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Uniqueness of meromorphic functions sharing three values

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Abstract. We prove a result on the uniqueness of meromorphic functions sharing three values with weights and as a consequence of this result we improve a recent result of W. R. Lü and H. X. Yi.

1. Introduction, definitions and results. Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we say that f and g *share the value* a CM (counting multiplicities) if f and g have the same set of a-points with the same multiplicities. If we do not take the multiplicities into account, we say that f, g *share the value* a IM (ignoring multiplicities). For the standard notations and definitions of the value distribution theory we refer to [1].

We denote by $N(r,a;f\,|\,{\le}k)$ the counting function of a-points of f with multiplicities not exceeding k, where $a \in \mathbb{C} \cup \{\infty\}$ and k is a positive integer or infinity. Also we define

$$\delta_{k)}(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f\,|\,{\le}k)}{T(r,f)}.$$

In this paper I denotes a set of nonnegative real numbers of infinite linear measure, not necessarily the same in each of its occurrences.

In 1976 M. Ozawa [8] proved the following result.

Theorem A. *Let f and g be two nonconstant entire functions of finite order sharing* 0, 1 *CM. If* $\delta(0;f) > 1/2$ *then either* $f \equiv g$ *or* $fg \equiv 1$.

Improving Theorem A, H. Ueda [9] proved the following result.

Theorem B. *Let f and g be two nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. If*

$$\limsup_{r\to\infty} \frac{N(r,0;f) + N(r,\infty;f)}{T(r,f)} < \frac{1}{2}$$

then either $f \equiv g$ *or* $fg \equiv 1$.

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In 1990 H. X. Yi [10] further improved Theorem B as follows:

Theorem C. *Let f and g be two nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. If*

$$N(r,0;f\,|\leq 1)+N(r,\infty;f\,|\leq 1)<\{\lambda+o(1)\}T(r)$$

for $r\in I$, where $0<\lambda<1/2$ and $T(r)=\max\{T(r,f),T(r,g)\}$, then either $f\equiv g$ or $fg\equiv 1$.

Recently W. R. Lü and H. X. Yi [7] investigated the situation when the bound $1/2$ in the above theorems is replaced by 1 and proved the following result.

Theorem D. *Let f and g be two distinct nonconstant meromorphic functions sharing* 0, 1, ∞ *CM. If*

$$\limsup_{r\to\infty,\, r\in I}\frac{N(r,0;f\,|\leq 1)+N(r,\infty;f\,|\leq 1)}{T(r,f)}<1$$

then

$$f=\frac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1}\quad \textit{and}\quad g=\frac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1},$$

where s and k are relatively prime positive integers with $1\leq s\leq k$ and γ is a nonconstant entire function.

Considering $f=(e^{\gamma}-1)^2$ and $g=e^{\gamma}-1$, where γ is a nonconstant entire function, we see that in Theorem D it is not possible to relax the nature of sharing the value 0 from CM to IM. So one may naturally ask: *Is it possible in Theorem D to relax the nature of sharing the value* 0*?*

In this paper we answer this question with the help of the notion of weighted sharing of values which measures how close a shared value is to being shared CM or to being shared IM.

Definition 1.1 ([2, 3]). Let k be a nonnegative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m\leq k$ and $k+1$ times if $m>k$. If $E_k(a;f)=E_k(a;g)$, we say that f, g *share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f-a$ with multiplicity m $(\leq k)$ if and only if it is a zero of $g-a$ with multiplicity m $(\leq k)$, and z_0 is a zero of $f-a$ with multiplicity m $(>k)$ if and only if it is a zero of $g-a$ with multiplicity $n\,(>k)$ where m is not necessarily equal to n.

We write f, g *share* (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for all integers p with $0\leq p<k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

We prove the following result which enables us to improve Theorem D.

THEOREM 1.1. *Let f and g be two distinct nonconstant meromorphic functions sharing $(0,1)$, $(1,m)$, (∞,k), where*

$$(m-1)(mk-1) > (1+m)^2$$

and

$$\limsup_{r\to\infty,\, r\in I} \frac{N(r,0;f\,|\,{\le}1) + N(r,\infty;f\,|\,{\le}1)}{T(r,f)} < 1. \tag{1.1}$$

Then f and g satisfy the following relations:

$$\left(1+\frac{\alpha}{f}-\frac{1}{f}\right)^s \equiv \alpha^{s+t}, \tag{1.2}$$

$$\left(1+\frac{1}{g\alpha}-\frac{1}{g}\right)^s \equiv \alpha^{-(s+t)}, \tag{1.3}$$

where α is a nonconstant meromorphic function such that $\overline{N}(r,0;\alpha) + \overline{N}(r,\infty;\alpha) = S(r,f)$ and s, t are relatively prime nonzero integers with $s > 0$ and $s+t \neq 0$.

The following corollary improves Theorem D.

COROLLARY 1.1. *The assertion of Theorem D holds if f and g share $(0,1)$, $(1,\infty)$, (∞,∞).*

Considering the example mentioned earlier we can easily verify that in Corollary 1.1 sharing $(0,1)$ cannot be relaxed to sharing $(0,0)$.

2. Lemmas. In this section we present some lemmas which are required to prove the theorem and the corollary.

LEMMA 2.1 ([4]). *Let f and g be two nonconstant meromorphic functions sharing $(0,0)$, $(1,0)$, $(\infty,0)$. Then*

$$T(r,f) \le 3T(r,g) + S(r,f) \quad \textit{and} \quad T(r,g) \le 3T(r,f) + S(r,g).$$

Hence it follows that $S(r,f) = S(r,g)$ and we denote them by $S(r)$.

LEMMA 2.2 ([4]). *Let f and g be two distinct nonconstant meromorphic functions sharing $(0,1)$, $(1,m)$, (∞,k), where $(m-1)(mk-1) > (1+m)^2$. Then for $a = 0, 1, \infty$,*

(i) $\overline{N}(r,a;f\,|\,{\ge}2) = S(r)$,
(ii) $\overline{N}(r,a;g\,|\,{\ge}2) = S(r)$,

where $\overline{N}(r,a;f\,|\,{\ge}2)$ denotes the reduced counting function of multiple a-points of f.

Lemma 2.3 ([6]). *Let f and g be two distinct nonconstant meromorphic functions sharing $(0,0)$, $(1,0)$, $(\infty,0)$. If f is a bilinear transformation of g then f and g satisfy one of the following*:

(i) $fg \equiv 1$,
(ii) $(f-1)(g-1) \equiv 1$,
(iii) $f+g \equiv 1$,
(iv) $f \equiv cg$,
(v) $f-1 \equiv c(g-1)$,
(vi) $[(c-1)f+1][(c-1)g-c]+c \equiv 0$, *where c $(\neq 0,1)$ is a constant.*

Lemma 2.4 ([11]). *Let f_1 and f_2 be nonconstant meromorphic functions satisfying $\overline{N}(r,0;f_i)+\overline{N}(r,\infty;f_i)=S_0(r)$ for $i=1,2$. Then either $\overline{N}_0(r,1;f_1,f_2)=S_0(r)$ or there exist two integers s,t $(|s|+|t|>0)$ such that $f_1^s f_2^t \equiv 1$, where $\overline{N}_0(r,1;f_1,f_2)$ denotes the reduced counting function of f_1 and f_2 related to the common 1-points and $T(r)=T(r,f_1)+T(r,f_2)$, $S_0(r)=o(T(r))$ as $r\to\infty$ possibly outside a set of finite linear measure.*

Lemma 2.5 ([5]). *Let f and g be two distinct nonconstant meromorphic functions sharing $(0,1)$, $(1,m)$, (∞,k), where $(m-1)(mk-1) > (1+m)^2$. If $\alpha=(f-1)/(g-1)$ and $h=f/g$ then $\overline{N}(r,a;\alpha)=S(r)$ and $\overline{N}(r,a;h)=S(r)$ for $a=0,\infty$.*

Lemma 2.6 ([4]). *Let f and g be two nonconstant meromorphic functions sharing $(0,1)$, $(1,m)$, (∞,k), where $(m-1)(mk-1)>(1+m)^2$. If*

$$2\delta_{1)}(0;f)+2\delta_{1)}(\infty;f)+\min\Big\{\sum_{a\neq 0,1,\infty}\delta_{2)}(a;f),\sum_{a\neq 0,1,\infty}\delta_{2)}(a;g)\Big\}>3$$

then either $f\equiv g$ or $fg\equiv 1$. If f has at least one zero or pole then the case $fg\equiv 1$ does not arise.

Lemma 2.7 ([5]). *Let f and g be two distinct nonconstant meromorphic functions sharing $(0,1)$, $(1,m)$ and (∞,k), where $(m-1)(mk-1)>(1+m)^2$. If f is not a bilinear transformation of g then each of the following holds*:

(i) $$T(r,f)+T(r,g)=N(r,0;g\,|\leq 1)+N(r,1;g\,|\leq 1)+N(r,\infty;g\,|\leq 1)+N_0(r)+S(r),$$

(ii) $$T(r,f)+T(r,g)=N(r,0;f\,|\leq 1)+N(r,1;f\,|\leq 1)+N(r,\infty;f\,|\leq 1)+N_0(r)+S(r),$$

where $N_0(r)$ denotes the counting function of those simple zeros of f–g which are not the zeros of $g(g-1)$, $1/g$ and so are not the zeros of $f(f-1)$, $1/f$.

Lemma 2.8 ([11]). *Let s and t be relatively prime integers with $s>0$. Then x^s-1 and x^t-c have one and only one common factor, where c is a constant satisfying $c^s=1$.*

3. Proofs of the theorem and the corollary

Proof of Theorem 1.1. Let $\alpha = (f-1)/(g-1)$ and $h = f/g$. Then clearly $\alpha \not\equiv 1$ and $h \not\equiv 1$. Also we get

$$f = h\,\frac{1-\alpha}{h-\alpha} \quad \text{and} \quad g = \frac{1-\alpha}{h-\alpha}. \tag{3.1}$$

We now consider the following cases.

Case I. Let $\delta_{1)}(0;f) + \delta_{1)}(\infty;f) > 3/2$. Then by Lemma 2.6 we get $fg \equiv 1$ and so (1.2) and (1.3) hold for $s = 1$, $t = -2$ and $\alpha = e^{\beta}$, where β is a nonconstant entire function.

Case II. Let $\delta_{1)}(0;f) + \delta_{1)}(\infty;f) \le 3/2$. If possible, suppose that f is a bilinear transformation of g. Then the possibilities (i)–(vi) of Lemma 2.3 will occur.

If $fg \equiv 1$ then $0, \infty$ are exceptional values of f in the sense of Picard (evP) and so $\delta_{1)}(0;f) + \delta_{1)}(\infty;f) = 2$, which is a contradiction.

If $(f-1)(g-1) \equiv 1$ then $1, \infty$ are evP of f. So by the second fundamental theorem and Lemma 2.2 we get

$$\begin{aligned} T(r,f) &\le \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,1;f) + S(r,f) \\ &= N(r,0;f\,|\,{\le}1) + S(r,f), \end{aligned}$$

which contradicts (1.1).

If $f+g \equiv 1$ then $0, 1$ are evP of f. So by the second fundamental theorem and Lemma 2.2 we get

$$T(r,f) \le N(r,\infty;f\,|\,{\le}1) + S(r,f),$$

which contradicts (1.1).

If $f \equiv cg$ then $1, c$ are evP of f. So by the second fundamental theorem and Lemma 2.2 we get

$$\begin{aligned} T(r,f) &\le \overline{N}(r,1;f) + \overline{N}(r,c;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &= N(r,\infty;f\,|\,{\le}1) + S(r,f), \end{aligned}$$

which contradicts (1.1).

If $f-1 \equiv c(g-1)$ then $0, 1-c$ are evP of f. So by the second fundamental theorem and Lemma 2.2 we get

$$T(r,f) \le N(r,\infty;f\,|\,{\le}1) + S(r,f),$$

which contradicts (1.1).

If $[(c-1)f+1][(c-1)g-c] + c \equiv 0$ then $\infty, 1/(1-c)$ are evP of f. So by the second fundamental theorem and Lemma 2.2 we get

$$T(r,f) \le N(r,0;f\,|\,{\le}1) + S(r,f),$$

which contradicts (1.1).

Therefore f is not a bilinear transformation of g. Noting that f, g share $(1, m)$, it follows from Lemma 2.7(ii) that

$$T(r,f) \le N(r,0;f\,|\le 1) + N(r,\infty;f\,|\le 1) + N_0(r) + S(r)$$

and so by (1.1) we get $N_0(r) \ne S(r)$.

Again since $T(r,\alpha) + T(r,h) \le 2T(r,f) + 2T(r,g) + O(1)$ and $N_0(r) \le \overline{N}_0(r,1;\alpha,h)$, it follows from Lemma 2.4 that there exist integers s and t ($|s| + |t| > 0$) such that $\alpha^t h^s \equiv 1$. Without loss of generality we may assume that $s > 0$ and s, t are relatively prime. Since f is not a bilinear transformation of g, we see that $t \ne 0$ and $s + t \ne 0$. Now from (3.1) we get $h^s(f - 1 + \alpha)^s \equiv \alpha^s f^s$ and $h^s g^s \equiv (\alpha g + 1 - \alpha)^s$. Since $\alpha^t h^s \equiv 1$, we can deduce (1.2) and (1.3). Since f and g are nonconstant, clearly α is nonconstant. Also by Lemma 2.5 we get $\overline{N}(r,0;\alpha) + \overline{N}(r,\infty;\alpha) = S(r,f)$. This proves the theorem. ∎

Proof of Corollary 1.1. Since f, g share $(1,\infty)$, (∞,∞), we can put $\alpha = (f-1)/(g-1) = e^{\beta}$, where β is a nonconstant entire function. Then from (1.2) and (1.3) we get

$$f = \frac{e^{\gamma s} - 1}{e^{\gamma(s+t)} - 1} \quad \text{and} \quad g = \frac{e^{-\gamma s} - 1}{e^{-\gamma(s+t)} - 1}, \quad \text{where} \quad \beta = \gamma s.$$

We now consider the following cases.

Case I. Let $t > 0$ so that $s + t \ge 2$. Since s, t are relatively prime and so are s, $s + t$, by Lemma 2.8 we get

$$\begin{aligned} T(r,f) &= (s+t-1)T(r,e^{\gamma}) + S(r), \\ N(r,\infty;f\,|\le 1) &= (s+t-1)T(r,e^{\gamma}) + S(r), \\ N(r,0;f\,|\le 1) &= (s-1)T(r,e^{\gamma}) + S(r). \end{aligned}$$

Then

$$\limsup_{r\to\infty,\, r\in I} \frac{N(r,0;f\,|\le 1) + N(r,\infty;f\,|\le 1)}{T(r,f)} = 1 + \frac{s-1}{s+t-1} \ge 1,$$

which contradicts the given condition.

Case II. Let $t < 0$. If $s + t = 1$ then $s \ge 2$ and

$$f = \frac{e^{s\gamma} - 1}{e^{\gamma} - 1} = 1 + e^{\gamma} + e^{2\gamma} + \cdots + e^{(s-1)\gamma}.$$

Hence $T(r,f) = (s-1)T(r,e^{\gamma}) + S(r)$, $N(r,0;f\,|\le 1) = (s-1)T(r,e^{\gamma}) + S(r)$ and $N(r,\infty;f\,|\le 1) \equiv 0$. Therefore

$$\limsup_{r\to\infty,\, r\in I} \frac{N(r,0;f\,|\le 1) + N(r,\infty;f\,|\le 1)}{T(r,f)} = 1,$$

which contradicts the given condition.

Let $s+t \geq 2$. Then as in Case I we arrive at a contradiction. Therefore $s+t \leq -1$. We now put $k=-1-t$. Then $k>0$ and $k-s=-t-1-s \geq 0$ so that $1 \leq s \leq k$. Also s and $1+k$ are relatively prime because s and t are so. Therefore we get

$$f = \frac{e^{s\gamma}-1}{e^{-(k+1-s)\gamma}-1} \quad \text{and} \quad g = \frac{e^{-s\gamma}-1}{e^{(k+1-s)\gamma}-1}.$$

This proves the corollary. ∎

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