ANNALES
POLONICI MATHEMATICI
102.3 (2011)

On prolongation of higher order connections

by Miroslav Doupovec (Brno)
and Włodzimierz M. Mikulski (Kraków)

Abstract. We describe all bundle functors G admitting natural operators transforming rth order holonomic connections on a fibered manifold $Y \to M$ into rth order holonomic connections on $GY \to M$. For second order holonomic connections we classify all such natural operators.

1. Introduction. Given a fibered manifold $Y \to M$, the bundle J^rY of all r-jets of local sections of Y is called the *rth holonomic prolongation* of Y. The *rth nonholonomic prolongation* $\widetilde{J}^rY$ is defined by iteration: $\widetilde{J}^1Y = J^1Y$, $\widetilde{J}^rY = J^1(\widetilde{J}^{r-1}Y \to M)$. One can also define the rth semiholonomic prolongation $\overline{J}^rY$ (see e.g. [19]). Then we have $J^rY \subset \overline{J}^rY \subset \widetilde{J}^rY$.

An *rth order nonholonomic connection* on a fibered manifold $Y \to M$ is a section $\Theta : Y \to \widetilde{J}^rY$. Such a connection is called *semiholonomic* or *holonomic* if it has values in $\overline{J}^rY$ or J^rY, respectively. We recall that higher order connections were introduced in the groupoid form by Ehresmann [7], and Kolář [12] extended the Ehresmann theory to the case of any fibered manifold. In particular, for $r = 1$ we obtain the concept of a *general connection*, which can be equivalently interpreted as the lifting map

$$Y \times_M TM \to TY. \tag{1}$$

If $Y = TM$ is the tangent bundle, then the linear morphism $TM \to J^1TM$ is exactly the classical linear connection on M, which can also be defined as a covariant derivative $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$. Finally, the right-invariant connection on a principal bundle $P \to M$ is called *principal*.

By *prolongation* of connections we understand geometric constructions (more precisely natural operators $\mathcal{D}$) transforming a connection Θ on $Y \to M$ into a connection $\mathcal{D}(\Theta)$ of the same type on $GY \to M$, where G is some bundle functor. Such geometric constructions have motivation in

---

2010 *Mathematics Subject Classification*: 58A05, 58A20.
Key words and phrases: jet prolongation, holonomic connection, natural operator.

DOI: 10.4064/ap102-3-7

mathematical physics, mainly in quantum mechanics and higher order dynamics [11], [27]. In [3] we classified all natural operators transforming first order connections on $Y \to M$ into first order connections on $GY \to M$ (see Proposition 1 below). This paper will be devoted to the similar problem for higher order connections. In Example 2 we generalize the vertical prolongation $\mathcal{V}_1^F$ of first order connections to the vertical prolongation $\mathcal{V}_r^F$ of rth order holonomic connections for any r. The first main result is Theorem 1, where we classify all bundle functors G that admit prolongation of rth order connections. The rest of the paper will be devoted to second order holonomic connections, which are most important in applications. In contrast to part (ii) of Proposition 1, in Theorem 2 we show that prolongation of second order connections depends on linear liftings of vector fields.

By [12], higher order connections are useful in the theory of higher order absolute differentiation. In [5] we showed applications of such connections in the geometric description of higher order geometric object fields. Using an rth order holonomic connection, the second author [23] recently generalized (1) to the lifting map $Y \times_M FM \to FY$ for any bundle functor F of order r on the category of all smooth manifolds and all smooth maps. This allows us to define the lifting of geometric objects on M (i.e. sections of $FM \to M$) to geometric objects on Y. For other applications of higher order connections see e.g. [1], [17], [26]. We also recall that linear holonomic connections $TM \to J^rTM$ (i.e. principal connections on the rth order frame bundle $P^rM = \operatorname{inv} J_0^r(\mathbb{R}^m, M)$) were also studied in [9] and [10].

Let $G : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor, where $\mathcal{FM}$ is the category of fibered manifolds and their fibered maps and $\mathcal{FM}_{m,n}$ is the subcategory of fibered manifolds with n-dimensional fibers, m-dimensional bases and their fibered embeddings. Prolongation of holonomic connections can be expressed as a natural operator in the sense of [14] in the following way. An *$\mathcal{FM}_{m,n}$-natural operator* $\mathcal{D} : J^r \rightsquigarrow J^rG$ transforming rth order holonomic connections Θ on $\mathcal{FM}_{m,n}$-objects $Y \to M$ to rth order holonomic connections $\mathcal{D}(\Theta)$ on $GY \to M$ is a family of $\mathcal{FM}_{m,n}$-invariant regular operators (functions) $\mathcal{D} : \operatorname{Con}^r(Y \to M) \to \operatorname{Con}^r(GY \to M)$ for all $\mathcal{FM}_{m,n}$-objects $Y \to M$ from the space $\operatorname{Con}^r(Y \to M)$ of all rth order holonomic connections on $Y \to M$ into the space $\operatorname{Con}^r(GY \to M)$ of all rth order holonomic connections on $GY \to M$. *Invariance* means that if $\Theta_1 \in \operatorname{Con}^r(Y_1 \to M_1)$ and $\Theta_2 \in \operatorname{Con}^r(Y_2 \to M_2)$ are f-related for an $\mathcal{FM}_{m,n}$-map $f : Y_1 \to Y_2$, then $\mathcal{D}(\Theta_1)$ and $\mathcal{D}(\Theta_2)$ are Gf-related. *Regularity* means that $\mathcal{D}$ transforms smoothly parametrized families of connections into smoothly parametrized ones.

In what follows, we denote by $\mathcal{M}f$ the category of smooth manifolds and all smooth maps, by $\mathcal{M}f_m$ the subcategory of m-dimensional manifolds and local diffeomorphisms and by $\mathcal{FM}_m$ the category of fibered manifolds with m-dimensional bases and fiber respecting mappings over local diffeomorphisms. We recall that the rth order jet functor $J^r : \mathcal{FM}_m \to \mathcal{FM}_m \subset \mathcal{FM}$ sends any $\mathcal{FM}_m$-object $Y \to M$ into the fibered manifold $J^rY \to M$ and any $\mathcal{FM}_m$-map $f : Y \to Y_1$ covering a local diffeomorphism $\underline{f} : M \to M_1$ into the map $J^rf : J^rY \to J^rY_1$ given by $J^rf(j^r_x\sigma) = j^r_{\underline{f}(x)}(f \circ \sigma \circ \underline{f}^{-1})$, $j^r_x\sigma \in J^rY$. All manifolds and maps are assumed to be infinitely differentiable.

2. Prolongation of higher order connections. Let $F : \mathcal{M}f_n \to \mathcal{FM}$ be a natural bundle. Then the *F-vertical functor* $V^F : \mathcal{FM}_{m,n} \to \mathcal{FM}$ is defined fiberwise by $V^FY = \bigcup_{x\in M} F(Y_x)$ and analogously for morphisms [15]. If F is defined on $\mathcal{M}f$, then V^F can be defined on $\mathcal{FM}_m$ (even on the whole category $\mathcal{FM}$). Next, if $F = T^A : \mathcal{M}f \to \mathcal{FM}$ is a Weil functor determined by a Weil algebra A, then $V^{T^A} : \mathcal{FM}_m \to \mathcal{FM}_m \subset \mathcal{FM}$ is called the *vertical Weil functor*, which will be briefly denoted by V^A. In particular, for the tangent functor $F = T$ we obtain the classical vertical bundle VY.

Given a connection $\Gamma : Y \to J^1Y$ on $Y \to M$, we can define the *first order connection* $\mathcal{V}^F_1\Gamma$ on $V^FY \to M$ by the lifting map

$$\mathcal{V}^F_1\Gamma : V^FY \times_M TM \to TV^FY, \quad \mathcal{V}^F_1\Gamma(v, X_x) = \mathcal{V}^F(\Gamma X)_v, \quad v \in (V^FY)_x,$$

$x \in M$, where $X \in \mathcal{X}(M)$ is a vector field on M, $\Gamma X \in \mathcal{X}_{\mathrm{proj}}(Y)$ is the Γ-horizontal lift of X and $\mathcal{V}^F(\Gamma X)$ denotes its flow prolongation to V^FY [15].

For prolongation of first order connections we have proved in [3]:

Proposition 1.

(i) *A bundle functor $G : \mathcal{FM}_{m,n} \to \mathcal{FM}$ admits an $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D}$ transforming connections Γ on $Y \to M$ into connections $\mathcal{D}(\Gamma)$ on $GY \to M$ if and only if G is isomorphic to the vertical functor V^F for some natural bundle $F : \mathcal{M}f_n \to \mathcal{FM}$.*

(ii) *For $G = V^F$ such an operator $\mathcal{D}$ is unique and $\mathcal{D}(\Gamma) = \mathcal{V}^F_1\Gamma$.*

In this section, part (i) of Proposition 1 will be extended to rth order holonomic connections for any r. We start with the following example.

Example 1. Let m be a natural number. By [2], for every rth order jet functor $J : \mathcal{FM}_m \to \mathcal{FM}$ we have an exchange $\mathcal{FM}_m$-natural equivalence $\kappa^{A,J} : V^AJ \to JV^A$. So for any rth order holonomic connection $\Theta : Y \to J^rY$ on an $\mathcal{FM}_m$-object $Y \to M$ we have $V^A\Theta : V^AY \to V^AJ^rY$, and the composition

$$\mathcal{V}^A_r\Theta := \kappa^{A,J^r} \circ V^A\Theta : V^AY \to J^rV^AY \tag{2}$$

is an rth order holonomic connection on $V^AY \to M$.

Proposition 2. *Let $\mu : A_1 \to A_2$ be a homomorphism of Weil algebras, $\mu : T^{A_1} \to T^{A_2}$ (denoted by the same symbol) be the corresponding $\mathcal{M}f$-natural transformation between Weil functors $T^{A_1}, T^{A_2} : \mathcal{M}f \to \mathcal{FM}$, and $\tilde{\mu} : V^{A_1} \to V^{A_2}$ be the corresponding $\mathcal{FM}_m$-natural transformation (fiber-wise extension of μ) between the vertical Weil functors $V^{A_1}, V^{A_2} : \mathcal{FM}_m \to \mathcal{FM}$. Let $\Theta : Y \to J^rY$ be an rth order holonomic connection on an $\mathcal{FM}_m$-object $Y \to M$. Then $\mathcal{V}_r^{A_1}\Theta$ is $\tilde{\mu}$-related to $\mathcal{V}_r^{A_2}\Theta$. More precisely, $J^r\tilde{\mu} \circ \mathcal{V}_r^{A_1}\Theta = \mathcal{V}_r^{A_2}\Theta \circ \tilde{\mu}$.*

Proof. By [2], the $\mathcal{FM}_m$-natural transformations κ^{A,J^r} and $\tilde{\mu}$ commute for any natural transformation $\mu : T^{A_1} \to T^{A_2}$. Then we have $\mathcal{V}_r^{A_2}\Theta \circ \tilde{\mu} = \kappa^{A_2,J^r} \circ V^{A_2}\Theta \circ \tilde{\mu} = \kappa^{A_2,J^r} \circ \tilde{\mu} \circ V^{A_1}\Theta = J^r\tilde{\mu} \circ \kappa^{A_1,J^r} \circ V^{A_1}\Theta = J^r\tilde{\mu} \circ \mathcal{V}_r^{A_1}\Theta$. ∎

As a direct consequence we immediately obtain

Lemma 1. *Let $\Theta : Y \to J^rY$ be an rth order holonomic connection on an $\mathcal{FM}_m$-object $Y \to M$. Then $\mathcal{V}_r^{\mathbb{D}_n^k}\Theta : V^{\mathbb{D}_n^k}Y \to J^rV^{\mathbb{D}_n^k}Y$ is a G_n^k-invariant rth order holonomic connection on $V^{\mathbb{D}_n^k}Y \to M$, where $\mathbb{D}_n^k = J_0^k(\mathbb{R}^n, \mathbb{R})$ is the Weil algebra of k-jets at $0 \in \mathbb{R}^n$ of maps $\mathbb{R}^n \to \mathbb{R}$, $G_n^k = \mathrm{inv}\, J_0^k(\mathbb{R}^n, \mathbb{R}^n)_0$ is the kth order differential group in dimension n, and the action of G_n^k on $\mathbb{D}_n^k$ is given by $j_0^k\psi \cdot j_0^kf = j_0^k(f \circ \psi^{-1})$.*

Proof. Indeed, G_n^k-invariance means exactly that $\mathcal{V}_r^{\mathbb{D}_n^k}\Theta$ is related to itself by the extension $\tilde{g} : V^{\mathbb{D}_n^k}Y \to V^{\mathbb{D}_n^k}Y$ of the automorphism $g : \mathbb{D}_n^k \to \mathbb{D}_n^k$ of Weil algebras (left translation by $g \in G_n^k$ of the action of G_n^k on $\mathbb{D}_n^k$) for any $g \in G_n^k$. ∎

We recall that two $\mathcal{FM}$-morphisms $f, g : Y \to \overline{Y}$ determine the same $(0,k,0)$-jet at $y \in Y$ if $j_y^k(f|Y_x) = j_y^k(g|Y_x)$ (see [14]). Using Lemma 1 we can generalize Example 1 as follows.

Example 2. Let $F : \mathcal{M}f_n \to \mathcal{FM}$ be a natural bundle of order k. Let $\Theta : Y \to J^rY$ be an rth order holonomic connection on an $\mathcal{FM}_{m,n}$-object $Y \to M$. We define an *rth order holonomic connection* $\mathcal{V}_r^F\Theta : V^FY \to J^rV^FY$ on $V^FY \to M$ as follows. Let $P_{m,n}^{0,k,0}Y = \mathrm{inv}\, J_{(0,0)}^{0,k,0}(\mathbb{R}^{m,n}, Y)$ be the space of $(0,k,0)$-jets at $(0,0) \in \mathbb{R}^{m,n}$ of $\mathcal{FM}_{m,n}$-maps $\mathbb{R}^{m,n} \to Y$. It is a principal G_n^k-bundle over Y (the projection $P_{m,n}^{0,k,0}Y \to Y$ is the target one) with the right action of G_n^k given by $j_{(0,0)}^{0,k,0}\Phi \cdot j_0^k\psi = j_{(0,0)}^{0,k,0}(\Phi \circ (\mathrm{id}_{\mathbb{R}^m} \times \psi))$. By an order argument, $V^FY = P_{m,n}^{0,k,0}Y[F_0\mathbb{R}^n]$ is the associated bundle with G_n^k-fiber $F_0\mathbb{R}^n$ (we have the well-defined identification $[j_{(0,0)}^{0,k,0}\Psi, v] = V^F\Psi(v)$), where the left action of G_n^k on $F_0\mathbb{R}^n$ is the standard one. By Lemma 1, the restriction $\mathcal{P}_{m,n;r}^{0,k,0}\Theta : P_{m,n}^{0,k,0}Y \to J^rP_{m,n}^{0,k,0}Y$ of $\mathcal{V}_r^{\mathbb{D}_n^k}\Theta : V^{\mathbb{D}_n^k}Y \to J^rV^{\mathbb{D}_n^k}Y$ to the open subbundle $P_{m,n}^{0,k,0}Y \subset V^{\mathbb{D}_n^k}Y$ (the inclusion is given by the identification

$j^{0,k,0}_{(0,0)}\Psi = j^k_0(\Psi_{|\{0\}\times\mathbb{R}^n}))$ is a G^k_n-invariant rth order holonomic connection on $P^{0,k,0}_{m,n}Y \to M$. We define

$$\mathcal{V}^F_r\Theta(w) := j^r_x([\sigma, v]), \qquad w = [p, v],\, p \in (P^{0,k,0}_{m,n}Y)_y,\, v \in F_0\mathbb{R}^n,\, y \in Y_x,$$

$x \in M$, where $\sigma : M \to P^{0,k,0}_{m,n}Y$ is a local section with $j^r_x\sigma = \mathcal{P}^{0,k,0}_{m,n;r}\Theta(p)$ and $[\sigma, v] : M \to V^F Y$ is a section of $V^F Y \to M$ given by $x \mapsto [\sigma(x), v]$. Using G^k_n-invariance of $\mathcal{P}^{0,k,0}_{m,n;r}\Theta$ and a local coordinate argument, one can easily show that the definition of $\mathcal{V}^F_r\Theta(w)$ is correct (i.e. independent of the choice of p and v with $w = [p, v]$ and of the choice of σ with $\mathcal{P}^{0,k,0}_{m,n;r}\Theta(p) = j^r_x\sigma$). By the canonical character of the construction of $\mathcal{V}^F_r\Theta$ we have the corresponding $\mathcal{FM}_{m,n}$-natural operator $\mathcal{V}^F_r$ transforming rth order holonomic connections Θ on $Y \to M$ into rth order holonomic connections $\mathcal{V}^F_r\Theta$ on $V^F Y \to M$.

Quite analogously to Proposition 2 we have

Proposition 3. *Let $\mu : F_1 \to F_2$ be an $\mathcal{M}f$-natural transformation between bundle functors $F_1, F_2 : \mathcal{M}f \to \mathcal{FM}$ of order k, and $\tilde\mu : V^{F_1} \to V^{F_2}$ be the corresponding $\mathcal{FM}_m$-natural transformation (fiberwise extension of μ) between the vertical bundle functors $V^{F_1}, V^{F_2} : \mathcal{FM}_m \to \mathcal{FM}$. Let $\Theta : Y \to J^rY$ be an rth order holonomic connection on an $\mathcal{FM}_{m,n}$-object $Y \to M$. Then $\mathcal{V}^{F_1}_r\Theta$ is $\tilde\mu$-related to $\mathcal{V}^{F_2}_r\Theta$. More precisely, $J^r\tilde\mu \circ \mathcal{V}^{F_1}_r\Theta = \mathcal{V}^{F_2}_r\Theta \circ \tilde\mu$.*

Proof. Let $w = [p, v] \in V^{F_1}Y$, $p \in P^{0,k,0}_{m,n}Y$, $v \in F_{10}\mathbb{R}^n$. Then $\tilde\mu(w) = [p, \mu(v)]$. Denoting $\mathcal{P}^{0,k,0}_{m,n;r}\Theta(p) = j^r_x\sigma$, we have $\mathcal{V}^{F_2}_r\Theta(\tilde\mu(w)) = j^r_x([\sigma, \mu(v)])$ $= j^r_x(\tilde\mu \circ [\sigma, v]) = J^r\tilde\mu(j^r_x([\sigma, v])) = J^r\tilde\mu(\mathcal{V}^{F_1}_r\Theta(w))$. ∎

Now we show that for $F = T^A$, the connection $\mathcal{V}^F_r\Theta$ from Example 2 is exactly $\mathcal{V}^A_r\Theta$ from Example 1.

Proposition 4. *For any Weil algebra A and any rth order holonomic connection $\Theta : Y \to J^rY$ on an $\mathcal{FM}_{m,n}$-object $Y \to M$ we have $\mathcal{V}^{T^A}_r\Theta = \mathcal{V}^A_r\Theta$, where $\mathcal{V}^{T^A}_r\Theta$ is as in Example 2 for $F = T^A$ and $\mathcal{V}^A_r\Theta$ is as in Example 1.*

Proof. Let $v \in (V^AY)_y = (V^{T^A}Y)_y$, $y \in Y$. We prove $(\mathcal{V}^{T^A}_r\Theta)_v = (\mathcal{V}^A_r\Theta)_v$. By $\mathcal{FM}_{m,n}$-invariance we can assume that $Y = Y^{\mathbb{R}^n}$ is the trivial bundle over $\mathbb{R}^m$ with the fiber $\mathbb{R}^n$, $y = (0, 0)$ and $v \in (V^AY^{\mathbb{R}^n})_{(0,0)} = (V^{T^A}Y^{\mathbb{R}^n})_{(0,0)} = T^A_0\mathbb{R}^n$. Let k be the order of A (i.e. of T^A). It is a simple observation that $(\mathcal{V}^{T^k_n}_r\Theta)_{j^k_0 \mathrm{id}_{\mathbb{R}^n}} = (\mathcal{P}^{0,k,0}_{m,n;r}\Theta)_{j^k_0 \mathrm{id}_{\mathbb{R}^n}} = (\mathcal{V}^{\mathbb{D}^k_n}_r\Theta)_{j^k_0 \mathrm{id}_{\mathbb{R}^n}}$, where $j^k_0 \mathrm{id}_{\mathbb{R}^n} = j^{0,k,0}_{(0,0)}(\mathrm{id}_{\mathbb{R}^m} \times \mathrm{id}_{\mathbb{R}^n}) \in (V^{\mathbb{D}^k_n}Y^{\mathbb{R}^n})_{(0,0)} = (T^{\mathbb{D}^k_n}\mathbb{R}^n)_0 = (T^k_n\mathbb{R}^n)_0$. For, write $(\mathcal{P}^{0,k,0}_{m,n;r}\Theta)_{j^k_0 \mathrm{id}_{\mathbb{R}^n}} = (\mathcal{V}^{\mathbb{D}^k_n}_r\Theta)_{j^k_0 \mathrm{id}_{\mathbb{R}^n}} = j^r_0\sigma$, where $\sigma : \mathbb{R}^m \to P^{0,k,0}_{m,n}Y^{\mathbb{R}^n}$

is a section. Then we have $(\mathcal{V}_r^{T^{\mathbb{D}_n^k}}\Theta)_{j_0^k\,\mathrm{id}_{\mathbb{R}^n}} = j_0^r([\sigma, j_0^k\,\mathrm{id}_{\mathbb{R}^n}]) = j_0^r\sigma = (\mathcal{P}^{0,k,0}_{m,n;r}\Theta)_{j_0^k\,\mathrm{id}_{\mathbb{R}^n}} = (\mathcal{V}_r^{\mathbb{D}_n^k}\Theta)_{j_0^k\,\mathrm{id}_{\mathbb{R}^n}}$. Moreover, there is an $\mathcal{M}f$-natural transformation $\mu : T_n^k = T^{\mathbb{D}_n^k} \to T^A$ defined by $\mu(j_0^k\gamma) = T^A\gamma(v)$ such that $\mu(j_0^k\,\mathrm{id}_{\mathbb{R}^n}) = v$. By Propositions 2 and 3 we have $(\mathcal{V}_r^{T^A}\Theta)_v = (\mathcal{V}_r^{T^A}\Theta)_{\tilde{\mu}(j_0^k\,\mathrm{id}_{\mathbb{R}^n})}$ $= J^r\tilde{\mu}((\mathcal{V}_r^{T_n^k}\Theta)_{j_0^k\,\mathrm{id}_{\mathbb{R}^n}}) = J^r\tilde{\mu}((\mathcal{V}_r^{\mathbb{D}_n^k}\Theta)_{j_0^k\,\mathrm{id}_{\mathbb{R}^n}}) = (\mathcal{V}_r^A\Theta)_{\tilde{\mu}(j_0^k\,\mathrm{id}_{\mathbb{R}^n})} = (\mathcal{V}_r^A\Theta)_v$. ∎

Now we generalize part (i) of Proposition 1 to higher order connections.

THEOREM 1. *A bundle functor $G : \mathcal{FM}_{m,n} \to \mathcal{FM}$ admits an $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D}$ transforming rth order holonomic connections Θ on $Y \to M$ into rth order holonomic connections $\mathcal{D}(\Theta)$ on $GY \to M$ if and only if $G \cong V^F$ for some bundle functor $F : \mathcal{M}f_n \to \mathcal{FM}$.*

Proof. The "if" implication is an immediate consequence of Example 2. The converse implication can be proved by using almost the same arguments as for Theorem 3 in [4]. More precisely, the converse implication follows immediately from Lemmas 2 and 3 below. ∎

LEMMA 2. *Let $G : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor. Suppose G admits an $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D}$ transforming rth order holonomic connections Θ on $Y \to M$ into rth order holonomic connections $\mathcal{D}(\Theta)$ on $GY \to M$. Then the natural bundle $G^1 : \mathcal{M}f_m \to \mathcal{FM}$ corresponding to G defined by $G^1M = G(M \times \mathbb{R}^n)$ and $G^1f = G(f \times \mathrm{id}_{\mathbb{R}^n})$ is of order 0.*

Proof. Suppose G^1 is of minimal order $s \geq 1$. Let M be an m-manifold. Let Θ_M be the trivial rth order holonomic connection on the trivial bundle $M \times \mathbb{R}^n \to M$ (i.e. $\Theta_M(x,y) = j_x^r(\tilde{y})$, where $\tilde{y} : M \to M \times \mathbb{R}^n$ is the constant section $x \mapsto (x,y)$). Applying $\mathcal{D}$, we have the rth order holonomic connection $\mathcal{D}(\Theta_M)$ on $G^1M \to M$. Denote by $\mathcal{D}^o(\Theta_M)$ the underlying first order connection of $\mathcal{D}(\Theta_M)$ on $G^1M \to M$. Since Θ_M is invariant with respect to $\mathcal{FM}_{m,n}$-maps of the form $f \times \mathrm{id}_{\mathbb{R}^n}$, the first order connection $\mathcal{D}^o(\Theta_M)$ on $G^1M \to M$ is $\mathcal{M}f_m$-invariant. Denoting by $X^{\mathcal{D}^o(\Theta_M)} \in \mathcal{X}(G^1M)$ the horizontal lift of a vector field X on M with respect to $\mathcal{D}^o(\Theta_M)$, we have the $\mathcal{M}f_m$-natural operator $B : T \rightsquigarrow TG^1$ transforming vector fields X on m-manifolds M into vector fields $B(X) := X^{\mathcal{D}^o(\Theta_M)}$ on G^1M (the notion of such operators will be recalled in the next section). Clearly, the order of B is zero. Further, as $B(X)$ is projectable over X, we have $B(X) = \mathcal{G}^1X + \mathcal{V}(X)$ for some vertical type $\mathcal{M}f_m$-natural operator $\mathcal{V} : T \rightsquigarrow TG^1$, where $\mathcal{G}^1 : T \rightsquigarrow TG^1$ is the well-known flow operator corresponding to G^1. As G^1 is of minimal order s, so is the operator $\mathcal{G}^1$. By Lemma 1 from [22], $\mathcal{V}$ is of order $\leq s-1$. Then B is of minimal order $s \geq 1$, which is a contradiction. ∎

LEMMA 3. *Let $G : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be a bundle functor of order s. Suppose that the natural bundle $G^1 : \mathcal{M}f_m \to \mathcal{FM}$ (as in Lemma 2) corresponding to G is of order* 0. *Then $G \cong V^F$ for some natural bundle $F : \mathcal{M}f_n \to \mathcal{FM}$.*

Proof. This follows directly from Proposition 9 and Corollary 3 in [3]. ∎

REMARK 1. By Theorem 1, if G is not isomorphic to V^F (e.g. $G = J^k$), then prolongation of connections from $Y \to M$ to $GY \to M$ requires the use of some additional geometric object. Given an arbitrary bundle functor G on $\mathcal{FM}_{m,n}$, the second author [23] recently constructed an rth order holonomic connection $\mathcal{G}(\Theta, \nabla)$ on $GY \to M$ from an rth order holonomic connection Θ on $Y \to M$ by means of a torsion free classical linear connection ∇ on M. This generalizes the classical geometric construction from [14] to first order connections.

3. Prolongation of second order holonomic connections. We recall that the product of two connections $\Theta_1 : Y \to \widetilde{J}^r Y$ and $\Theta_2 : Y \to \widetilde{J}^s Y$ is a connection $\Theta_1 * \Theta_2 : Y \to \widetilde{J}^{r+s} Y$ defined by $\Theta_1 * \Theta_2 := \widetilde{J}^s \Theta_1 \circ \Theta_2$. Then the *rth order Ehresmann prolongation* of $\Gamma : Y \to J^1 Y$ is a semiholonomic connection $\Gamma^{(r-1)} : Y \to \overline{J}^r Y$ defined by induction:

$$\Gamma^{(1)} := \Gamma * \Gamma : Y \to \overline{J}^2 Y, \quad \Gamma^{(r-1)} := \Gamma^{(r-2)} * \Gamma : Y \to \overline{J}^r Y.$$

In what follows, if $\Theta : Y \to J^2 Y$ is a second order connection on $Y \to M$, then $\Theta^o : Y \to J^1 Y$ will denote its underlying connection.

EXAMPLE 3. Let $\mathcal{V}_1^F$ be the operator from Proposition 1. We have an $\mathcal{FM}_{m,n}$-natural operator $\mathcal{V}^{F,2} : J^2 \rightsquigarrow J^2 V^F$ transforming second order holonomic connections Θ on $Y \to M$ into second order holonomic connections,

$$\mathcal{V}^{F,2}\Theta := C^{(2)}(\mathcal{V}_1^F \Theta^o * \mathcal{V}_1^F \Theta^o) : V^F Y \to J^2 V^F Y$$

on $V^F Y \to M$, where $\mathcal{V}_1^F \Theta^o * \mathcal{V}_1^F \Theta^o : V^F Y \to \overline{J}^2 V^F Y$ is the second order semiholonomic Ehresmann prolongation of $\mathcal{V}_1^F \Theta^o : V^F Y \to J^1 V^F Y$ and $C^{(2)} : \overline{J}^2 Y \to J^2 Y$ is the well-known symmetrization of second order semiholonomic jets (see e.g. [5]).

In [5] we have proved that the symmetrization $\overline{J}^r Y \to J^r Y$ exists only for $r \le 2$. This means that the operator $\mathcal{V}^{F,2}$ cannot be generalized to $r > 2$.

By [6], second order holonomic connections $\Theta : Y \to J^2 Y$ on $Y \to M$ are in bijection with couples (Γ, Δ) of first order connections Γ on $Y \to M$ and tensor fields $\Delta : Y \to S^2 T^* M \otimes VY$. This bijection is given by

$$\Gamma = \Theta^o \quad \text{and} \quad \Delta = \Theta - C^{(2)}(\Gamma * \Gamma),$$

where the difference is on the affine bundle $J^2 Y \to J^1 Y$ with the vector bundle $S^2 T^* M \otimes VY$. Indeed, Θ and $C^{(2)}(\Gamma * \Gamma)$ have the same underlying first order connection, so that we can take the difference. The inverse bijection is given by $(\Gamma, \Delta) \mapsto C^{(2)}(\Gamma * \Gamma) + \Delta$. In what follows, if Θ is a second order holonomic connection on $Y \to M$, then the equality $\Theta = (\Gamma, \Delta)$ means that (Γ, Δ) corresponds to Θ in the above bijection.

Clearly, if $\Theta = (\Gamma, \Delta)$, then $\mathcal{V}^{F,2}\Theta = (\mathcal{V}_1^F \Gamma, 0)$, where $\mathcal{V}_1^F \Gamma$ is from Proposition 1 and $0 : V^F Y \to S^2 T^* M \otimes V V^F Y$ is the zero tensor field.

Let $F : \mathcal{M}f_n \to \mathcal{FM}$ be a natural bundle. We recall that a *natural operator* $A : T \rightsquigarrow TF$ is an $\mathcal{M}f_n$-invariant family of regular operators $A : \mathcal{X}(N) \to \mathcal{X}(FN)$ for any n-manifold N, where $\mathcal{X}(N)$ is the set of vector fields on N. An operator A is called *linear* if $A : \mathcal{X}(N) \to \mathcal{X}(FN)$ is $\mathbb{R}$-linear for any n-manifold N. A simple example of such an $A : T \rightsquigarrow TF$ is the flow operator $\mathcal{F}$ transforming vector fields $X \in \mathcal{X}(N)$ with the flow φ_t into vector fields $\mathcal{F}X \in \mathcal{X}(FN)$ defined by the flow $F\varphi_t$. Clearly, the flow operator $\mathcal{F}$ is linear. There are many papers where classifications of all natural (linear) operators $A : T \rightsquigarrow TF$ for some F are presented (see e.g. [8], [13], [14], [18], [20], [21], [25]).

Example 4. Given an $\mathcal{M}f_n$-natural linear operator $A : T \rightsquigarrow TF$, we construct an $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D}^A : J^2 \rightsquigarrow J^2 V^F$ transforming second order holonomic connections Θ on $Y \to M$ into second order holonomic connections $\mathcal{D}^A(\Theta)$ on $V^F Y \to M$. Let $\Theta = (\Gamma, \Delta)$ be a second order holonomic connection on $Y \to M$, where Γ is a first order connection on $Y \to M$ and $\Delta : Y \to S^2 T^* M \otimes VY$ is a tensor field. We define

$$\mathcal{D}^A(\Theta) := (\mathcal{V}_1^F \Gamma, \Delta^A),$$

where $\mathcal{V}_1^F \Gamma : V^F Y \to J^1 V^F Y$ is from Proposition 1 and

$$\Delta^A : V^F Y \to S^2 T^* M \otimes V V^F Y$$

is defined as follows. Given $u, v \in T_x M$, $x \in M$, we have the vector field $\Delta(u, v) \in \mathcal{X}(Y_x)$, where Y_x is the fiber of Y over x. Applying A we obtain the vector field $A(\Delta(u, v))$ on $F(Y_x) = (V^F Y)_x$. Write $\Delta^A(u, v) := A(\Delta(u, v))$. By linearity of A, $\Delta^A(u, v)$ depends linearly on u and v. Since $\Delta(u, v)$ is symmetric in u, v, so is $\Delta^A(u, v)$.

Clearly, if $\Theta = (\Gamma, \Delta)$, then $\mathcal{D}^A(\Theta) = \mathcal{V}^{F,2}\Theta + \Delta^A$, where $\mathcal{V}^{F,2}\Theta$ is from Example 3, Δ^A is from Example 4 and addition is on the affine bundle $J^2 V^F Y \to J^1 V^F Y$.

In the next section we prove the following second main result of this paper.

Theorem 2.

(i) *A bundle functor $G : \mathcal{FM}_{m,n} \to \mathcal{FM}$ admits an $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D} : J^2 \rightsquigarrow J^2 G$ transforming second order connections Θ on $Y \to M$ into second order connections $\mathcal{D}(\Theta)$ on $GY \to M$ if and* *only if G is isomorphic to the vertical bundle functor V^F for some natural bundle $F : \mathcal{M}f_n \to \mathcal{FM}$.*

(ii) *For $G = V^F$ the space of all such natural operators $\mathcal{D}$ is an affine space with the corresponding vector space of all $\mathcal{M}f_n$-natural linear operators $A : T \rightsquigarrow TF$ transforming vector fields X on n-manifolds N to vector fields $A(X)$ on FN.*

Part (ii) of Theorem 2 follows directly from

Theorem $2'$. *Let $F : \mathcal{M}f_n \to \mathcal{FM}$ be a natural bundle. Let $\mathcal{D} : J^2 \rightsquigarrow J^2V^F$ be an $\mathcal{FM}_{m,n}$-natural operator. Then there exists a unique $\mathcal{M}f_n$-natural linear operator $A : T \rightsquigarrow TF$ such that $\mathcal{D} = \mathcal{D}^A$.*

Remark 2. By Theorem 1, part (i) of Theorem 2 is true also for rth order holonomic connections for any r. On the other hand, the problem is open whether one can generalize Theorem $2'$ to rth order holonomic connections, $r > 2$. It is a simple observation that for $r > 2$, holonomic connections $\Theta : Y \to J^rY$ on $\mathcal{FM}_{m,n}$-objects $Y \to M$ cannot be in canonical bijection with systems $(\Gamma, \Delta_1, \ldots, \Delta_{r-1})$ of first order connections $\Gamma : Y \to J^1Y$ and tensor fields $\Delta_i : Y \to S^{i+1}T^*M \otimes VY$ for $i = 1, \ldots, r-1$. Indeed, if we had such a bijection, then we would have an $\mathcal{FM}_{m,n}$-natural operator A given by $A(\Gamma) = (\Gamma, 0, \ldots, 0)$ transforming general connections Γ on $Y \to M$ into rth order holonomic connections $A(\Gamma) : Y \to J^rY$ on $Y \to M$. But this is impossible for $r > 2$ (see [5]).

4. Proof of Theorem 2. Part (i) of Theorem 2 is exactly Theorem 1 for $r = 2$. We also present the following alternative (independent of Theorem 1) proof of part (i). The "only if" implication can be deduced as follows. Suppose we have an $\mathcal{FM}_{m,n}$-natural operator $\mathcal{D} : J^2 \rightsquigarrow J^2G$. Let $\Gamma : Y \to J^1Y$ be a connection on $Y \to M$. Applying symmetrization $C^{(2)} : \overline{J}^2Y \to J^2Y$ to the Ehresmann prolongation $\Gamma * \Gamma : Y \to \overline{J}^2Y$ we obtain a second order holonomic connection $C^{(2)}(\Gamma * \Gamma) : Y \to J^2Y$. Using $\mathcal{D}$ we get a second order holonomic connection $\mathcal{D}(C^{(2)}(\Gamma * \Gamma))$ on $GY \to M$. Write $D(\Gamma) = (\mathcal{D}(C^{(2)})(\Gamma * \Gamma))^o : GY \to J^1GY$ for the underlying connection of $\mathcal{D}(C^{(2)}(\Gamma * \Gamma))$. So we have an $\mathcal{FM}_{m,n}$-natural operator $D : J^1 \rightsquigarrow J^1G$. By Proposition 1, $G \cong V^F$ for some natural bundle $F : \mathcal{M}f_n \to \mathcal{FM}$. The "if" implication is an immediate consequence of Example 3. Of course, the above alternative proof of Theorem 1 for $r = 2$ cannot be extended to other r, because there is no symmetrization $\overline{J}^rY \to J^rY$ if $r > 2$.

The rest of this section will be devoted to the proof of Theorem $2'$, which also proves (ii) of Theorem 2. Let $x^1, \ldots, x^m, y^1, \ldots, y^n$ be the usual coordinates on $\mathbb{R}^m \times \mathbb{R}^n$.

Proof of Theorem 2′. Let $\Theta = (\Gamma, \Delta)$ be a second order holonomic connection on $Y \to M$ (the notation is taken from the previous section). We can write

$$\mathcal{D}(\Theta) = (\mathcal{V}_1^F \Gamma + \Delta_1(\Gamma, \Delta), \Delta_2(\Gamma, \Delta))$$

for unique $\mathcal{FM}_{m,n}$-natural operators Δ_1 and Δ_2 transforming pairs (Γ, Δ) as above into tensor fields $\Delta_1(\Gamma, \Delta) : V^F Y \to T^*M \otimes VV^F Y$ and $\Delta_2(\Gamma, \Delta) : V^F Y \to S^2 T^*M \otimes VV^F Y$.

Using Δ_2 we define an $\mathcal{M}f_n$-natural linear operator $A : T \rightsquigarrow TF$ as follows. Let X be a vector field on an n-manifold N. We have the trivial connection $\Gamma_N : Y^N \to J^1 Y^N$ on the trivial bundle $Y^N = \mathbb{R}^m \times N$ over $\mathbb{R}^m$. We also have a tensor field $\Delta^X := (dx^1 \odot dx^1) \otimes X : Y^N \to S^2 T^* \mathbb{R}^m \otimes VY^N$. Write

$$A(X) := \langle \Delta_2(\Gamma_N, \Delta^X), u^o \odot u^o \rangle : FN = (V^F Y^N)_0 \to (VV^F Y^N)_0 = TFN,$$

where $u^o = \frac{\partial}{\partial x^1}\big|_0 \in T_0 \mathbb{R}^m$. Because of the canonical character of the construction, the family $A : T \rightsquigarrow TF$ is an $\mathcal{M}f_n$-natural operator. By Proposition 42.5 in [14], A is of finite order. Using invariance of Δ_2 with respect to the base homotheties on $\mathbb{R}^m$ we get the homogeneity condition $A(t^2 X) = t^2 A(X)$ for $t > 0$. Then A is linear by the homogeneous function theorem (Theorem 24.1 in [14]).

Now we show that $\Delta_1(\Gamma, \Delta) = 0$ and $\Delta_2(\Gamma, \Delta) = \Delta^A$, where Δ^A is from Example 4. Let $y \in Y$ be a point. It is sufficient to show that $\Delta_1(\Gamma, \Delta) = 0$ and $\Delta_2(\Gamma, \Delta) = \Delta^A$ over y. Choose a sufficiently large natural number r and a torsion free classical linear connection on M. By Proposition 2.2 in [24] there exists a "special" fibered chart ψ on Y with $\psi(y) = (0, 0)$ such that $j_0^r(\psi_* \Gamma(0, -)) = j_0^r(\Gamma_{\mathbb{R}^n}(0, -))$ and

$$j^1_{(0,0)}(\psi_* \Gamma) = j^1_{(0,0)}\left(\Gamma_{\mathbb{R}^n} + \sum_{k=1}^{n} \sum_{i,j=1}^{m} a^k_{ij} x^i dx^j \otimes \frac{\partial}{\partial y^k}\right)$$

for some numbers $a^k_{ij} \in \mathbb{R}$ with $a^k_{ij} = -a^k_{ji}$. So, taking into account invariance of Δ_1 and Δ_2 with respect to the "special" fibered charts we may assume that $Y = Y^{\mathbb{R}^n}$, $y = (0, 0) \in \mathbb{R}^m \times \mathbb{R}^n$, $j_0^r(\Gamma(0, -) = j_0^r(\Gamma_{\mathbb{R}^n}(0, -))$ and

$$j^1_{(0,0)}(\Gamma) = j^1_{(0,0)}\left(\Gamma_{\mathbb{R}^n} + \sum_{k=1}^{n} \sum_{i,j=1}^{m} a^k_{ij} x^i dx^j \otimes \frac{\partial}{\partial y^k}\right)$$

for some numbers $a^k_{ij} \in \mathbb{R}$ with $a^k_{ij} = -a^k_{ji}$. Let $w \in (V^F Y^{\mathbb{R}^n})_{(0,0)}$. It remains to verify that $\Delta_1(\Gamma, \Delta)_w = 0$ and $\Delta_2(\Gamma, \Delta)_w = \Delta^A_w$.

Using invariance of Δ_1, Δ_2 and Δ^A with respect to fiber homotheties for $t > 0$ (they preserve w) we can additionally assume that (Γ, Δ) are sufficiently close (in the compact open C^∞-topology) to $(\Gamma_{\mathbb{R}^n}, 0)$. So, assuming that the above r is sufficiently large, by the nonlinear Peetre theorem (more precisely by Theorem 19.10 in [14] for $K = \{w\}$ and $f = (\Gamma_{\mathbb{R}^n}, 0)$), we can additionally assume that the coefficients of Γ and Δ (in the usual coordinates on $\mathbb{R}^m \times \mathbb{R}^n$) are polynomials of degree $\le r$. Now, applying invariance of Δ_1 with respect to the base homotheties $t\,\mathrm{id}_{\mathbb{R}^m} \times \mathrm{id}_{\mathbb{R}^n}$ for $t > 0$ (they preserve w) and the homogeneous function theorem we get $\Delta_1(\Gamma, \nabla)_w = 0$. Similarly, using invariance of Δ_2 with respect to the base homotheties and the homogeneous function theorem and next using invariance of Δ_2 with respect to maps from $\mathrm{GL}(m) \times \{\mathrm{id}_{\mathbb{R}^n}\}$ (they preserve w) and the invariant tensor theorem (Theorem 24.4 in [14]) we deduce that $\Delta_2(\Gamma, \Delta)_w$ depends linearly on the value $\Delta_{|(0,0)}$ only. By polarization, it suffices to verify $\langle \Delta_2(\Gamma, \Delta)_w, u \odot u\rangle = \langle \Delta^A_w, u \odot u\rangle$ for any $u \in T_0\mathbb{R}^m$ and any Γ, Δ, w as above. Consequently, using invariance of Δ_2 with respect to linear maps from $\mathrm{Gl}(m) \times \mathrm{Gl}(n)$ it suffices to show that

$$\langle \Delta_2(\Gamma, \Delta)_w, u^o \odot u^o\rangle = \langle \Delta^A_w, u^o \odot u^o\rangle$$

for $\Gamma = \Gamma_{\mathbb{R}^n}$ and $\Delta = (dx^1 \odot dx^1) \otimes \frac{\partial}{\partial y^1}$, where $u^o = \frac{\partial}{\partial x^1}\big|_0 \in T_0\mathbb{R}^m$. But this equality is an immediate consequence of the definitions of A and Δ^A.

If $\Delta_2(\Gamma, \Delta) = \Delta^B$ for another $\mathcal{M}f_n$-natural linear operator $B : T \rightsquigarrow TF$, then $A\big(\frac{\partial}{\partial x^1}\big)_w = \langle \Delta^A_w, u^o \odot u^o\rangle = \langle \Delta^B_w, u^o \odot u^o\rangle = B\big(\frac{\partial}{\partial x^1}\big)_w$, so that $A = B$. ∎

Open problem. By [6], second order nonholonomic connections Θ on $Y \to M$ are in bijection with triples $(\Gamma, \Gamma_1, \Delta)$ of first order connections Γ, Γ_1 on $Y \to M$ and tensor fields $\Delta : Y \to \otimes^2 T^*M \otimes VY$. It seems that using a similar (but more technically complicated) proof to the one of Theorem 2′ one can also completely describe all $\mathcal{FM}_{m,n}$-natural operators $\mathcal{D} : \tilde{J}^2 \to \tilde{J}^2 V^F$ transforming second order nonholonomic connections Θ on $Y \to M$ into second order nonholonomic connections $\mathcal{D}(\Theta)$ on $V^F Y \to M$.

5. Applications of Theorem 2. If F is the identity functor on $\mathcal{M}f_n$, then $V^F Y = Y$. By [14], the vector space of all $\mathcal{M}f_n$-natural linear operators $A : T \rightsquigarrow T$ is one-dimensional (generated by $A = \mathrm{id}$). Then we have the following corollary of Theorem 2′:

Corollary 1. *Let $\mathcal{D} : J^2 \rightsquigarrow J^2$ be an $\mathcal{FM}_{m,n}$-natural operator. Then there exists a unique number $\alpha \in \mathbb{R}$ such that $\mathcal{D}(\Theta) = \mathcal{V}^{F,2}\Theta + \alpha\Delta$ for all second order connections $\Theta = (\Gamma, \Delta)$ on $Y \to M$, where F is the identity functor on $\mathcal{M}f_n$.*

If $F = T^r_p$ is the Ehresmann functor of (p, r)-velocities, then $V^F Y = V^r_p Y$ is the vertical bundle of (p, r)-velocities. The basis of the vector space of all $\mathcal{M}f_n$-linear operators $A : T \rightsquigarrow TT^r_p$ is formed by Morimoto lifts $\mathcal{L}^{(\lambda)} : T \rightsquigarrow T^r_p$ for all p-tuples of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_p)$ with $0 \le |\lambda| \le r$ (see [8], [13]). So we have

COROLLARY 2. *Let $\mathcal{D} : J^2 \rightsquigarrow J^2 V_p^r$ be an $\mathcal{FM}_{m,n}$-natural operator. Then there exist unique real numbers a_λ such that $\mathcal{D}(\Theta) = \mathcal{V}^{T_p^r,2}\Theta + \sum_\lambda a_\lambda \Delta^{\mathcal{L}^{(\lambda)}}$ (where the sum is over all p-tuples λ of nonnegative integers with $0 \le |\lambda| \le r$) for all second order connections $\Theta = (\Gamma, \Delta)$ on $Y \to M$.*

If $F = T^A$ is the Weil functor corresponding to a Weil algebra A, then $V^F Y = V^A Y$. All $\mathcal{M}f_n$-linear natural operators $B : T \rightsquigarrow TT^A$ are $\mathrm{op}(a) \circ \mathcal{T}^A : T \rightsquigarrow TT^A$ for all $a \in A$, where $\mathcal{T}^A : T \rightsquigarrow TT^A$ is the flow operator and $\mathrm{op}(a) : TT^A N \to TT^A N$ is the natural affinor on $T^A N$ corresponding to $a \in A$ (see [13]). Thus the vector space of all $\mathcal{M}f_n$-natural linear operators $B : T \rightsquigarrow TT^A$ is $\dim_{\mathbb{R}}(A)$-dimensional. So we have the following corollary of Theorem 2′, which generalizes Corollaries 1, 2.

COROLLARY 3 ([16]). *Let A be a Weil algebra. Let $\mathcal{D} : J^2 \rightsquigarrow J^2 V^A$ be an $\mathcal{FM}_{m,n}$-natural operator. Then there is a unique $a \in A$ such that $\mathcal{D}(\Theta) = \mathcal{V}^{T^A,2}\Theta + \Delta^{\mathrm{op}(a)\circ\mathcal{T}^A}$ for all second order connections $\Theta = (\Gamma, \Delta)$ on $Y \to M$.*

Let $T^{r*}N = J^r(N, \mathbb{R})_0$ be the space of all r-jets from an n-manifold N into $\mathbb{R}$ with target 0. Since $\mathbb{R}$ is a vector space, $T^{r*}N$ has a canonical structure of a vector bundle over N, which is called the *rth order cotangent bundle*. The dual bundle $T^{(r)}N = (T^{r*}N)^*$ is called the *rth order tangent bundle*. For every map $f : N \to N_1$ the jet composition $A \mapsto A \circ (j^r_x f)$, $x \in N$, $A \in (T^{r*}N_1)_{f(x)}$, defines a linear map $(T^{r*}N_1)_{f(x)} \to (T^{r*}N)_x$. The dual map $T^{(r)}_x f : (T^{(r)}N)_x \to (T^{(r)}N_1)_{f(x)}$ is called the *rth tangent map* of f at x. This yields a vector bundle functor $T^{(r)}$, which is defined on the whole category $\mathcal{M}f$ of all manifolds and maps. Clearly, for $r = 1$ we obtain the classical tangent functor T and for $r > 1$ the functor $T^{(r)}$ does not preserve products. Obviously we have the canonical inclusion $TN \subset T^{(r)}N$. Using fiber translations on $T^{(r)}N$, we can extend every section $X : N \to TN$ to a vector field $V(X)$ on $T^{(r)}N$. This defines a linear $\mathcal{M}f_n$-natural operator $V : T \rightsquigarrow TT^{(r)}$. In [20] the second author classified all $\mathcal{M}f_n$-natural operators $T \rightsquigarrow TT^{(r)}$. From this result we obtain directly that all linear $\mathcal{M}f_n$-natural operators $T \rightsquigarrow TT^{(r)}$ are of the form $c_1\mathcal{T}^{(r)} + c_2 V$, $c_i \in \mathbb{R}$. If $F = T^{(r)}$ then we have $V^F = V^{(r)} : \mathcal{FM}_{m,n} \to \mathcal{FM}$.

COROLLARY 4. *Let $\mathcal{D} : J^2 \rightsquigarrow J^2 V^{(r)}$ be an $\mathcal{FM}_{m,n}$-natural operator. Then there exist unique real numbers c_1, c_2 such that $\mathcal{D}(\Theta) = \mathcal{V}^{T^{(r)},2}\Theta + c_1\Delta^{\mathcal{T}^{(r)}} + c_2\Delta^V$ for all second order connections $\Theta = (\Gamma, \Delta)$ on $Y \to M$.*

If $F = T^*$ is the cotangent functor, then $V^F Y = V^* Y$. By [14], all linear $\mathcal{M}f_n$-natural operators $A : T \rightsquigarrow TT^*$ are linear combinations (with real coefficients) of the flow operator $\mathcal{T}^*$ and the operator V defined by

$V(X)_\omega = \langle \omega, X_x \rangle^{\cdot} C_\omega$, where C is the Liouville vector field of the cotangent bundle and $X \in \mathcal{X}(N)$, $\omega \in T^*_x N$, $x \in N$. Thus we have

COROLLARY 5. *Let $\mathcal{D} : J^2 \rightsquigarrow J^2 V^*$ be an $\mathcal{FM}_{m,n}$-natural operator. Then there exist unique real numbers c_1, c_2 such that $\mathcal{D}(\Theta) = \mathcal{V}^{T^*,2}\Theta + c_1 \Delta^{T^*} + c_2 \Delta^V$ for all second order connections $\Theta = (\Gamma, \Delta)$ on $Y \to M$.*

Quite analogously to Corollary 4, one can generalize Corollary 5 to the rth order cotangent bundle $F = T^{r*}$. Indeed, the second author [21] described all linear natural operators $T \rightsquigarrow T^{r*}$, which enables us to describe all natural operators $\mathcal{D} : J^2 \rightsquigarrow J^2 V^{r*}$.

Acknowledgements. The first author was supported by a grant of the GA ČR, no. 201/09/0981.

References

[1] A. Cabras and I. Kolář, *Prolongation of second order connections to vertical Weil bundles*, Arch. Math. (Brno) 37 (2001), 333–347.

[2] M. Doupovec and I. Kolář, *Iteration of fiber product preserving bundle functors*, Monatsh. Math. 134 (2001), 39–50.

[3] M. Doupovec and W. M. Mikulski, *On the existence of prolongation of connections*, Czechoslovak Math. J. 56 (131) (2006), 1323–1334.

[4] —, —, *On involutions of iterated bundle functors*, Colloq. Math. 106 (2006), 135–145.

[5] —, —, *Holonomic extension of connections and symmetrization of jets*, Rep. Math. Phys. 60 (2007), 299–316.

[6] —, —, *Some geometric constructions of second order connections*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 61 (2007), 15–22.

[7] C. Ehresmann, *Sur les connexions d'ordre supérieur*, in: Atti del V. Cong. dell'Un. Mat. Ital. (1995), Cremonese, Roma, 1956, 344–346.

[8] J. Gancarzewicz, *Liftings of functions and vector fields to natural bundles*, Dissertationes Math. 212 (1983).

[9] —, *Connections of order r*, Ann. Polon. Math. 34 (1977), 69–83.

[10] J. Gancarzewicz, N. Rahmani and M. Salgado, *Connections of higher order and product preserving functors*, Czechoslovak Math. J. 52 (127) (2002), 889–896.

[11] J. Janyška and M. Modugno, *Relations between linear connections on the tangent bundle and connections on the jet bundle of a fibred manifold*, Arch. Math. (Brno) 32 (1996), 281–288.

[12] I. Kolář, *Higher order absolute differentiation with respect to generalized connections*, in: Differential Geometry, Banach Center Publ. 12, PWN, Warszawa, 1984, 153–161.

[13] —, *On the natural operators on vector fields*, Ann. Global Anal. Geom. 6 (1988), 109–117.

[14] I. Kolář, P. W. Michor and J. Slovák, *Natural Operations in Differential Geometry,* Springer, Berlin, 1993.

[15] I. Kolář and W. M. Mikulski, *Natural lifting of connections to vertical bundles*, Rend. Circ. Mat. Palermo (2) Suppl. 63 (2000), 97–102.

[16] J. Kurek and W. M. Mikulski, *Constructions on second order connections*, Ann. Polon. Math. 92 (2007), 215–223.

[17] M. Kureš, *Generalizations of jets and reflections in the theory of connections*, Publ. Math. Debrecen 59 (2001), 339–352.
[18] M. Kureš and W. M. Mikulski, *Natural operators lifting vector fields to bundles of Weil contact elements*, Czechoslovak Math. J. 54 (129) (2004), 855–867.
[19] P. Libermann, *Introduction to the theory of semi-holonomic jets*, Arch. Math. (Brno) 33 (1997), 173–189.
[20] W. M. Mikulski, *Some natural operations on vector fields*, Rend. Mat. Appl. (7) 12 (1992), 783–803.
[21] —, *Some natural constructions on vector fields and higher order cotangent bundles*, Monatsh. Math. 117 (1994), 107–119.
[22] —, *Non-existence of natural operators transforming connections on $Y \to M$ into connections on $FY \to Y$*, Arch. Math. (Brno) 41 (2005), 1–4.
[23] —, *On prolongation of connections*, Ann. Polon. Math. 97 (2010), 101–121.
[24] —, *On "special" fibred coordinates for general and classical connections*, ibid. 99 (2010), 99–105.
[25] J. Tomáš, *Natural operators on vector fields on the cotangent bundles of the bundles of (k, r)-velocities*, Rend. Circ. Mat. Palermo (2) Suppl. 54 (1998), 113–124.
[26] J. Virsik, *On the holonomity of higher order connections*, Cahiers Topol. Géom. Diff. 12 (1971), 197–212.
[27] A. Vondra, *Higher-order differential equations represented by conections on prolongations of a fibered manifold*, Extracta Math. 15 (2000), 421–512.

Miroslav Doupovec
Department of Mathematics
Brno University of Technology
FSI VUT
Technická 2
616 69 Brno, Czech Republic
E-mail: doupovec@fme.vutbr.cz

Włodzimierz M. Mikulski
Institute of Mathematics
Jagiellonian University
Łojasiewicza 6
30-348 Kraków, Poland
E-mail: mikulski@im.uj.edu.pl

Received 6.12.2010
and in final form 10.2.2011 (2337)