with modulus c* if

$$f(tx+(1-t)y)+f((1-t)x+ty) \le f(x)+f(y)-2ct(1-t)\|x-y\|^2 \tag{1.2}$$

  for all $x, y \in D$ and $t \in [0,1]$;
- *strongly midconvex* (or *strongly Jensen convex*) *with modulus c* if (1.1) is assumed only for $t = 1/2$, that is,

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2, \quad x, y \in D. \tag{1.3}$$

We say that f is *strongly convex*, *strongly Wright-convex*, or *strongly midconvex* if it satisfies the condition (1.1), (1.2) or (1.3), respectively, with some $c > 0$. Note that every strongly convex function is strongly Wright-convex, and every strongly Wright-convex function is strongly midconvex (with the same modulus c), but not the converse (see Example 1.1 below).

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The usual notions of convexity, Wright-convexity and midconvexity correspond to the case $c = 0$. A comprehensive review on this subject can be found, for instance, in [Ku], [RV], [N-P]. Strongly convex functions have been introduced by Polyak [P] and they play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature (see, for instance, [J], [MN], [M], [P], [RW], [RV], [V]). Strongly midconvex functions were considered in [AGNS], [V], [NP].

The aim of this note is to present some properties of strongly Wright-convex functions. First we prove that a function $f : D \to \mathbb{R}$ is strongly Wright-convex with modulus c if and only if $f = f_1 + a$, where f_1 is a function strongly convex with modulus c and a is an additive function. This is a counterpart to the known result of Ng [Ng1]. Next we show that if a strongly midconvex function f is majorized by a strongly midconcave function then f is strongly Wright-convex. Finally we prove that in inner product spaces every function f strongly Wright-convex with modulus c can be represented in the form $f = h + c\|\cdot\|^2$, where h is Wright-convex. Moreover, we show that this condition characterizes inner product spaces among normed spaces.

As was mentioned above, strong convexity with modulus c implies strong Wright-convexity with modulus c, which in turn implies strong midconvexity with modulus c. The following examples show that the converse implications are not true.

Example 1.1. Let $a : \mathbb{R} \to \mathbb{R}$ be an additive discontinuous function and $f_1(x) = a(x) + x^2$, $x \in \mathbb{R}$. By simple calculation one can check that f_1 is strongly Wright-convex with modulus 1. However, f_1 is not strongly convex (even it is not convex) because it is not continuous.

Now, take the function $f_2(x) = |a(x)| + x^2$, $x \in \mathbb{R}$. Clearly, f_2 is strongly midconvex, but it is not strongly Wright-convex (it is not even Wright-convex) because it is discontinuous and bounded from below (see [N2, Prop.2]).

2. A representation. In [Ng1] Ng proved that a function f defined on a convex subset of $\mathbb{R}^n$ is Wright-convex if and only if it can be represented in the form $f = f_1 + a$, where f_1 is a convex function and a is an additive function (see also [N2]). Kominek [K1] extended that result to functions defined on algebraically open subsets of a vector space. In this section we present a similar representation theorem for strongly Wright-convex functions. We start with the following useful fact.

Lemma 2.1. *Let D be a convex subset of a normed space and $c > 0$. If a function $f : D \to \mathbb{R}$ is convex and strongly midconvex with modulus c, then it is strongly convex with modulus c.*

Proof. Fix arbitrary $x, y \in D$, $x \neq y$, and $t \in (0,1)$. Since f is strongly midconvex with modulus c, it satisfies the condition

$$f(qx + (1-q)y) \leq qf(x) + (1-q)f(y) - cq(1-q)\|x-y\|^2 \tag{2.1}$$

for all dyadic $q \in (0,1)$ (see [AGNS]). Consider the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(s) = f(sx + (1-s)y), \quad s \in [0,1].$$

By (2.1) we have

$$g(q) \leq qg(1) + (1-q)g(0) - cq(1-q)\|x-y\|^2 \tag{2.2}$$

for all dyadic $q \in (0,1)$. Since f is convex, also g is convex and hence it is continuous on the open interval $(0,1)$. Take a sequence (q_n) of dyadic numbers in $(0,1)$ tending to t. Using (2.2) for $q = q_n$ and the continuity of g at t, we obtain

$$g(t) \leq tg(1) + (1-t)g(0) - ct(1-t)\|x-y\|^2.$$

Now, by the definition of g, we get

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2,$$

which finishes the proof. ∎

THEOREM 2.2. *Let D be an open convex subset of a normed space X and $c > 0$. A function $f : D \to \mathbb{R}$ is strongly Wright-convex with modulus c if and only if there exist a function $f_1 : D \to \mathbb{R}$ strongly convex with modulus c and an additive function $a : X \to \mathbb{R}$ such that*

$$f(x) = f_1(x) + a(x), \quad x \in D. \tag{2.3}$$

Proof. Assume first that f is strongly Wright-convex with modulus c. Then f is also Wright-convex and hence, by the result of Kominek [K1], f can be represented in the form $f = f_1 + a$, with some convex function f_1 and additive function a. Since f is strongly Wright-convex with modulus c, the function $f - a$ is also strongly Wright-convex with modulus c and, consequently, it is strongly midconvex with modulus c. Hence, by Lemma 2.1, $f_1 = f - a$ is strongly convex with modulus c, which proves that f has the representation (2.3). The reverse implication is obvious. ∎

3. Strongly midconvex functions with strongly midconcave bounds. It is known that if a midconvex function f is bounded from above by a midconcave function g then f is Wright-convex and g is Wright-concave. Moreover, there exist a convex function f_1, a concave function g_1 and an additive function a such that $f = f_1 + a$ and $g = g_1 + a$ (see [Ng2], [N1], [K2]). In this section we present a counterpart of that result for strongly midconvex functions. We say that a function f is *strongly concave* (*strongly midconcave*) with modulus c if $-f$ is strongly convex (strongly midconvex) with modulus c. In the proof of the theorem below we adopt the method used in [K2].

THEOREM 3.1. *Let D be an open convex subset of a normed space X and c be a positive constant. Assume that $f : D \to \mathbb{R}$ is strongly midconvex with modulus c, $g : D \to \mathbb{R}$ is strongly midconcave with modulus c and $f \leq g$ on D. Then there exist an additive function $a : X \to \mathbb{R}$, a continuous function $f_1 : D \to \mathbb{R}$ strongly convex with modulus c and a continuous function $g_1 : D \to \mathbb{R}$ strongly concave with modulus c such that*

$$f(x) = f_1(x) + a(x) \quad \text{and} \quad g(x) = g_1(x) + a(x) \tag{3.1}$$

for all $x \in D$.

Proof. Since f is strongly midconvex, it is also midconvex. Therefore, by the theorem of Rodé [R], there exists a Jensen function $a_1 : D \to \mathbb{R}$ such that $a_1(x) \leq f(x)$, $x \in D$. This function is of the form

$$a_1(x) = a(x) + b, \qquad x \in D,$$

where $a : X \to \mathbb{R}$ is an additive function and b is a constant (see [Ku]). The function $g_1 = g - a$ is midconcave and

$$g_1(x) = g(x) - a(x) \geq f(x) - a(x) \geq b, \qquad x \in D.$$

Therefore by the famous Bernstein–Doetsch theorem (see [Ku], [RV]), g_1 is continuous and concave. On the other hand, the function $f_1 = f - a$ is midconvex and $f_1 \leq g_1$ on D. Hence, applying the Bernstein–Doetsch theorem once more, we infer that f_1 is continuous and convex. Using Lemma 2.1 we deduce that f_1 is strongly convex with modulus c and g_1 is strongly concave with modulus c. Thus we get the representations (3.1), which completes the proof. ∎

4. A characterization of inner product spaces by strongly Wright-convex functions. In this section we show that in the case where D is a convex subset of an inner product space, a function $f : D \to \mathbb{R}$ is strongly Wright-convex with modulus c if and only if it is of the form

$$f(x) = h(x) + c\|x\|^2, \qquad x \in D, \tag{4.1}$$

where $h : D \to \mathbb{R}$ is a Wright-convex function. Moreover, we show that the fact that every strongly Wright-convex function has the representation (4.1) characterizes inner product spaces among normed spaces. Similar characterizations of inner product spaces by strongly convex and strongly midconvex functions are presented in [NP].

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a real normed space. The following conditions are equivalent:*

1. *$(X, \|\cdot\|)$ is an inner product space.*
2. *For every $c > 0$ and for every function $f : D \to \mathbb{R}$ defined on a convex subset D of X, f is strongly Wright-convex with modulus c if and only if $h = f - c\|\cdot\|^2$ is Wright-convex.*
3. *$\|\cdot\|^2 : X \to \mathbb{R}$ is strongly Wright-convex with modulus 1.*

Proof. To prove 1⇒2 assume that $(X, \|\cdot\|)$ is an inner product space and $f : D \to \mathbb{R}$ is strongly Wright-convex with modulus c. Using elementary properties of the inner product we get

$$\begin{aligned}
&h(tx + (1-t)y) + h((1-t)x + ty)\\
&\qquad = f(tx + (1-t)y) - c\|tx + (1-t)y\|^2\\
&\qquad\quad + f((1-t)x + ty) - c\|((1-t)x + ty)\|^2\\
&\qquad \le f(x) + f(y) - 2ct(1-t)\|x-y\|^2\\
&\qquad\quad - c\|tx + (1-t)y\|^2 - c\|((1-t)x + ty)\|^2\\
&\qquad = f(x) + f(y) - c(2t(1-t)(\|x\|^2 - 2\langle x|y\rangle + \|y\|^2)\\
&\qquad\quad + t^2\|x\|^2 + 2t(1-t)\langle x|y\rangle\\
&\qquad\quad + (1-t)^2\|y\|^2 + (1-t)^2\|x\|^2 + 2t(1-t)\langle x|y\rangle + t^2\|y\|^2)\\
&\qquad = f(x) - c\|x\|^2 + f(y) - c\|y\|^2 = h(x) + h(y),
\end{aligned}$$

which shows that h is Wright-convex.

Conversely, if h is Wright-convex and $f = h + c\|\cdot\|^2$, then

$$\begin{aligned}
&f(tx + (1-t)y) + f((1-t)x + ty)\\
&\quad = h(tx + (1-t)y) + c\|tx + (1-t)y\|^2\\
&\qquad + h((1-t)x + ty) + c\|((1-t)x + ty)\|^2\\
&\quad \le h(x) + h(y) + c(t^2\|x\|^2 + 4t(1-t)\langle x|y\rangle\\
&\qquad + (1-t)^2\|y\|^2 + (1-t)^2\|x\|^2 + t^2\|y\|^2)\\
&\quad = h(x) + c\|x\|^2 + h(y) + c\|y\|^2 - 2ct(1-t)(\|x\|^2 - 2\langle x|y\rangle + \|y\|^2)\\
&\quad = f(x) + f(y) - 2ct(1-t)\|x-y\|^2,
\end{aligned}$$

which proves that f is strongly Wright-convex with modulus c.

To see that 2⇒3 take $f = c\|\cdot\|^2$. Then f is strongly Wright-convex with modulus c because $h = f - c\|\cdot\|^2 = 0$ is Wright-convex. Consequently, $\|\cdot\|^2 = c^{-1}f$ is strongly Wright-convex with modulus 1.

To prove 3⇒1 observe that by the strong Wright-convexity with modulus 1 of $\|\cdot\|^2$ we have, for $t = 1/2$,

$$\left\|\frac{x+y}{2}\right\|^2 \le \frac{\|x\|^2+\|y\|^2}{2} - \frac{1}{4}\|x-y\|^2,$$

and hence

$$\|x+y\|^2 + \|x-y\|^2 \le 2\|x\|^2 + 2\|y\|^2 \tag{4.2}$$

for all $x, y \in X$. Now, putting $u = x+y$ and $v = x-y$ in (4.2), we get

$$2\|u\|^2 + 2\|v\|^2 \le \|u+v\|^2 + \|u-v\|^2, \qquad u, v \in X. \tag{4.3}$$

Conditions (4.2) and (4.3) mean that the norm $\|\cdot\|$ satisfies the parallelogram law. Hence, by the classical Jordan–von Neumann theorem, $(X, \|\cdot\|)$ is an inner product space. ∎

Using the above Theorem 4.1 and the representation of Wright-convex functions due to Ng [Ng1] (cf. also Kominek [K1]), or alternatively, using Theorem 2.2 and the representation of strongly convex functions in inner product spaces proved by Nikodem and Páles [NP], we obtain the following characterization of strongly Wright-convex functions in inner product spaces.

Corollary 4.2. *Let $(X, \|\cdot\|)$ be a real inner product space, D be an open convex subset of X and $c > 0$. A function $f : D \to \mathbb{R}$ is strongly Wright-convex with modulus c if and only if there exist a convex function $g : D \to \mathbb{R}$ and an additive function $a : X \to \mathbb{R}$ such that*

$$f(x) = g(x) + a(x) + c\|x\|^2, \qquad x \in D. \tag{4.4}$$

Remark 4.3. It is well known that convex functions defined on an open subset of a finite-dimensional space are continuous. Therefore, in the case where $X = \mathbb{R}^n$ (with the Euclidean norm), the function g appearing in the representation (4.4) is convex and continuous. In infinite-dimensional inner product spaces not every strongly Wright-convex function f can be represented in the form (4.4) with convex and continuous g (see Example 4.4 below). However, if f is strongly Wright-convex with modulus c and has a (strongly) midconcave bound then, in view of Theorem 3.1, it has the representation (4.4) with convex continuous g.

Example 4.4 (cf. [K1]). Assume that X is an infinite-dimensional inner product space and $l : X \to \mathbb{R}$ is a discontinuous linear functional. Let $f(x) = |l(x)| + \|x\|^2$, $x \in X$. By Theorem 4.1, f is strongly Wright-convex with modulus 1. Suppose that

$$f(x) = g(x) + a(x) + \|x\|^2, \qquad x \in X, \tag{4.5}$$

with an additive function a and a convex continuous function g. Then $|l(x)| = g(x) + a(x)$, $x \in X$. Consider $U = \{x \in X : g(x) < 1\}$. By the continuity of g, the set U is open and nonempty ($0 \in U$). Since a is additive and

$$a(x) = |l(x)| - g(x) > -1, \qquad x \in U,$$

it follows that a is continuous. Consequently, in view of (4.5), f is continuous, which contradicts the definition of f.

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