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A new approach to the existence results for orientor fields with Nicoletti's boundary conditions

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Abstract. Applying a global bifurcation theorem for convex-valued completely continuous mappings we prove some existence theorems for convex-valued differential inclusions of the form $x' \in F(t,x)$, where x satisfies the Nicoletti boundary conditions.

1. Introduction. In this paper we consider the Nicoletti boundary value problem for the first order differential inclusions

$$\begin{cases} x'(t) \in F(t,x(t)) & \text{for a.e. } t \in (a,b), \\ l(x) = 0, \end{cases} \tag{1.1}$$

where $F\colon [a,b]\times\mathbb{R}^k \to \mathrm{cf}(\mathbb{R}^k)$ is a convex-valued mapping and $l : C([a,b],\mathbb{R}^k) \to \mathbb{R}^k$ is given by

$$l(x_1,\dots,x_k) = (x_1(a), x_2(b), x_3(t_3), \dots, x_k(t_k)), \text{where } t_3,\dots,t_k \in [a,b].$$

The Nicoletti single-valued and multi-valued boundary value problem has been considered by several authors (see for instance [5], [6], [10]). In these papers it is assumed that the mapping F satisfies the Carathéodory conditions and the inequality

$$|F(t,x)| \le p(t)|x| + q(t) \quad \text{for } x \in \mathbb{R}^k \text{ and } t \in [a,b], \tag{1.2}$$

where $p,q : [a,b] \to \mathbb{R}_+$ are integrable functions, and p satisfies the inequality

$$\int_a^b p(t)\,dt < \frac{\pi}{2}. \tag{1.3}$$

Lasota and Olech [6] considered the single-valued Nicoletti problem. They showed that for F satisfying (1.2) the condition (1.3) is the best possible sufficient condition for the existence of solution of (1.1).

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Our assumptions refer to the behaviour of $F(t,x)$ for $|x|$ close to 0 and to ∞. The main tool we use is a global bifurcation theorem for convex-valued completely continuous mappings. In Section 2 we state the main existence theorems. In Section 3 we give auxiliary lemmas, and transform the Nicoletti boundary value problem to appropriate second order boundary value problems. Finally, in Section 4 we prove the existence theorems.

2. Main theorems. Let E be a real Banach space. We denote by $\mathrm{cf}(E)$ the family of all non-empty, closed, bounded and convex subsets of E, and by $\mathrm{D}(A,B)$ the Hausdorff distance between $A,B\in\mathrm{cf}(E)$. In particular we put $|A|=\mathrm{D}(A,\{0\})$.

Let X be a closed non-empty subset of E. A multi-valued mapping $\Phi: X\to\mathrm{cf}(E)$ is called *upper semicontinuous* (u.s.c.) if for each open set $U\subset E$ the set $\{x\in X:\Phi(x)\subset U\}$ is open in X.

Let $I\subset\mathbb{R}$ be a closed interval. A multi-valued mapping $\Phi: I\to\mathrm{cf}(\mathbb{R}^k)$ is called *measurable* if for every open set $U\subset\mathbb{R}^k$ the set $\{x\in I:\Phi(x)\cap U\neq\emptyset\}$ is Lebesgue measurable.

For $x=(x_1,\ldots,x_k)\in\mathbb{R}^k$ we write $|x|=(\sum_{i=1}^k x_i^2)^{1/2}$, and let $\Pi_i:\mathbb{R}^k\to\mathbb{R}$ be the linear projection given by $\Pi_i(x_1,\ldots,x_k)=x_i$ for $i=1,\ldots,k$.

Recall that the multi-valued mapping $F:[a,b]\times\mathbb{R}^k\to\mathrm{cf}(\mathbb{R}^k)$ satisfies the *Carathéodory conditions* if:

(i) for each $x\in\mathbb{R}^k$ the mapping $F(\cdot,x)$ is measurable;
(ii) for each $t\in[a,b]$ the mapping $F(t,\cdot)$ is u.s.c.;
(iii) for each $R>0$ there exists a function $m_R\in L^1(a,b)$ such that for each $x\in\mathbb{R}^k$ with $|x|\le R$ we have $|F(t,x)|\le m_R(t)$ a.e. on $[a,b]$.

THEOREM 1. *Assume that $F:[a,b]\times\mathbb{R}^k\to\mathrm{cf}(\mathbb{R}^k)$ satisfies the Carathéodory conditions and*

(2.1) *there exists $\delta>0$ and an integrable function $\psi:[a,b]\to\mathbb{R}_+$ such that*

$$|F(t,x)|\le\psi(t)|x|\quad \textit{for } t\in[a,b],\ |x|\le\delta\quad\textit{and}\quad\int_a^b\psi(t)\,dt<\frac{\pi}{2};$$

(2.2) *for every $\varepsilon>0$ there exists $R_0>0$ such that*

$$\mathrm{D}((\Pi_1\circ F)(t,x),\{x_2\})+\mathrm{D}((\Pi_2\circ F)(t,x),\{-M|x_1|\})\le\varepsilon|x|$$

for $t\in[a,b]$, $|x_1|+|x_2|\ge R_0$ and $M>(\pi/2(b-a))^2$;

(2.3) *there exists $R_1>0$ and integrable functions $\psi_i:[a,b]\to\mathbb{R}_+$ with*

$$|(\Pi_i\circ F)(t,x)|\le\psi_i(t)|x_i|\quad \textit{for } t\in[a,b],\ |x|\ge R_1\ \textit{and } i=3,\ldots,k.$$

Then there exists a non-trivial solution of the boundary value problem (1.1).

THEOREM 2. *Assume that $F:[a,b]\times\mathbb{R}^k\to\mathrm{cf}(\mathbb{R}^k)$ satisfies the Carathéodory conditions and conditions* (2.1), (2.2), *and*

(2.4) *there exists* $R_1 > 0$ *and an integrable function* $\psi : [a,b] \to \mathbb{R}_+$ *such that*

$$\Big(\sum_{i=3}^{k} |(\Pi_i \circ F)(t,x)|^2\Big)^{1/2} \le \psi(t)\Big(\sum_{i=3}^{k} x_i^2\Big)^{1/2} \quad \textit{for } t \in [a,b],\ |x| \ge R_1$$

and $\int_a^b \psi(t)\,dt < \pi/2$.

Then there exists a non-trivial solution of the boundary value problem (1.1).

3. Auxiliary lemmas. Let $\psi : [a,b] \to \mathbb{R}_+$ satisfy (2.1). Then there exists $\alpha > 1$ such that $\alpha \int_a^b \psi(t)\,dt < \pi/2$. Let $q_1, q_2 : (0,\infty) \to [0,1]$ be a continuous partition of unity subordinate to the open cover $\{(0,\alpha),(\beta,\infty)\}$ of the interval $(0,\infty)$, where $1 < \beta < \alpha$, and let $p : \mathbb{R}^k \to \mathbb{R}^k$ be given by $p(x_1, x_2, \ldots, x_k) = (-x_2, M|x_1|, 0, \ldots, 0)$.

We start with the following scalar boundary value problem:

$$\text{(3.1)} \qquad \begin{cases} x''(t) + \lambda x(t) = 0, \\ x(a) = x'(b) = 0. \end{cases}$$

It is well known (cf. [4]) that there exists exactly one eigenvalue $\mu_0 = (\pi/2(b-a))^2$ of (3.1) for which there exists an eigenvector $x_0 : [a,b] \to \mathbb{R}$ such that $x_0(t) > 0$ for $t \in (a,b)$.

The following fact is a consequence of the properties of Green's function (cf. [2]–[4]).

(3.2) If $\mu > (\pi/2(b-a))^2$, $\eta > 0$ and $x_0 : [a,b] \to \mathbb{R}$ is as above then the scalar problem

$$\begin{cases} x''(t) + \mu x(t) + \eta x_0(t) = 0, \\ x(a) = x'(b) = 0, \\ x(t) \ge 0, \end{cases}$$

has no solutions.

Let $x^0 : [a,b] \to \mathbb{R}^k$ be given by $x^0(t) = (0, x_0(t), 0, \ldots, 0)$. We now associate with the problem (1.1) the following two-parameter family of boundary value problems:

$$(2_{(\lambda,\tau)}) \qquad \begin{cases} x'(t) \in \lambda\tau q_1(\lambda) F(t,x(t)) - \lambda q_2(\lambda) p(x(t)) - (1-\tau) q_2(\lambda) x^0(t) \\ \hfill \text{for a.e. } t \in (a,b), \\ l(x) = 0, \end{cases}$$

for $\lambda \in (0,\infty)$ and $\tau \in [0,1]$. An absolutely continuous function $x : [a,b] \to \mathbb{R}^k$ satisfying $(2_{(\lambda,\tau)})$ is called a *solution* of $(2_{(\lambda,\tau)})$. For every pair $(\lambda,\tau) \in (0,\infty) \times [0,1]$ denote by $S_{(\lambda,\tau)}$ the set of all solutions of the problem $(2_{(\lambda,\tau)})$. In what follows we will make use of the following fact.

(3.3) (Lasota–Olech [5]) Let $p : [a,b] \to \mathbb{R}_+$ be an integrable function with $\int_a^b p(t)\,dt < \pi/2$, let $t_1,\ldots,t_k \in [a,b]$ and let $x = (x_1,\ldots,x_k) : [a,b] \to \mathbb{R}^k$ be an absolutely continuous mapping satisfying the system

$$\begin{cases} |x'(t)| \le p(t)|x(t)| & \text{for a.e. } t \in (a,b), \\ x_i(t_i) = 0 & \text{for } i = 1,\ldots,k. \end{cases}$$

Then $x(t) = 0$ for every $t \in [a,b]$.

Let $\|\cdot\|$ be the supremum norm in $C[a,b]$, let $\|\cdot\|_k$ be the norm in $C([a,b],\mathbb{R}^k)$ given by $\|x\|_k = \sum_{i=1}^k \|x_i\|$ for $x = (x_1,\ldots,x_k) \in C([a,b],\mathbb{R}^k)$ and let $B(0,r) \subset C([a,b],\mathbb{R}^k)$ be an open ball centred at 0 of radius $r > 0$.

LEMMA 1. *Assume that* $F : [a,b] \times \mathbb{R}^k \to \mathrm{cf}(\mathbb{R}^k)$ *satisfies all assumptions of Theorem* 1. *Then*:

(3.4) $\exists_{r>0}\ \forall_{\lambda\in(0,\beta]}\ \forall_{\tau\in[0,1]}\quad B(0,r) \cap S_{(\lambda,\tau)} = \{0\};$

(3.5) $\forall_{\lambda\ge\alpha}\quad S_{(\lambda,1)} = \{0\};$

(3.6) $\forall_{\tau\in[0,1)}\ \forall_{\lambda\ge\alpha}\quad S_{(\lambda,\tau)} = \emptyset;$

(3.7) $\exists_{K>0}\ \forall_{\lambda>0}\ \forall_{x\in C([a,b],\mathbb{R}^k)}\quad x \in S_{(\lambda,1)} \Rightarrow \sum_{i=3}^k \|x_i\| \le K;$

(3.8) $\exists_{K>0}\ \forall_{\lambda\ge1}\ \forall_{x\in C([a,b],\mathbb{R}^k)}\quad x \in S_{(\lambda,1)} \Rightarrow \|x\|_k \le K.$

Proof of (3.4). By (2.1) there exists $r > 0$ such that $|F(t,x)| \le \psi(t)|x|$ for each $t \in [a,b]$ and $|x| \le r$. Let $\lambda \in (0,\beta]$ and $x \in B(0,r) \cap S_{(\lambda,1)}$. Then

$$\begin{cases} x'(t) \in \tau\lambda F(t,x(t)) & \text{for a.e. } t \in (a,b), \\ l(x) = 0. \end{cases}$$

Hence

$$|x'(t)| \le \tau\lambda|F(t,x(t))| \le \tau\lambda\psi(t)|x(t)| \le \beta\psi(t)|x(t)| \quad \text{for a.e. } t \in (a,b),$$

so according to (3.3), $x(t) = 0$ for each $t \in [a,b]$.

Proof of (3.5). Let $\lambda \ge \alpha$ and $x \in S_{(\lambda,1)}$. Then

$$\begin{cases} x'(t) = -\lambda p(x(t)), \\ l(x) = 0. \end{cases}$$

From the definition of p we obtain

$$\begin{cases} x_1'(t) = \lambda x_2(t), \\ x_2'(t) = -\lambda M|x_1(t)|, \\ x_1(a) = 0, \quad x_2(b) = 0, \end{cases}$$

so

$$\begin{cases} x_1''(t) = -\lambda^2 M|x_1(t)|, \\ x_1(a) = 0, \\ x_1'(b) = 0. \end{cases}$$

By the maximum principle (cf. [8]), $x_1(t) \geq 0$ for every $t \in [a,b]$. Since $\lambda \geq 1$, (3.1) yields $x_1(t) = 0$ for each $t \in [a,b]$. Therefore $x(t) = 0$ for each $t \in [a,b]$, so (3.5) is proved.

Proof of (3.6). Let $x \in S_{(\lambda,\tau)}$ for some $\lambda \geq \alpha$ and $\tau \in [0,1)$. Then

$$\begin{cases} x'(t) = -\lambda p(x(t)) - (1-\tau)x^0(t), \\ l(x) = 0, \end{cases}$$

hence

$$\begin{cases} x_1''(t) = -\lambda^2 M|x_1(t)| - \lambda(1-\tau)x_0(t), \\ x_1(a) = 0, \\ x_1'(b) = 0. \end{cases}$$

From (3.2) we conclude that the above problem has no solution. This completes the proof.

Proof of (3.7). Let $x \in S_{(\lambda,1)}$. By (iii) and (2.3) there exists $m_R \in L^1(a,b)$ such that

$$|(x_i)'(t)| \leq m_R(t) + \lambda\psi_i(t)|x_i(t)| \quad \text{for a.e. } t \in [a,b] \text{ and } i = 3,4,\ldots,k.$$

From (3.5) we obtain

$$|(x_i)'(t)| \leq m_R(t) + \alpha\psi_i(t)|x_i(t)| \quad \text{for a.e. } t \in [a,b] \text{ and } i = 3,4,\ldots,k.$$

Hence by the Gronwall inequality we have

$$|x_i(t)| \leq \int_a^b m_R(t) e^{\alpha \int_a^b \psi_i(t)\,dt}\,dt \quad \text{for } t \in [a,b] \text{ and } i = 3,4,\ldots,k.$$

This completes the proof of (3.7).

Proof of (3.8). Suppose that (3.8) is not satisfied, i.e. there exist sequences $\{\lambda_n\} \subset (1,\infty)$ and $\{x^n\} \subset S_{(\lambda_n,1)}$ such that $\sum_{i=1}^k \|x_i^n\| \to \infty$ and $\lambda_n \to \lambda_0 \geq 1$. From (3.7) we obtain $\|x_1^n\| + \|x_2^n\| \to \infty$. Since $x^n \in S_{(\lambda_n,1)}$, we have for a.e. $t \in (a,b)$,

$$\begin{cases} (x_1^n)'(t) \in \lambda_n q_1(\lambda_n)(\Pi_1 \circ F)(t, x_n(t)) + \lambda_n q_2(\lambda_n) x_2^n(t), \\ (x_2^n)'(t) \in \lambda_n q_1(\lambda_n)(\Pi_2 \circ F)(t, x_n(t)) - \lambda_n q_2(\lambda_n) M|x_1(t)|, \\ x_1^n(a) = 0, \quad x_2^n(b) = 0. \end{cases}$$

So there exists a sequence $\{w_i^n\} \subset L^1(a,b)$ such that $w_i^n(t) \in (\Pi_i \circ F)(t, x^n(t))$ a.e. on $[a,b]$ for $i = 1,2$ and

$$x_1^n(t) = \lambda_n q_1(\lambda_n) \int_a^t (w_1^n(s) - x_2^n(s))\,ds + \lambda_n \int_a^t x_2^n(s)\,ds,$$

$$x_2^n(t) = \lambda_n q_1(\lambda_n) \int_b^t (w_2^n(s) + M|x_1^n(s)|)\,ds - \lambda_n M \int_b^t |x_1^n(s)|\,ds.$$

Set $v_i^n(t) = x_i^n(t)/\|x^n\|_k$. From conditions (iii) and (2.2), and the Arzelà–Ascoli Theorem, there exists a subsequence of $\{v_i^n\}$ which is convergent to v_i for $i = 1, 2$ and the following conditions are satisfied:

$$\begin{cases} v_1'(t) = \lambda_0 v_2(t), \\ v_2'(t) = -\lambda_0 M|v_1(t)|, \\ v_1(a) = 0, \quad v_2(b) = 0. \end{cases}$$

Similarly to what we showed in (3.5), by (3.1) we have $\lambda_0 M = (\pi/2(b-a))^2$ so necessarily $\lambda_0 < 1$. This contradiction and finishes the proof of (3.8).

LEMMA 2. *Assume that $F : [a, b]\times\mathbb{R}^k \to \mathrm{cf}(\mathbb{R}^k)$ satisfies all assumptions of Theorem 2. Then there exists $K > 0$ such that*

$$\sum_{i=3}^{k} \|x_i\| \le K \quad \text{for } \lambda \in (0,\infty) \text{ and } x \in S_{(\lambda,1)}. \tag{3.9}$$

Proof of (3.9). Suppose (3.9) is not satisfied, i.e. there exist sequences $\{\lambda_n\}$ and $\{x^n\} \subset S_{(\lambda_n,1)}$ such that $\sum_{i=3}^k \|x_i^n\| \to \infty$ and $\lambda_n \to \lambda_0 < \alpha$. Set $y_n = (x_3^n, \dots, x_k^n)$. By (iii) and (2.4) there exists $m_R \in L^1(a, b)$ such that

$$\begin{cases} |y_n'(t)| \le m_R(t) + \lambda_n \psi(t)|y_n(t)| \quad \text{a.e. } t \in [a,b], \\ l(y_n) = 0. \end{cases}$$

Observe that the function $v_n = y_n/\|y_n\|_{k-2}$ is a solution of the problem

$$\begin{cases} |v_n'(t)| \le \dfrac{m_R(t)}{\|y_n\|_{k-2}} + \lambda_n \psi(t)|v_n(t)| \quad \text{a.e. } t \in [a,b], \\ l(v_n) = 0. \end{cases}$$

Therefore the sequence $\{v_n'\}$ of derivatives is bounded by the integrable function $\alpha\psi_1 + m_R$ for $n \in \mathbb{N}$ large enough. Then by the Pliś Lemma [7] there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ uniformly convergent to an absolutely continuous function $v_0 : [a, b] \to \mathbb{R}^{k-2}$ which is a non-trivial solution of the problem (3.3). This contradiction finishes the proof of (3.9).

4. Proofs of Theorems 1 and 2. To the Carathéodory mapping $F : [a, b]\times\mathbb{R}^k \to \mathrm{cf}(\mathbb{R}^k)$ we associate the Nemytskiĭ operator $\mathcal{F} : C([a, b],\mathbb{R}^k) \to \mathrm{cf}(L^1((a, b),\mathbb{R}^k))$, given by

$$\mathcal{F}(x) = \{w \in L^1((a,b),\mathbb{R}^k) : w(t) \in F(t, x(t)) \text{ for a.e. } t \in (a,b)\}.$$

Let $P : C([a, b],\mathbb{R}^k) \to L^1((a, b),\mathbb{R}^k)$ be the Nemytskiĭ operator for the mapping $p : \mathbb{R}^k \to \mathbb{R}^k$, and let $T = (T_1, \dots, T_k) : L^1((a, b),\mathbb{R}^k) \to C([a, b],\mathbb{R}^k)$ be the integral operator given by

$$T(x) = (T_1(x_1), \dots, T_k(x_k)) \quad \text{where} \quad T_i(x_i)(t) = \int_{t_i}^{t} x_i(s)\,ds.$$

With the family of boundary value problems $(2_{(\lambda,\tau)})$ we associate the family of vector fields $f_\tau : (0,\infty) \times C([a,b],\mathbb{R}^k) \to \mathrm{cf}(C([a,b],\mathbb{R}^k))$ given by

$$(4_\tau) \quad f_\tau(\lambda,x) = x - \lambda\tau q_1(\lambda) T\mathcal{F}(x) + \lambda q_2(\lambda) TP(x) + (1-\tau) q_2(\lambda) T(x^0).$$

Observe that $x \in S_{(\lambda,\tau)}$ iff $0 \in f_\tau(\lambda,\tau)$. Moreover the vector field f_τ is completely continuous (cf. [9], [10]).

We call $(\mu,0) \in (0,\infty) \times C([a,b],\mathbb{R}^k)$ a *bifurcation point* of the mapping f_1 if for every open subset $U \subset (0,\infty) \times C([a,b],\mathbb{R}^k)$ with $(\mu,0) \in U$ there exists a point $(\lambda,x) \in U$ such that $x \neq 0$ and $0 \in f_1(\lambda,x)$. Denote by $\mathcal{B}_{f_1}$ the set of all bifurcation points of f_1. Let $\mathcal{R}_{f_1} \subset (0,\infty) \times C([a,b],\mathbb{R}^k)$ be the closure (in $(0,\infty) \times C^1([a,b],\mathbb{R}^k)$) of the set of non-trivial solutions of the inclusion $0 \in f_1(\lambda,x)$, i.e.

$$\mathcal{R}_{f_1} = \overline{\{(\lambda,x) \in (0,\infty) \times C([a,b],\mathbb{R}^k) : x \neq 0 \wedge 0 \in f_1(\lambda,x)\}}.$$

For each λ satisfying $(\lambda,0) \notin \mathcal{B}_{f_1}$ there exists $r_0 > 0$ such that $0 \notin f_1(\lambda,x)$ for $\|u\| = r \in (0,r_0]$, so the value $\deg(f_1(\lambda,\cdot),B(0,r),0)$ is defined.

Assume that for an interval $[c,d] \subset (0,\infty)$ there exists $\delta > 0$ such that

$$(([c-\delta,c) \cup (d,d+\delta]) \times \{0\}) \cap \mathcal{B}_{f_1} = \emptyset.$$

Then we may define the *bifurcation index* $s[f_1,c,d]$ of the mapping f_1 with respect to the interval $[c,d]$ as

$$s[f_1,c,d] = \lim_{\lambda\to d^+} \deg(f_1(\lambda,\cdot),B(0,r),0) - \lim_{\lambda\to c^-} \deg(f_1(\lambda,\cdot),B(0,r),0),$$

where $r = r(\lambda) > 0$ is small enough.

The main tool used in this section, Theorem A below, is a global bifurcation theorem for convex-valued completely continuous mappings which is a consequence of a generalization of the Rabinovitz global bifurcation alternative (see [1], [11]).

THEOREM A (see [3]). *Let $f_1 : (0,\infty) \times C([a,b],\mathbb{R}^k) \to \mathrm{cf}(C([a,b],\mathbb{R}^k))$ be given by (4_1), and assume that there exists an interval $[c,d] \subset (0,\infty)$ such that $\mathcal{B}_{f_1} \subset [c,d] \times \{0\}$ and $s[f_1,c,d] \neq 0$. Then there exists a non-compact component $\mathcal{C} \subset \mathcal{R}_{f_1}$ satisfying $\mathcal{C} \cap \mathcal{B}_{f_1} \neq \emptyset$.*

Proof of Theorem 1. According to (3.4) and (3.5), $\mathcal{B}_{f_1} \subset [\beta,\alpha]$. Observe that by (3.4) for $\lambda < \beta$ there exists $r > 0$ such that $f_1(\lambda,\cdot) : B(0,r) \to \mathrm{cf}(C[a,b],\mathbb{R}^k)$ is homotopic to the identity mapping. Hence by the homotopy property of the topological degree we have $\deg(f_1(\lambda,\cdot),B(0,r),0) = 1$. According to (3.5) and (3.6), for $\lambda > \alpha$ there exists $r > 0$ such that $f_1(\lambda,\cdot) : B(0,r) \to \mathrm{cf}(C[a,b],\mathbb{R}^k)$ is homotopic to the mapping $f_0(\lambda,\cdot)$ which has no zeros and $\deg(f_1(\lambda,\cdot),B(0,r),0) = 0$. Therefore $s[f_1,\beta,\alpha] = -1$. According to Theorem A there exists a non-compact component $\mathcal{C} \subset \mathcal{R}_{f_1}$ satisfying $\mathcal{C} \cap \mathcal{B}_{f_1} \neq \emptyset$. Since $\mathcal{C}$ is not compact there exists a sequence $\{(\lambda_n,x_n)\} \subset \mathcal{C}$ such that $\|x_n\|_k \to \infty$, or $\lambda_n \to \infty$, or $\lambda_n \to 0$. Observe that by (3.5) the case $\lambda_n \to \infty$ is impossible. Now consider the case $\|x_n\|_k \to \infty$. Then it follows from (3.8) that $\lambda_n < 1$ for $n \in \mathbb{N}$ large enough. So the connected set $\mathcal{C}$ contains pairs (λ_1, x_1) and (λ_2, x_2) with $\lambda_1 < 1$ and $\lambda_2 > 1$. Hence there exists $(1, x) \in \mathcal{C}$. This solution of the inclusion $0 \in f_1(1, x)$ must have $x \neq 0$ because $(1, 0) \notin \mathcal{R}_{f_1}$. The proof is complete.

Proof of Theorem 2. Since F satisfies assumptions (2.1) and (2.2) of Theorem 1, there exists a non-compact component $\mathcal{C} \subset \mathcal{R}_{f_1}$ satisfying $\mathcal{C} \cap \mathcal{B}_{f_1} \neq \emptyset$. Similarly to what we showed in the proof of Theorem 1, from (3.9) and (3.8) it follows that there exist $\lambda_1 < 1$ and $x \in C([a, b], \mathbb{R}^k)$ such that $(\lambda_1, x) \in \mathcal{C}$. Because $\mathcal{B}_{f_1} \subset [\beta, \alpha]$ we can see that the connected set $\mathcal{C}$ contains pairs (λ_1, x_1) and (λ_2, x_2) with $\lambda_1 < 1$ and $\lambda_2 > 1$. Hence $(1, x) \in \mathcal{C}$ for some $x \neq 0$ as before. The proof is complete.

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