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On weakly symmetric manifolds with a type of semi-symmetric non-metric connection

by Hülya Bağdatli Yilmaz (Istanbul)

Abstract. The object of the present paper is to study weakly symmetric manifolds admitting a type of semi-symmetric non-metric connection.

1. Introduction. The notion of weakly symmetric manifolds was introduced by Tamássy and Binh [TB1]. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called *weakly symmetric* if its curvature tensor R of type $(0, 4)$ satisfies the condition

$$(1.1)\qquad (\nabla_X R)(Y,Z,U,V) = A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) + C(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) + E(V)R(Y,Z,U,X)$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, where ∇ denotes the Levi-Civita connection on (M^n, g) and A, B, C, D and E are 1-forms which are not simultaneously zero. The 1-forms are called the *associated* 1-*forms* of the manifold and such an n-dimensional manifold is denoted by $(WS)_n$.

De and Bandyopadhyay [DB] proved that the associated 1-forms C and E of a $(WS)_n$ are identical with B and D, respectively. So the defining condition of a $(WS)_n$ reduces to

$$(1.2)\qquad (\nabla_X R)(Y,Z,U,V) = A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) + B(Z)R(Y,X,U,V) + D(U)R(Y,Z,X,V) + D(V)R(Y,Z,U,X)$$

where A, B and D are three non-zero 1-forms defined by

$$(1.3)\qquad A(X) = g(X,\rho),\quad B(X) = g(X,Q),\quad D(X) = g(X,P).$$

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In spite of the fact that the definition of a $(WS)_n$ $(n > 2)$ is similar to that of a *generalized pseudo symmetric manifold* [C1, CM] given by

$$(\nabla_X R)(Y,Z,U,V) = 2A(X)R(Y,Z,U,V) + B(Y)R(X,Z,U,V) + B(Z)R(Y,X,U,V) + A(U)R(Y,Z,X,V) + A(V)R(Y,Z,U,X), \tag{1.4}$$

the defining condition of a $(WS)_n$ is weaker than that of a generalized pseudo symmetric manifold. If we take $B = D = \frac{1}{2}A$, then a $(WS)_n$ $(n > 2)$ is just a *pseudo symmetric manifold* [C2] defined by

$$(\nabla_X R)(Y,Z,U,V) = 2A(X)R(Y,Z,U,V) + A(Y)R(X,Z,U,V) + A(Z)R(Y,X,U,V) + A(U)R(Y,Z,X,V) + A(V)R(Y,Z,U,X). \tag{1.5}$$

An n-dimensional pseudo symmetric manifold is denoted by $(PS)_n$. Hence, the notion of a $(PS)_n$ is a particular case of that of a $(WS)_n$.

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called *weakly Ricci symmetric* [TB2] if its Ricci tensor S of type $(0,2)$ is not identically zero and if it satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(X,Z) + D(Z)S(Y,X) \tag{1.6}$$

where A, B, D and ∇ have the meaning already stated. Such an n-dimensional manifold is denoted by $(WRS)_n$. Due to the definitions, it is seen that, in general, a $(WS)_n$ is not necessarily a $(WRS)_n$.

Prvanović [P] proved the existence of a $(WS)_n$ and De and Bandyopadhyay [DB] obtained an example of a $(WS)_n$ by choosing φ suitably in the metric

$$ds^2 = \varphi(dx^1)^2 + k_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1 dx^n.$$

Also, De and Sengupta [DS] proved that if a $(WS)_n$ admits a type of semi-symmetric metric connection, then it reduces to a particular kind of a $(WRS)_n$.

The present paper deals with weakly symmetric manifolds $(WS)_n$ $(n>3)$ admitting a type of semi-symmetric non-metric connection $\overline{\nabla}$ whose torsion tensor T is given by

$$T(X,Y) = A(Y)X - A(X)Y \tag{1.7}$$

and whose curvature tensor $\overline{R}$ and torsion tensor T satisfy the conditions

$$\overline{R}(X,Y)Z = 0 \tag{1.8}$$

and

$$(\overline{\nabla}_X T)(Y,Z) = B(X)T(Y,Z) + D(X)g(Y,Z)P. \tag{1.9}$$

In Section 4 we enquire under what condition a $(WS)_n$ will be a $(WRS)_n$ and it is shown that a $(WS)_n$ $(n > 3)$ admitting a semi-symmetric non-metric connection whose torsion tensor T is given by (1.7) and whose curvature tensor $\overline{R}$ and torsion tensor T satisfy the conditions (1.8) and (1.9) is a $(WRS)_n$ of constant curvature whose scalar curvature is non-zero.

In Section 5 we show that if a $(WS)_n$ $(n > 3)$ admits a semi-symmetric non-metric connection of the type mentioned above, then it is a particular kind of a special conformally flat manifold, namely, it is a subprojective manifold. Finally, it is shown that a simply connected $(WS)_n$ $(n > 3)$ admitting such a semi-symmetric non-metric connection can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface.

2. Preliminaries. L denotes the symmetric endomorphism of the tangent space at each point of a $(WS)_n$ corresponding to the Ricci tensor, that is,

$$g(LX, Y) = S(X, Y). \tag{2.1}$$

Now, putting $Y = V = e_i$ in (1.2) where $\{e_i\}$ $(1 \le i \le n)$ is an orthonormal basis of the tangent space at each point of the manifold and summing over i $(1 \le i \le n)$, we obtain

$$\begin{aligned}(\nabla_X S)(Z, U) = A(X)S(Z, U) + B(R(X, Z)U) + B(Z)S(X, U)\\ + D(U)S(Z, X) + D(R(X, U)Z).\end{aligned} \tag{2.2}$$

Contracting (2.2) with respect to Z and U, we have

$$dr(X) = A(X)r + 2S(X, Q) + 2S(X, P). \tag{2.3}$$

3. Semi-symmetric non-metric connection. A semi-symmetric non-metric connection $\overline{\nabla}$ is defined in the following form by Agashe and Chafle [AC]:

$$\overline{\nabla}_X Y = \nabla_X Y + A(Y)X \tag{3.1}$$

for all vector fields X, Y.

Let us denote the curvature tensors with respect to the connections $\overline{\nabla}$ and ∇ by $\overline{R}$ and R, respectively. Then, due to (3.1), we obtain [AC]

$$\overline{R}(X, Y)Z = R(X, Y)Z + \alpha(X, Z)Y - \alpha(Y, Z)X \tag{3.2}$$

where α is a tensor field of type $(0, 2)$ defined by

$$\alpha(X, Y) = (\nabla_X A)(Y) - A(X)A(Y). \tag{3.3}$$

By virtue of (3.1), we have

$$(\overline{\nabla}_X A)(Y) = (\nabla_X A)(Y) - A(X)A(Y). \tag{3.4}$$

From (3.3) and (3.4), it follows that

$$\alpha(X, Y) = (\overline{\nabla}_X A)(Y). \tag{3.5}$$

Contracting (3.2), we have

$$\overline{S}(Y,Z) = S(Y,Z) + (1-n)\alpha(Y,Z) \tag{3.6}$$

where $\overline{S}$ and S denote the Ricci tensors of the semi-symmetric non-metric connection and the Levi-Civita connection, respectively.

4. $(WS)_n$ $(n>3)$ **admitting a special type of semi-symmetric non-metric connection.** In this section we consider a Riemannian manifold admitting a semi-symmetric non-metric connection whose torsion tensor T is given by (1.7) and whose curvature tensor $\overline{R}$ and torsion tensor T satisfy (1.8) and (1.9), respectively. Then, from (1.7), contracting over X, we get

$$(C_1^1 T)(Y) = (n-1)A(Y). \tag{4.1}$$

From (4.1), it follows that

$$(\overline{\nabla}_X C_1^1 T)(Y) = (n-1)(\overline{\nabla}_X A)(Y). \tag{4.2}$$

Contracting (1.9) and using (4.1), we get

$$(\overline{\nabla}_X C_1^1 T)(Z) = (n-1)B(X)A(Z) + D(X)D(Z). \tag{4.3}$$

Using (4.3), from (4.2), we have

$$(\overline{\nabla}_X A)(Y) = B(X)A(Y) + \frac{1}{n-1}D(X)D(Y). \tag{4.4}$$

Thus, due to (4.4), (3.4) can be written as

$$(\nabla_X A)(Y) = B(X)A(Y) + A(X)A(Y) + \frac{1}{n-1}D(X)D(Y). \tag{4.5}$$

In virtue of (3.5) and (4.4), we get

$$\alpha(X,Y) = B(X)A(Y) + \frac{1}{n-1}D(X)D(Y). \tag{4.6}$$

Now, using (3.2), the expression of the curvature tensor $\overline{R}$ with respect to the connection $\overline{\nabla}$ can be written as

$$\begin{aligned}\overline{R}(X,Y)Z = R(X,Y)Z &+ \left\{B(X)A(Z) + \frac{1}{n-1}D(X)D(Z)\right\}Y \\ &- \left\{B(Y)A(Z) + \frac{1}{n-1}D(Y)D(Z)\right\}X.\end{aligned} \tag{4.7}$$

On account of the condition (1.8), it follows that

$$\begin{aligned}R(X,Y)Z = &\left\{B(Y)A(Z) + \frac{1}{n-1}D(Y)D(Z)\right\}X \\ &- \left\{B(X)A(Z) + \frac{1}{n-1}D(X)D(Z)\right\}Y.\end{aligned} \tag{4.8}$$

Contracting (4.8), we obtain

$$S(Y,Z) = (n-1)B(Y)A(Z) + D(Y)D(Z). \tag{4.9}$$

Again contracting (4.9), we get a scalar curvature as

$$r = (n-1)B(\rho) + D(P) \tag{4.10}$$

where the vector fields ρ and P are defined by (1.3). We know that the Ricci tensor S is symmetric. Therefore, it follows from (4.9) that

$$A(Y) = \theta B(Y) \tag{4.11}$$

where θ is a non-zero scalar function. By (4.11), (4.8) can be written as

$$\begin{aligned} R(X,Y)Z = &\left\{\theta B(Y)B(Z) + \frac{1}{n-1}D(Y)D(Z)\right\}X \\ &- \left\{\theta B(X)B(Z) + \frac{1}{n-1}D(X)D(Z)\right\}Y. \end{aligned} \tag{4.12}$$

Hence, we have

$$\begin{aligned} R(X,Y,Z,W) = &\left\{\theta B(Y)B(Z) + \frac{1}{n-1}D(Y)D(Z)\right\}g(X,W) \\ &- \left\{\theta B(X)B(Z) + \frac{1}{n-1}D(X)D(Z)\right\}g(Y,W). \end{aligned} \tag{4.13}$$

Let us put $W = Q$ and $W = P$. Thus, we get the expressions

$$R(X,Y,Z,Q) = \frac{1}{n-1}D(Z)\{D(Y)B(X) - D(X)B(Y)\}, \tag{4.14}$$

$$R(X,Y,Z,P) = \theta B(Z)\{B(Y)D(X) - B(X)D(Y)\}. \tag{4.15}$$

We see that the expressions (4.14) and (4.15) vanish provided that

$$B(Y) = \lambda D(Y) \tag{4.16}$$

where λ is a non-zero scalar function. Therefore, from (4.14)–(4.16) we have

$$B(R(X,Y)Z) = 0, \tag{4.17}$$

$$D(R(X,Y)Z) = 0. \tag{4.18}$$

Substituting (4.17) and (4.18) in (2.2), we get

$$(\nabla_X S)(Z,U) = A(X)S(Z,U) + B(Z)S(X,U) + D(U)S(Z,X). \tag{4.19}$$

This shows that in virtue of (1.6) a $(WS)_n$ under consideration is a $(WRS)_n$.

Due to (4.11) and (4.16), we have

$$B(Y) = \varphi A(Y) \quad \text{and} \quad D(Y) = \phi A(Y) \tag{4.20}$$

where φ and ϕ are non-zero scalar functions. From (4.5), using (4.20) we get

$$(\nabla_X A)(Y) = \left\{1 + \varphi + \frac{1}{n-1}\phi^2\right\}A(X)A(Y). \tag{4.21}$$

From (4.21), we find that the associated 1-form A is closed and from (4.20) it follows that the 1-forms B and D are also closed. Agashe and Chafle [AC] proved that if a Riemannian manifold (M^n, g) $(n > 3)$ admits a semi-symmetric non-metric connection whose curvature tensor vanishes, then the manifold is projectively flat and hence a manifold of constant curvature. Moreover, as B and D are non-zero, from (4.10), the scalar curvature r is non-zero. Summing up, we can state the following theorem.

THEOREM 4.1. *If a* $(WS)_n$ $(n > 3)$ *admits a semi-symmetric non-metric connection whose torsion tensor* T *is given by* (1.7) *and whose curvature tensor* $\overline{R}$ *and torsion tensor* T *satisfy* (1.8) *and* (1.9), *then the manifold is of constant curvature and a* $(WRS)_n$ *with non-zero scalar curvature whose associated* 1*-forms* A, B *and* D *are closed.*

5. Special conformally flat $(WS)_n$ $(n > 3)$ admitting a special type of semi-symmetric non-metric connection. Chen and Yano [CY] introduced the notion of a special conformally flat manifold which generalizes the notion of a subprojective manifold. A conformally flat manifold is called a *special conformally flat manifold* if the tensor H of type $(0, 2)$ defined by

$$(5.1)\qquad H(X,Y) = -\frac{1}{n-2}S(X,Y) + \frac{r}{2(n-1)(n-2)}g(X,Y)$$

is expressible in the form

$$(5.2)\qquad H(X,Y) = -\frac{\alpha^2}{2}g(X,Y) + \beta(\nabla_X\alpha)(\nabla_Y\alpha)$$

where α and β are two scalars such that α is positive. In particular, if β is a function of α then the special conformally flat manifold is called a *subprojective manifold* [S].

Let us consider $(WS)_n$ $(n > 3)$ admitting a semi-symmetric non-metric connection whose torsion tensor T is given by (1.7) and whose curvature tensor $\overline{R}$ and torsion tensor T satisfy (1.8) and (1.9).

Substituting (4.9) and (4.20) in (5.1), we get

$$(5.3)\qquad H(X,Y) = \frac{r}{2(n-1)(n-2)}g(X,Y) - \frac{(n-1)}{n-2}(\varphi + \phi^2)A(X)A(Y).$$

Now, put

$$(5.4)\qquad \alpha^2 = -\frac{r}{(n-1)(n-2)}.$$

Since $r \neq 0$, it follows that α^2 will be positive provided that $r < 0$.

Using (1.3) and (2.1), from (2.3) it follows that

$$(5.5)\qquad dr(X) = rA(X) + 2B(LX) + 2D(LX).$$

From (4.17), (4.18) and (5.5) we find that

$$dr(X) = rA(X). \tag{5.6}$$

Let us take the covariant derivative of both sides of (5.4) with respect to X and use (5.6) to obtain

$$\nabla_X \alpha = -\frac{r}{2(n-1)(n-2)\alpha} A(X). \tag{5.7}$$

Then, from (5.7), we get

$$A(X) = -\frac{2(n-1)(n-2)\alpha}{r} \nabla_X \alpha. \tag{5.8}$$

Thus, due to (5.4) and (5.8), (5.3) can be expressed in the form

$$H(X,Y) = -\frac{\alpha^2}{2} g(X,Y) + \beta(\nabla_X \alpha)(\nabla_Y \alpha) \tag{5.9}$$

where

$$\beta = -\frac{4\alpha^2(n-1)^3(n-2)}{r^2}(\varphi + \phi^2). \tag{5.10}$$

Since $r \neq 0$, it follows that α cannot be zero. Hence α can be taken positive. Thus, from (5.9), we can say that the $(WS)_n$ $(n > 3)$ under consideration is a special conformally flat manifold. In virtue of (5.10), we deduce that β is a function of α. This means that the manifold under consideration is a particular kind of a special conformally flat manifold, namely a subprojective manifold. Thus, we can state the following theorem.

THEOREM 5.1. *If a $(WS)_n$ $(n > 3)$ admits a semi-symmetric non-metric connection whose torsion tensor T is given by* (1.7) *and whose curvature tensor $\overline{R}$ and torsion tensor T satisfy* (1.8) *and* (1.9), *then the manifold is a particular kind of a special conformally flat manifold, namely a subprojective manifold.*

COROLLARY 5.2 ([CY, Corollary 1]). *Every simply connected subprojective space can be isometrically immersed in a Euclidean space as a hypersurface.*

Moreover, using this corollary, we can also state the following theorem.

THEOREM 5.3. *If a simply connected $(WS)_n$ $(n > 3)$ admits a semi-symmetric non-metric connection whose torsion tensor T is given by* (1.7) *and whose curvature tensor $\overline{R}$ and torsion tensor T satisfy* (1.8) *and* (1.9), *then the manifold can be isometrically immersed in a Euclidean space E^{n+1} as a hypersurface.*

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Hülya Bağdatlı Yılmaz
Department of Mathematics
Faculty of Sciences and Arts
Marmara University
34722 Istanbul, Turkey
E-mail: hbagdatli@marmara.edu.tr

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