ANNALES
POLONICI MATHEMATICI
86.1 (2005)

Univalence, strong starlikeness and integral transforms

by M. Obradović (Belgrad), S. Ponnusamy (Chennai),
and P. Vasundhra (Chennai)

Abstract. Let $\mathcal{A}$ represent the class of all normalized analytic functions f in the unit disc Δ. In the present work, we first obtain a necessary condition for convex functions in Δ. Conditions are established for a certain combination of functions to be starlike or convex in Δ. Also, using the Hadamard product as a tool, we obtain sufficient conditions for functions to be in the class of functions whose real part is positive. Moreover, we derive conditions on f and μ so that the non-linear integral transform $\int_0^z(\zeta/f(\zeta))^\mu\,d\zeta$ is univalent in Δ. Finally, we give sufficient conditions for functions to be strongly starlike of order α.

1. Introduction. Let $\mathcal{H}$ denote the class of all functions analytic in the unit disc $\Delta=\{z:|z|<1\}$, and $\mathcal{A}$ the class of all normalized functions f ($f(0)=f'(0)-1=0$) in $\mathcal{H}$. Let $\mathcal{S}$ denote the univalent subclass of $\mathcal{A}$, and $\mathcal{S}^*$ denote the subclass of $f\in\mathcal{S}$ for which $f(\Delta)$ is starlike with respect to the origin. Recall the prominent subclasses studied in the theory of univalent functions (see [7]), for $0\le\beta<1$:

$$\begin{aligned}
\mathcal{P}(\beta)&=\left\{f\in\mathcal{A}:\operatorname{Re}\left(\frac{f(z)}{z}\right)>\beta,\ z\in\Delta\right\},\\
\mathcal{R}(\beta)&=\{f\in\mathcal{A}:zf'\in\mathcal{P}(\beta)\},\\
\mathcal{S}^*(\beta)&=\left\{f\in\mathcal{A}:\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\beta,\ z\in\Delta\right\},\\
\mathcal{S}^*_\beta&=\left\{f\in\mathcal{A}:\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\beta\pi}{2},\ z\in\Delta\right\},\\
\mathcal{K}(\beta)&=\{f\in\mathcal{A}:zf'\in\mathcal{S}^*(\beta)\}.
\end{aligned}$$

It is well known that $\mathcal{K}\equiv\mathcal{K}(0)\subsetneq\mathcal{S}^*(1/2)$. Functions in $\mathcal{S}^*_\beta$ are called *strongly starlike of order* β, while those in $\mathcal{S}^*(\beta)$ are *starlike of order* β. For $\beta<0$, $\mathcal{S}^*(\beta)\not\subseteq\mathcal{S}$, while for $0<\beta<1$, $\mathcal{S}^*(\beta)\subsetneq\mathcal{S}^*\subsetneq\mathcal{S}$, and functions

---

2000 *Mathematics Subject Classification*: 30C45, 30C55.
Key words and phrases: univalent, starlike, strongly starlike and convex functions.
The authors thank the referee for his/her valuable comments.

in $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ are simply referred to as *starlike*. For $0 < \beta < 1$, clearly, $\mathcal{S}^*_\beta \subsetneq \mathcal{S}^*$ and $\mathcal{S}^*_1 \equiv \mathcal{S}^*$.

For $a, b, c \in \mathbb{C}$ and $c \neq 0, -1, -2, \ldots$, the Gaussian hypergeometric series $F(a, b; c; z)$ is defined as

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \qquad |z| < 1,$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ and $(a)_0 = 1$. This series represents an analytic function in Δ and has an analytic continuation throughout the finite complex plane except at most for the cut $[1, \infty)$.

Let $\mathcal{B}$ denote another important subclass, of all analytic functions $\omega \in \mathcal{H}$ such that $\omega(0) = 0$ and $\omega(\Delta) \subseteq \Delta$. A function $f \in \mathcal{H}$ is called *subordinate* to another function $g \in \mathcal{H}$, and one writes $f(z) \prec g(z)$, if there exists an $\omega \in \mathcal{B}$ such that $f(z) = g(\omega(z))$ for all $z \in \Delta$. It is well known that this implies in particular $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$, and that these two conditions are also sufficient for $f(z) \prec g(z)$ whenever g is univalent in Δ. Next, we remark that if $f \in \mathcal{H}$, $f(0) = 0$ and $|f(z)| \leq M$ on Δ, then this can be equivalently expressed in the form

$$f(z) = M\omega(z), \quad \omega \in \mathcal{B},$$

and so $f(z) \prec Mz$.

In [8], R. Singh and S. Paul showed that for all real λ and μ with $0 \leq \mu \leq \lambda/2$ one has the following implication:

$$f \in \mathcal{K} \;\Rightarrow\; \mathrm{Re}\left(\lambda \frac{f(z)}{zf'(z)} + \mu \frac{1}{f'(z)}\right) > 0, \quad z \in \Delta. \tag{1.1}$$

We observe that the well known strict inclusion result, namely $\mathcal{K} \subsetneq \mathcal{S}^*(1/2)$, does not follow from the above one way implication. In view of this, in Theorem 2.1 we use a different approach and determine $R = R(\lambda, \mu)$ such that

$$f \in \mathcal{K} \;\Rightarrow\; G(\Delta) \subset \{w \in \mathbb{C} : |w - R| < R\}, \qquad G(z) = \lambda \frac{f(z)}{zf'(z)} + \mu \frac{1}{f'(z)},$$

for all real values of λ and μ with $|\mu| \leq \lambda/2$.

Trimble [11] showed that if $f \in \mathcal{K}$, then F defined by

$$F(z) = \lambda z + (1 - \lambda) f(z)$$

is in $\mathcal{S}^*(\beta)$, where $\beta = (1 - 3\lambda)/(2(2 + \lambda))$ with $0 \leq \lambda \leq 1/3$. Related problems were considered in [2, 12], by imposing an additional condition on f.

In Theorem 2.3, we impose conditions on $f \in \mathcal{A}_n := \{f \in \mathcal{A} : f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k\}$ different from those of [2, 12] and obtain the starlikeness of

$$F(z) = \lambda z + \frac{1-\lambda}{\alpha}\int_0^1 t^{1/\alpha-2} f(tz)\,dt \tag{1.2}$$

for all $\lambda < 1$. It follows that the integral (1.2) is well defined or convergent only for $\operatorname{Re}\alpha > 0$ and also at $\alpha = 0$ as a limiting case, because

$$\begin{aligned}\frac{1}{\alpha}\int_0^1 t^{1/\alpha-2+k}\,dt &= \frac{1}{(k-1)\alpha+1}\left[1 - \lim_{t\to 0^+} \exp\left(\left(\frac{1}{\alpha} - 1 + k\right)\ln t\right)\right]\\ &= \frac{1}{(k-1)\alpha+1},\end{aligned}$$

for $k = 1, n+1, n+2, \dots$, where the principal branches of possible multiple-valued power functions are considered. We remark that the relation (1.2) looks much simpler in the following differential form:

$$\alpha z F'(z) + (1-\alpha)F(z) = \lambda z + (1-\lambda) f(z) \tag{1.3}$$

since

$$f(z) \equiv \int_0^1 \frac{\partial}{\partial t}(t^{1/\alpha-1} f(tz))\,dt.$$

Thus, for a given $f \in \mathcal{A}_n$, there is exactly one solution $F \in \mathcal{A}_n$ of the equation (1.3) if and only if $\alpha \in \mathbb{C} \setminus \{-1/j : j = n, n+1, n+2, \dots\}$:

$$F(z) \equiv z + (1-\lambda)\sum_{k=n+1}^{\infty} \frac{a_k}{(k-1)\alpha+1}\, z^k \tag{1.4}$$

whenever $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$. We use this observation in the proof of Theorem 2.3.

Also, we provide a condition on β such that $\operatorname{Re} z f''(z) > -\beta(1-\lambda)$ implies that $\operatorname{Re}(f(z)/z) > \lambda$ (see Theorem 2.6). In addition to these results, in Theorem 2.7, we obtain conditions so that the non-linear operator

$$g(z) = \int_0^z \left(\frac{\zeta}{f(\zeta)}\right)^{\mu} d\zeta$$

is univalent. Finally, we derive a sufficient condition for f to be strongly starlike of order α.

2. Main results

THEOREM 2.1. *If $f \in \mathcal{K}$ then*

$$\left|\lambda \frac{f(z)}{zf'(z)} + \mu \frac{1}{f'(z)} - \frac{\lambda(\lambda+2\mu)}{\lambda - 2\mu}\right| < \frac{\lambda(\lambda+2\mu)}{\lambda-2\mu}, \quad z \in \Delta, \tag{2.1}$$

for all real λ *and* μ *with* $0 < \mu \leq \lambda/2$, *and*

$$\left|\lambda \frac{f(z)}{zf'(z)} + \mu \frac{1}{f'(z)} - \lambda\right| < \lambda, \quad z \in \varDelta, \tag{2.2}$$

for all real λ *and* μ *with* $-\lambda/2 \leq \mu < 0$.

Proof. Let $f \in \mathcal{K}$. Since $\mathcal{K} \subsetneq \mathcal{S}^*(1/2)$, we exclude the trivial case $\mu = 0 < |\lambda|$ as this may be obtained as a limiting case. Then, for all z and w in $\varDelta$, it is known that

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z) - f(w)} - \frac{w}{z - w}\right) > \frac{1}{2}, \tag{2.3}$$

where the expression is defined by its limit when $z = w$. Further, for $f \in \mathcal{K}$ it is also known that $\mathrm{Re}(f(z)/z) > 1/2$ in $\varDelta$ and hence, for $0 < \mu \leq \lambda/2$, this shows that

$$0 < \mathrm{Re}\left(\frac{\mu z}{\mu z + \lambda f(z)}\right) \leq \frac{2\mu}{\lambda + 2\mu}. \tag{2.4}$$

Since $f \in \mathcal{K}$, the image of f covers the disc $|\zeta| < 1/2$ and therefore, it can be readily seen that there exists $w \in \varDelta$ such that

$$f(w) = -(\mu/\lambda)z.$$

From (2.3) and (2.4),

$$\begin{aligned}
\mathrm{Re}\left(\frac{\lambda zf'(z)}{\lambda f(z) + \mu z}\right) &= \mathrm{Re}\left(\frac{zf'(z)}{f(z) - f(w)}\right) \\
&> \frac{1}{2} + \mathrm{Re}\left(\frac{w}{z - w}\right) = \frac{1}{2} - \mathrm{Re}\left(\frac{\mu w}{\mu w + \lambda f(w)}\right) \\
&> \frac{1}{2} - \frac{2\mu}{\lambda + 2\mu} = \frac{\lambda - 2\mu}{2(\lambda + 2\mu)},
\end{aligned}$$

which proves the first assertion (2.1) for $0 < \mu < \lambda/2$. If $\mu = \lambda/2$, then the last inequality becomes

$$\mathrm{Re}\left(\lambda \frac{f(z)}{zf'(z)} + \frac{1}{2} \frac{1}{f'(z)}\right) > 0,$$

which is same as (2.1) in the limiting case.

Next, we observe that for $-\lambda/2 \leq \mu < 0$,

$$\mathrm{Re}\left(1 + \frac{\lambda f(z)}{\mu z}\right) < \frac{\lambda + 2\mu}{2\mu} \leq 0$$

so that

$$\frac{2\mu}{\lambda + 2\mu} < \frac{1}{\mathrm{Re}(1 + \lambda f(z)/\mu z)} \leq \mathrm{Re}\left(\frac{1}{1 + \lambda f(z)/\mu z}\right) < 0.$$

This observation shows that

$$\mathrm{Re}\left(\frac{\lambda z f'(z)}{\lambda f(z)+\mu z}\right) > \frac{1}{2}, \qquad z\in\Delta,$$

which proves the second assertion (2.2). ■

COROLLARY 2.2. *Let $f\in\mathcal{K}$. For $z,w\in\Delta$, define*

$$G(z,w)=\lambda\,\frac{[f(z)-f(w)](1-|w|^2)}{(z-w)f'(z)(1-\overline{w}z)}+\mu\,\frac{f'(w)(1-|w|^2)^2}{f'(z)(1-\overline{w}z)^2}. \tag{2.5}$$

Then, for all real λ and μ such that $0<\mu\le\lambda/2$, we have

$$\left|G(z,w)-\frac{\lambda(\lambda+2\mu)}{\lambda-2\mu}\right| < \frac{\lambda(\lambda+2\mu)}{\lambda-2\mu},$$

and for $-\lambda/2\le\mu<0$, we have $|G(z,w)-\lambda|<\lambda$.

Proof. Since $f'(w)\neq 0$ in Δ, we consider a disc automorphism of Δ and define g by

$$g(\zeta)=\frac{f((\zeta+w)/(1+\overline{w}\zeta))-f(w)}{f'(w)(1-|w|^2)}.$$

As the convexity is preserved under disc automorphisms, we have $g\in\mathcal{K}$ if and only if $f\in\mathcal{K}$. Writing $z=(w+\zeta)/(1+\zeta\overline{w})$, it can be shown that

$$\frac{\lambda g(\zeta)+\mu\zeta}{\zeta g'(\zeta)}=G(z,w)$$

where $G(z,w)$ is given by (2.5). Since $g\in\mathcal{K}$, the desired conclusion follows from Theorem 2.1 and the last equality. ■

THEOREM 2.3. *Let $n\in\mathbb{N}$, $\alpha\in\mathbb{C}\setminus\{-1/j : j=n,n+1,n+2,\ldots\}$ with $\mathrm{Re}\,\alpha>-1/n$ and let $f\in\mathcal{A}_n$ satisfy the condition*

$$|zf''(z)|<\frac{\mu}{1-\lambda}, \qquad z\in\Delta, \tag{2.6}$$

for some $\lambda<1$. Then, for F defined by (1.3), we have

(a) $\left|\dfrac{zF'(z)}{F(z)}-1\right|\le 1$ *for* $0<\mu\le n\,\mathrm{Re}\,\alpha+1$,

(b) $\left|\dfrac{zF''(z)}{F'(z)}\right|\le 1$ *for* $0<\mu\le (n\,\mathrm{Re}\,\alpha+1)/2$.

Proof. From the representation (1.4), we easily see that

$$zF''(z)=(1-\lambda)\sum_{k=n}^{\infty}\frac{(k+1)ka_{k+1}z^k}{1+k\alpha}=(1-\lambda)\left[zf''(z)*\left(\sum_{k=n}^{\infty}\frac{z^k}{1+k\alpha}\right)\right],$$

and thus,

$$zF''(z) = (1-\lambda)\int_0^1 t^{\alpha} z f''(t^{\alpha} z)\, dt. \tag{2.7}$$

Suppose that f satisfies condition (2.6), which may be rewritten as

$$zf''(z) = \frac{\mu}{1-\lambda}\,\omega(z), \quad \omega \in \mathcal{B}_n,$$

where $\mathcal{B}_n = \{\omega \in \mathcal{H} : |\omega(z)| < 1 \text{ and } \omega^{(k)}(0) = 0 \text{ for } k = 0, 1, \ldots, n-1\}$. Schwarz' lemma then shows that $|\omega(z)| \le |z|^n$ for $z \in \Delta$. Therefore, (2.7) becomes

$$zF''(z) = \mu \int_0^1 \omega(t^{\alpha} z)\, dt$$

and hence, by the condition on α, it follows that

$$|zF''(z)| \le \frac{\mu |z|^n}{n \operatorname{Re} \alpha + 1} < \frac{\mu}{n \operatorname{Re} \alpha + 1}, \quad z \in \Delta.$$

Then (see [7, 10]) we have

$$\left| \frac{zF'(z)}{F(z)} - 1 \right| \le \frac{\mu/[2n \operatorname{Re} \alpha + 2]}{1 - \mu/[2n \operatorname{Re} \alpha + 2]} \tag{2.8}$$

and

$$\left| \frac{zF''(z)}{F'(z)} \right| \le \frac{\mu/[n \operatorname{Re} \alpha + 1]}{1 - \mu/[n \operatorname{Re} \alpha + 1]}. \tag{2.9}$$

In particular, F is starlike for $0 < \mu \le n \operatorname{Re} \alpha + 1$ and convex if $0 < \mu \le (n \operatorname{Re} \alpha + 1)/2$. ■

The case $n = 1$ of Theorem 2.3 gives

COROLLARY 2.4. *Let* $\operatorname{Re} \alpha > -1$ *and let* $f \in \mathcal{A}$ *satisfy the condition*

$$|zf''(z)| < \frac{\mu}{1-\lambda}, \quad z \in \Delta, \tag{2.10}$$

for some $\lambda < 1$. *Then, for* F *defined by* (1.2), *we have*

(a) $\left| \dfrac{zF'(z)}{F(z)} - 1 \right| \le 1$ *for* $0 < \mu \le \operatorname{Re} \alpha + 1$,

(b) $\left| \dfrac{zF''(z)}{F'(z)} \right| \le 1$ *for* $0 < \mu \le (\operatorname{Re} \alpha + 1)/2$.

Note that $z + (c/2)z^2 \notin \mathcal{S}$ whenever $|c| > 1$. Define

$$f(z) = z + (\mu/2(1-\lambda))z^2.$$

Now, if we let $1-\mu < \lambda \le 1$, then $\mu/(1-\lambda) > 1$ and hence f is not univalent but satisfies (2.10). On the other hand, the corresponding F defined by (1.2) is starlike for $0 < \mu \le \operatorname{Re} \alpha + 1$ and is in fact convex for $0 < \mu \le (\operatorname{Re} \alpha + 1)/2$.

LEMMA 2.5. *Let p be analytic in Δ and $p(0)=1$. Suppose that*

$$\mathrm{Re}(z^2p''(z)+\alpha zp'(z)) > -\beta(1-\lambda), \qquad z\in\Delta,$$

for some $\alpha>1$, $\lambda<1$ and $0<\beta\le\beta(\alpha)$, where

$$\beta(\alpha) := \frac{\alpha(\alpha-1)}{2[\alpha\log 2 - F(1,\alpha;\alpha+1;-1)]}.$$

Then $\mathrm{Re}\, p(z)>\lambda$ for $z\in\Delta$. In particular, if

$$\mathrm{Re}(z^2p''(z)+\alpha zp'(z)) > -\beta$$

for $0<\beta\le\beta(\alpha)$, then $\mathrm{Re}\, p(z)>0$ for $z\in\Delta$.

Proof. We consider a more general differential equation

$$z^2p''(z)+\alpha zp'(z) = \beta(1-\lambda)(\phi(z)-1) \tag{2.11}$$

where $\mathrm{Re}\,\phi(z)>0$ in Δ, and $\phi(0)=1$. If p and ϕ are of the form

$$p(z)=1+\sum_{n=1}^{\infty}p_nz^n \quad\text{and}\quad \phi(z)=1+\sum_{n=1}^{\infty}\phi_nz^n,$$

respectively, then, by comparing the coefficients of z^n on both sides of (2.11), it follows that

$$n(n-1+\alpha)p_n = \beta(1-\lambda)\phi_n, \qquad n\ge 1,$$

which gives

$$p(z) = 1+\beta(1-\lambda)\sum_{n=1}^{\infty}\frac{\phi_n}{n(n-1+\alpha)}\,z^n.$$

It can be easily seen that $p(z)$ has the integral representation (see [5, Proposition 1])

$$p(z) = 1+\beta(1-\lambda)\int_0^1\!\!\int_0^1 u^{-1}v^{\alpha-2}(\phi(uvz)-1)\,du\,dv.$$

As $\mathrm{Re}\,\phi(z) > (1-|z|)/(1+|z|)$ for $z\in\Delta$, we have

$$\mathrm{Re}(\phi(uvz)-1) \ge -\frac{2|uvz|}{1+uv|z|} \ge -\frac{2uv}{1+uv}, \quad z\in\Delta,$$

and therefore,

$$\begin{aligned}\mathrm{Re}\, p(z) &> 1-2\beta(1-\lambda)\int_0^1\!\!\int_0^1 \frac{v^{\alpha-1}}{1+uv}\,du\,dv\\ &= 1-2\beta(1-\lambda)\int_0^1 v^{\alpha-2}\log(1+v)\,dv\end{aligned}$$

$$= 1 - 2\beta(1-\lambda)\left[\log(1+v)\,\frac{v^{\alpha-1}}{\alpha-1}\Big|_0^1 - \frac{1}{\alpha-1}\int_0^1 \frac{v^{\alpha-1}}{1+v}\,dv\right]$$

$$= 1 - 2\beta(1-\lambda)\left[\frac{\log 2}{\alpha-1} - \frac{F(1,\alpha;\alpha+1;-1)}{\alpha(\alpha-1)}\right]$$

$$\geq 1 - 2\beta(\alpha)(1-\lambda)\left[\frac{\alpha\log 2 - F(1,\alpha;\alpha+1;-1)}{\alpha(\alpha-1)}\right] = \lambda.$$

The desired conclusion follows. ■

THEOREM 2.6. *Let $f \in \mathcal{A}$ satisfy the condition*

$$\operatorname{Re} zf''(z) > -\beta(1-\lambda), \qquad 0 < \beta \leq \frac{1}{2(2\log 2 - 1)} \approx 1.29435.$$

Then $f \in \mathcal{P}(\lambda)$. In particular,

$$\operatorname{Re} zf''(z) > -\beta \;\Rightarrow\; \operatorname{Re}\left(\frac{f(z)}{z}\right) > \frac{1-\log 2}{\log 2} = 0.4427\ldots$$

$$\text{for } 0 < \beta \leq 1/\log 4.$$

Proof. Define $p(z) = f(z)/z$. Then $z^2p''(z) + 2zp'(z) = zf''(z)$ and therefore, the desired conclusion follows from Lemma 2.5, since $F(1,2;3;-1) = 2(1-\log 2)$. ■

REMARK. From [1], we recall that if $\operatorname{Re} zf''(z) > -\beta$ for $0 < \beta \leq 1/\log 4 \approx 0.721348$, then $f \in \mathcal{S}^*$. We observe that $\mathcal{S}^*(1/2) \subsetneq \mathcal{P}(1/2)$. From Theorem 2.6, it follows that if $f \in \mathcal{A}$ satisfies the differential inequality

$$\operatorname{Re}(z^2 f'''(z) + 2zf''(z)) > -\beta, \tag{2.12}$$

then $\operatorname{Re} f'(z) > 0$ whenever $0 < \beta \leq 1/[4\log 2 - 2] = \beta_0 \approx 1.29435$. It is interesting to recall that if $f \in \mathcal{A}$ satisfies (2.12) then f is convex whenever

$$0 < \beta \leq \beta_{\mathrm{c}} = 1/\log 4.$$

Note that $\beta_0 > \beta_{\mathrm{c}}$ and we know that a convex function $f \in \mathcal{A}$ does not necessarily satisfy $\operatorname{Re} f'(z) > 0$ for $z \in \Delta$, and conversely, a function f satisfying the last condition does not always have the convexity property. Indeed, even the assumption that $|f'(z) - 1| < \lambda$ in Δ does not necessarily imply that f is starlike unless $\lambda \leq 2/\sqrt{5}$ (see [3, 9]).

Our next result, which is of independent interest, is a reformulated version of a result from [6] in our setting.

THEOREM 2.7. *Let $f \in \mathcal{A}_n = \{f \in \mathcal{A} : f(z) = z + a_{n+1}z^{n+1} + \cdots\}$ satisfy the condition*

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} - 1\right| < \lambda \quad (\lambda > 0)$$

and let

$$g(z)=\int_0^z\left(\frac{\zeta}{f(\zeta)}\right)^{\mu}d\zeta.$$

(i) *For* $0<\mu<n$,

$$g\in\mathcal{R}\left(1-\frac{\lambda\mu}{n-\mu}\right).$$

In particular, $\operatorname{Re} g'(z)>0$ *whenever* $0<\mu\le n/(1+\lambda)$.

(ii) *For* $\mu=n$,

$$g\in\mathcal{R}\left(1-\frac{n|f^{(n+1)}(0)|}{(n+1)!}-n\lambda\right).$$

In particular,

$$\operatorname{Re} g'(z)>0\quad\textit{whenever}\quad 0<\lambda\le\frac{1}{n}-\frac{|f^{(n+1)}(0)|}{(n+1)!}.$$

Proof. For $\mu\in(0,n)$ and $f(z)\neq 0$ in $0<|z|<1$, we see that $g'(z)=(z/f(z))^{\mu}$ and

$$zg''(z)=\mu\left(\frac{z}{f(z)}\right)^{\mu-1}\left[-\left(\frac{z}{f(z)}\right)^{2}f'(z)+\frac{z}{f(z)}\right]$$

so that

$$g'(z)-\frac{1}{\mu}\,zg''(z)=\left(\frac{z}{f(z)}\right)^{\mu+1}f'(z).$$

By hypothesis, we can write

$$g'(z)-\frac{1}{\mu}\,zg''(z)=1+\lambda w(z) \tag{2.13}$$

where $w\in\mathcal{B}_n$. Suppose that $g'(z)=1+\sum_{k=n}^{\infty}p_kz^k$ and $w(z)=\sum_{k=n}^{\infty}b_kz^k$. Then

$$g'(z)-\frac{1}{\mu}\,zg''(z)=1+\sum_{k=n}^{\infty}\left(1-\frac{k}{\mu}\right)p_kz^k.$$

A comparison of the coefficient of z^k on both sides of (2.13) shows that

$$\left(1-\frac{k}{\mu}\right)p_k=\lambda b_k\quad(k\ge n)$$

so that

$$g'(z)=1+\lambda\sum_{k=n}^{\infty}\frac{b_k}{1-k/\mu}\,z^k.$$

Since $0<\mu<n$, we can rewrite the last equality in integral form

$$g'(z)=1-\lambda\int_1^{\infty}w(t^{-1/\mu}z)\,dt$$

and therefore (using $|w(z)| \le |z|^n$ for $z \in \Delta$), it follows that

$$|g'(z) - 1| < \lambda \int_1^\infty t^{-n/\mu}\, dt = \frac{\lambda\mu}{n-\mu},$$

which gives the required conclusion. In particular, for $0 < \mu \le n/(1+\lambda)$, we have $\operatorname{Re} g'(z) > 0$ for $z \in \Delta$.

For the case $\mu = n$, proceeding as above but with $w(z) = \sum_{k=n+1}^\infty b_k z^k$, we get the required result. ∎

THEOREM 2.8. *Let $f \in \mathcal{A}$, $0 < \alpha \le 1$, and $\lambda > (1-\alpha)\sin(\pi\alpha/2)$. Suppose that $f'(z)f(z)/z \ne 0$ on Δ and*

$$\left| \operatorname{Im}\left[\lambda \frac{zf''(z)}{f'(z)} + (1-\lambda)\frac{zf'(z)}{f(z)}\right]\right| < \beta(\alpha,\lambda), \tag{2.14}$$

where

$$\beta(\alpha,\lambda) = \frac{\lambda}{2}\left[(\alpha+1)\frac{1}{t_0} + (\alpha-1)t_0\right]$$

and t_0 is the pointwise solution of the equation

$$2t^{1+\alpha}\sin(\alpha\pi/2) - \lambda(1-t^2) = 0.$$

Then $f \in \mathcal{S}^*_\alpha$.

Proof. Define

$$\frac{zf'(z)}{f(z)} = \left(\frac{1+w(z)}{1-w(z)}\right)^\alpha. \tag{2.15}$$

It suffices to prove that $|w(z)| < 1$ for $z \in \Delta$. Logarithmic differentiation of (2.15) gives

$$1 + \frac{zf''(z)}{f'(z)} = \left(\frac{1+w(z)}{1-w(z)}\right)^\alpha + \alpha\frac{2zw'(z)}{1-w^2(z)}$$

and therefore,

$$\lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\lambda)\frac{zf'(z)}{f(z)} = \left(\frac{1+w(z)}{1-w(z)}\right)^\alpha + \alpha\lambda\frac{2zw'(z)}{1-w^2(z)}. \tag{2.16}$$

Suppose it is not true that $|w(z)| < 1$, $z \in \Delta$. Then there exists a $z_0 \in \Delta$ such that $|w(z_0)| = 1$ and, by Jack's well known lemma, $z_0 w'(z_0) = kw(z_0)$ with $k \ge 1$. If we put $w(z_0) = e^{i\theta}$, then from (2.16), we obtain

$$\begin{aligned}\lambda\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) + (1-\lambda)\frac{z_0 f'(z_0)}{f(z_0)} &= \left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^\alpha + \alpha\lambda\frac{2ke^{i\theta}}{1-e^{2i\theta}} \\ &= (i\cot(\theta/2))^\alpha + i\frac{\lambda k\alpha}{\sin\theta}.\end{aligned} \tag{2.17}$$

We consider first the case $0 < \theta < \pi$. Then taking the imaginary part on both sides of (2.17), we get

$$\begin{aligned}\mathrm{Im}\left(\lambda\,\frac{z_0 f''(z_0)}{f'(z_0)} + (1-\lambda)\,\frac{z_0 f'(z_0)}{f(z_0)}\right) &= \cot^\alpha(\theta/2)\sin(\alpha\pi/2) + \frac{\alpha\lambda k}{\sin\theta} \\ &\geq \cot^\alpha(\theta/2)\sin(\alpha\pi/2) + \frac{\alpha\lambda}{\sin\theta} \\ &= t^\alpha \sin(\alpha\pi/2) + \frac{\alpha\lambda}{2}\left(t + \frac{1}{t}\right) \\ &=: g(t), \qquad \text{where } t = \cot(\theta/2) > 0.\end{aligned}$$

We have

$$g'(t) = \alpha t^{\alpha-1}\sin(\alpha\pi/2) + \alpha\lambda/2 - \alpha\lambda/(2t^2)$$

and

$$g''(t) = \alpha(\alpha-1)t^{\alpha-2}\sin(\alpha\pi/2) + \alpha\lambda/t^3 = \frac{\alpha}{t^3}\,[(\alpha-1)t^{1+\alpha}\sin(\alpha\pi/2) + \lambda].$$

Since $\lim_{t\to 0+} g'(t) = -\infty$, $g'(1) = \alpha\sin(\alpha\pi/2) > 0$ and $g''(t) > 0$ for $0 < t \leq 1$ and $\lambda > (1-\alpha)\sin(\pi\alpha/2)$, we conclude that the function $g(t)$ attains its minimum

$$\beta(\alpha,\lambda) = g(t_0) = \frac{1}{2}[(\alpha+1)/t_0 + (\alpha-1)t_0],$$

where $t_0 \in (0,1)$ is the smallest positive root of the equation $g'(t) = 0$, i.e.

$$2t^{1+\alpha}\sin(\alpha\pi/2) + \lambda t^2 - \lambda = 0.$$

Thus

$$\mathrm{Im}\left(\lambda\,\frac{z_0 f''(z_0)}{f'(z_0)} + (1-\lambda)\,\frac{z_0 f'(z_0)}{f(z_0)}\right) \geq \beta(\alpha,\lambda).$$

Similarly, for $-\pi < \theta < 0$, we obtain

$$\mathrm{Im}\left(\lambda\,\frac{z_0 f''(z_0)}{f'(z_0)} + (1-\lambda)\,\frac{z_0 f'(z_0)}{f(z_0)}\right) \leq -\beta(\alpha,\lambda).$$

A combination of these two inequalities shows that

$$\left|\mathrm{Im}\left(\lambda\,\frac{z_0 f''(z_0)}{f'(z_0)} + (1-\lambda)\,\frac{z_0 f'(z_0)}{f(z_0)}\right)\right| \geq \beta(\alpha,\lambda),$$

which contradicts the assumption of the theorem.

So, $|w(z)| < 1$ for $z \in \Delta$, and from (2.15), this is equivalent to the assertion that $f \in \mathcal{S}^*_\alpha$. ∎

For $\lambda = 1$, we have

Corollary 2.9. *Let $f \in \mathcal{A}$ be such that $f'(z)f(z)/z \neq 0$ on Δ and*

$$\left|\mathrm{Im}\,\frac{zf''(z)}{f'(z)}\right| < \beta(\alpha), \quad z \in \Delta,$$

where $0 < \alpha \le 1$,

$$\beta(\alpha) = \frac{1}{2}\left[(\alpha+1)\frac{1}{t_0} + (\alpha-1)t_0\right]$$

and t_0 *is the pointwise solution of the equation*

$$2t^{1+\alpha}\sin(\alpha\pi/2) - (1-t^2) = 0.$$

Then $f \in \mathcal{S}^*_\alpha$.

Example 2.1. For $\alpha = 1$, we have the equation $(2+\lambda)t^2 - \lambda = 0$ with positive root $t_0 = \sqrt{\lambda/(2+\lambda)}$ and $\beta(1,\lambda) = \sqrt{\lambda(2+\lambda)}$. Now, we have the following implication (see [4, p. 115]) for $f \in \mathcal{A}$ with $f'(z)f(z)/z \neq 0$ on Δ:

$$\left|\mathrm{Im}\left[\lambda\frac{zf''(z)}{f'(z)} + (1-\lambda)\frac{zf'(z)}{f(z)}\right]\right| < \sqrt{\lambda(2+\lambda)} \;\Rightarrow\; \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2},$$

i.e. $f \in \mathcal{S}^*$.

A simple computation shows that $\beta(\alpha,\lambda)$ in Theorem 2.8 is larger than $\alpha\lambda$, and $\beta(\alpha,\lambda)$ is independent of the root t_0 of the appropriate equation. Namely, if we let

$$\phi(t) := \beta(\alpha,\lambda) = \frac{\lambda}{2}\left[(\alpha+1)/t + (\alpha-1)t\right]$$

then

$$\phi'(t_0) = \frac{\lambda}{2t_0^2}\left[-(\alpha+1) + (\alpha-1)t_0^2\right] = \frac{1}{2t_0^2}\left[(t_0^2-1)\alpha - (1+t_0^2)\right] < 0,$$

since $0 < t_0 < 1$, $0 < \alpha \le 1$ and $\lambda > 0$. It means that $\phi(t)$ is a decreasing function of $t_0 \in [0,1]$ and we have

$$\phi(t_0) > \phi(1) = \alpha\lambda.$$

References

[1] R. M. Ali, S. Ponnusamy and V. Singh, *Starlikeness of functions satisfying a differential inequality*, Ann. Polon. Math. 61 (1995), 135–140.

[2] P. N. Chichra and R. Singh, *Complex sum of univalent functions*, J. Austral. Math. Soc. 14 (1972), 503–507.

[3] R. Fournier, *On integrals of bounded analytic functions in the closed unit disc*, Complex Var. Theory Appl. 11 (1989), 125–133.

[4] S. Ponnusamy, *Some applications of differential subordination and convolution techniques to univalent functions theory*, Ph.D. thesis, I.I.T. Kanpur, 1988.

[5] S. Ponnusamy and S. Sabapathy, *Polylogarithms in the theory of univalent functions*, Results Math. 30 (1996), 136–150.

[6] S. Ponnusamy and P. Sahoo, *Geometric properties of certain linear integral transforms*, Bull. Belg. Math. Soc. Simon Stevin 12 (2005), 95–108.

[7] S. Ponnusamy and V. Singh, *Criteria for univalent, starlike and convex functions*, ibid. 9 (2002), 511–531. (Also Preprint 265, 2001, Department of Mathematics, University of Helsinki.)

[8] R. Singh and S. Paul, *Linear sums of certain analytic functions*, Proc. Amer. Math. Soc. 99 (1987), 719–725.

[9] V. Singh, *Univalent functions with bounded derivative in the unit disc*, Indian J. Pure Appl. Math. 8 (1977), 1370–1377.

[10] —, *On some problems of Mocanu type*, Indian J. Pure Appl. Math. 32) (2001), 1859–1867.

[11] S. Y. Trimble, *The convex sums of convex functions*, Math Z. 109 (1969), 112–114.

[12] K.-J.Wirths, *Bemerkungen zu einem Satz von Fejér*, Anal. Math. 1 (1975), 313–318.

Department of Mathematics
Faculty of Technology and Metallurgy
4 Karnegijeva St.
11000 Belgrad
Serbia and Montenegro
E-mail: obrad@elab.tmf.bg.ac.yu

Department of Mathematics
Indian Institute of Technology Madras
Chennai-600 036, India
E-mail: samy@iitm.ac.in
vasu2kk@yahoo.com

Reçu par la Rédaction le 17.10.2003
Révisé le 30.6.2005 (1475)