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Bi-Legendrian connections

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Abstract. We define the concept of a bi-Legendrian connection associated to a bi-Legendrian structure on an almost $\mathcal{S}$-manifold M^{2n+r}. Among other things, we compute the torsion of this connection and prove that the curvature vanishes along the leaves of the bi-Legendrian structure. Moreover, we prove that if the bi-Legendrian connection is flat, then the bi-Legendrian structure is locally equivalent to the standard structure on $\mathbb{R}^{2n+r}$.

1. Introduction. Given a symplectic manifold (M,ω) of dimension $2n$, a foliation $\mathcal{F}$ of dimension n on M is said to be *Lagrangian* if $\omega(X,X')=0$ for any vectors X,X' tangent to $\mathcal{F}$. In [5] H. Hess, working on geometric quantization, proved that, given two complementary Lagrangian distributions L and Q on M, there exists a unique connection ∇ satisfying the following conditions:

(1) $\nabla\omega=0$;
(2) $\nabla(\Gamma L)\subset\Gamma L$ and $\nabla(\Gamma Q)\subset\Gamma Q$;
(3) $T(X,Y)=0$ if $X\in\Gamma L$ and $Y\in\Gamma Q$, where T is the torsion tensor of ∇.

This connection is called *bi-Lagrangian* and if L and Q are involutive subbundles of TM, i.e. if they are Lagrangian foliations on M, then ∇ is torsion free and it is flat along the leaves of the foliations (for more details, see also [10] and [11]).

Analogues of symplectic manifolds in odd dimensions are contact manifolds and analogues of Lagrangian foliations are the so-called Legendrian foliations (cf. [9], [8] or [6]). The aim of this paper is to give an answer to the natural question of defining an analogue, for contact manifolds, of the notion of bi-Lagrangian connection. More generally, we define the *bi-Legendrian connection* associated to a bi-Legendrian structure on an almost $\mathcal{S}$-manifold M^{2n+r} and we can regard bi-Lagrangian connections as a partic-

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ular case of our definition (namely, for $r = 0$). Moreover, we investigate the properties of this connection, in particular those involving its torsion and curvature tensors. Finally, we present some basic examples of bi-Legendrian structures and bi-Legendrian connections, recognizing that they are very familiar geometrical objects. More precisely, we prove that if the bi-Legendrian connection is flat, then the bi-Legendrian structure is locally equivalent to the standard structure on $\mathbb{R}^{2n+r}$, where $2n + r$ is the dimension of the almost $\mathcal{S}$-manifold M.

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2. Preliminaries

2.1. *Almost $\mathcal{S}$-manifolds.* An *f-structure* on a smooth manifold M is defined by a non-vanishing tensor field ϕ of type $(1, 1)$ of constant rank $2n$ which satisfies $\phi^3 + \phi = 0$. It can be proved that TM then splits into two complementary subbundles $\mathrm{Im}(\phi)$ and $\ker(\phi)$. When $\ker(\phi)$ is parallelizable we say that we have an f-structure with parallelizable kernel, briefly an *$f \cdot pk$-structure.* In this case there exist global sections $\xi_1, \dots, \xi_r$ of $\ker(\phi)$ and 1-forms $\eta_1, \dots, \eta_r$ such that $\eta_i(\xi_j) = \delta_{ij}$ and

$$\phi^2 = -I + \sum_{i=1}^{r} \eta_i \otimes \xi_i,$$

from which it follows that $\phi(\xi_i) = 0$ and $\eta_i \circ \phi = 0$ for all $i \in \{1, \dots, r\}$. Almost complex and almost contact structures are $f \cdot pk$-structures with $r = 0$ and $r = 1$, respectively. It is known that, given an $f \cdot pk$-structure (ϕ, ξ_i, η_i), there exists a Riemannian metric g on M such that

$$g(\phi V, \phi W) = g(V, W) - \sum_{i=1}^{r} \eta_i(V)\eta_i(W) \tag{1}$$

for all $V, W \in \Gamma(TM)$. Such a metric is not, in general, unique. If g is any metric satisfying (1) we say that (ϕ, ξ_i, η_i, g) is a *metric $f \cdot pk$-structure.* We denote by Φ the 2-form defined by $\Phi(V, W) = g(V, \phi W)$. A metric $f \cdot pk$-manifold M^{2n+r} with structure (ϕ, ξ_i, η_i, g) is called an *almost $\mathcal{S}$-manifold* if $d\eta_1 = \dots = d\eta_r = \Phi$. This definition reduces to that of contact metric manifold for $r = 1$ and of almost Hermitian manifold for the extreme case $r = 0$. In this paper we will assume that Φ is closed. This is always true for $r \geq 1$; for $r = 0$ this hypothesis implies that (M^{2n}, Φ) is a symplectic manifold and g is an associated metric with respect to the almost complex structure ϕ. We conclude these preliminaries with some useful properties of almost $\mathcal{S}$-manifolds.

LEMMA 2.1. *Let $(M^{2n+r}, \phi, \eta_i, \xi_i, g)$ be an almost $\mathcal{S}$-manifold and let $\mathcal{H}$ denote the $2n$-dimensional distribution on M given by $\mathcal{H} = \bigcap_{i=1}^{r} \ker(\eta_i)$. Then, for all $i, j \in \{1, \ldots, r\}$, we have*:

(i) $\Phi(W, \xi_i) = 0$ *for all* $W \in \Gamma(TM)$,
(ii) $[\xi_i, \xi_j] = 0$ *and* $[Z, \xi_i] \in \Gamma\mathcal{H}$ *for all* $Z \in \Gamma\mathcal{H}$,
(iii) $\mathcal{L}_{\xi_i}\eta_j = \mathcal{L}_{\xi_i} d\eta_j = 0$.

For more details good references are, for example, [1], [2] and [4].

2.2. *Legendrian foliations.* Let $(M^{2n+r}, \phi, \xi_i, \eta_i, g)$ be an almost $\mathcal{S}$-manifold. An n-dimensional distribution L on M is called a *Legendrian* if L is a subbundle of $\mathcal{H}$ and

$$\Phi(X, X') = 0 \tag{2}$$

for any $X, X' \in \Gamma L$. When L is involutive, the foliation $\mathcal{F}$ whose tangent bundle is L is called a *Legendrian foliation.* Note that for $r = 0$ and under the hypothesis $d\Phi = 0$, our definition of Legendrian distribution reduces to that of Lagrangian distribution on symplectic manifolds.

We denote by $L^{\perp}$ the orthogonal bundle of L. Then, setting $Q = \mathcal{H} \cap L^{\perp}$, we obtain another n-dimensional distribution on M and we get the decomposition $TM = L \oplus Q \oplus E_1 \oplus \cdots \oplus E_r = L \oplus Q \oplus E$, where E_i denotes the line bundle generated by ξ_i and $E = \bigoplus_{i=1}^{r} E_i$. It is not difficult to prove that $\phi(L) = Q$ and $\phi(Q) = L$, from which one can see that, for each $i \in \{1, \ldots, r\}$ and $Y \in \Gamma Q$, $\eta_i(Y) = \eta_i(\phi(X)) = 0$, where X is the section of L such that $\phi(X) = Y$. In general Q is not involutive, even if L is; precisely, $[Y, Y'] \in \Gamma\mathcal{H}$ for any $Y, Y' \in \Gamma Q$. Hence, when Q is integrable, we obtain another Legendrian foliation on M^{2n+r}, called the *conjugate Legendrian foliation* of $\mathcal{F}$. A *bi-Legendrian structure* on M is a pair $(\mathcal{F}, \mathcal{G})$ of two complementary Legendrian foliations on M. For instance, a typical example of bi-Legendrian structure is given by the pair of a Legendrian foliation and its conjugate whenever the conjugate Legendrian foliation exists.

Let $\overline{\xi}$ denote the vector field defined by $\overline{\xi} := \sum_{i=1}^{r} \xi_i$. A Legendrian foliation is said to be *flat* (respectively, *strongly flat*) if $\overline{\xi}$ (respectively, each $\xi_1, \ldots, \xi_r$) is projectable (or foliated) with respect to $\mathcal{F}$, i.e. if $[X, \overline{\xi}] \in \Gamma L$ whenever $X \in \Gamma L$.

Everywhere in this paper, we will denote by L and Q two Legendrian distributions on M and, when L and Q are integrable, by $\mathcal{F}$ and $\mathcal{G}$ the corresponding Legendrian foliations. We will make use of the following lemma, whose proof is given in [3].

LEMMA 2.2. *Let (M,ϕ,η_i,ξ_i,g) be an almost $\mathcal{S}$-manifold such that each ξ_i is a Killing vector field, and $\mathcal{F}$ a Legendrian foliation on M such that the conjugate Legendrian foliation of $\mathcal{F}$ exists. Then if $\mathcal{F}$ is strongly flat also its conjugate is strongly flat.*

3. Bi-Legendrian connections. Let (M,ϕ,η_i,ξ_i,g), $i\in\{1,\dots,r\}$, be an almost $\mathcal{S}$-manifold of dimension $2n+r$ and take two vector fields $V,W\in\Gamma(TM)$. We define a section $H(V,W)$ of $\mathcal{H}$ to be the unique section of $\mathcal{H}$ such that

$$i_{H(V,W)}\Phi|_{\mathcal{H}}=(\mathcal{L}_V i_W\Phi)|_{\mathcal{H}},$$

that is, $\Phi(H(V,W),Z)=V(\Phi(W,Z))-\Phi(W,[V,Z])$ for every $Z\in\Gamma\mathcal{H}$. The existence and uniqueness of this vector field depends on the fact that the 2-form Φ is non-degenerate on $\mathcal{H}$.

REMARK 3.1. Observe that the above definition yields $H(\xi_i,W)=p_{\mathcal{H}}([\xi_i,W])$ and $H(V,\xi_i)=0$ for all $V,W\in\Gamma(TM)$ and $i\in\{1,\dots,r\}$. Indeed, using Lemma 2.1(iii) we have, for all $Z\in\Gamma\mathcal{H}$,

$$\begin{aligned}\Phi(H(\xi_i,W),Z)&=\xi_i(\Phi(W,Z))-\Phi(W,[\xi_i,Z])\\&=(\mathcal{L}_{\xi_i}\Phi)(W,Z)+\Phi([\xi_i,W],Z)=\Phi([\xi_i,W],Z),\end{aligned}$$

so $H(\xi_i,W)=p_{\mathcal{H}}([\xi_i,W])$. Finally, since $\Phi(H(V,\xi_i),Z)=V(\Phi(\xi_i,Z))-\Phi(\xi_i,[V,Z])=0$ for all $Z\in\Gamma\mathcal{H}$, we get $H(V,\xi_i)=0$.

LEMMA 3.2. *For every $f\in C^\infty(M)$ and $V,V',W,W'\in\Gamma(TM)$ we have:*

(i) $H(V+V',W)=H(V,W)+H(V',W)$,
(ii) $H(V,W+W')=H(V,W)+H(V,W')$,
(iii) $H(V,fW)=fH(V,W)+V(f)W_{\mathcal{H}}$,
(iv) $H(fV,W)=fH(V,W)$ *if* $\Phi(V,W)=0$,

where $W_{\mathcal{H}}$ denotes the projection of W onto the subbundle $\mathcal{H}$ of TM.

Proof. We prove (iii) and (iv), (i) and (ii) being obvious. For every $Z\in\Gamma\mathcal{H}$, we have

$$\Phi(H(V,fW),Z)=\Phi(V(f)W+fH(V,W),Z)$$

so $H(V,fW)=fH(V,W)+V(f)W_{\mathcal{H}}$. Moreover,

$$\begin{aligned}\Phi(H(fV,W),Z)&=fV(\Phi(W,Z))-\Phi(W,[fV,Z])\\&=f\Phi(H(V,W),Z)-Z(f)\Phi(V,W)\end{aligned}$$

and (iv) follows. ■

Let (L,Q) be a pair of complementary distributions on the almost $\mathcal{S}$-manifold (M,ϕ,η_i,ξ_i,g). We want to associate to (L,Q) a canonical connection on M. For this purpose let V_L, V_Q and V_E denote the projections of a vector field $V\in\Gamma(TM)$ onto L, Q and E, respectively. Then we have

PROPOSITION 3.3. *For all $W \in \Gamma(TM)$, $X \in \Gamma L$, $Y \in \Gamma Q$ and $Z \in \Gamma E_i$, define*

$$\begin{aligned}
\nabla^L_W X &:= H(W_L, X)_L + [W_Q, X]_L + [W_E, X]_L,\\
\nabla^Q_W Y &:= H(W_Q, Y)_Q + [W_L, Y]_Q + [W_E, Y]_Q,\\
\nabla^{(i)}_W Z &:= W_E(\eta_i(Z))\xi_i + [W_L, Z]_{E_i} + [W_Q, Z]_{E_i} = W(\eta_i(Z))\xi_i.
\end{aligned}$$

Then ∇^L is a connection on the bundle L, ∇^Q a connection on Q, and $\nabla^{(i)}$ on E_i, $i \in \{1, \dots, r\}$.

Proof. Indeed, for all $f \in C^\infty(M)$, by Lemma 3.2 we have

$$\begin{aligned}
\nabla^L_{fW} X &= \nabla^L_{fW_L} X + \nabla^L_{fW_Q} X + \nabla^L_{fW_E} X\\
&= H(fW_L, X)_L + [fW_Q, X]_L + [fW_E, X]_L\\
&= fH(W_L, X)_L + (f[W_Q, X] - X(f)W_Q)_L + (f[W_E, X] - X(f)W_E)_L\\
&= f\nabla^L_W X.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\nabla^L_W(fX) &= \nabla^L_{W_L}(fX) + \nabla^L_{W_Q}(fX) + \nabla^L_{W_E}(fX)\\
&= H(W_L, fX)_L + [W_Q, fX]_L + [W_E, fX]_L\\
&= fH(W_L, X)_L + W_L(f)X + f[W_Q, X]_L + W_Q(f)X\\
&\quad + f[W_E, X]_L + W_E(f)X\\
&= f\nabla^L_W X + W(f)X,
\end{aligned}$$

so ∇^L is a connection on L. In the same way one can prove that ∇^Q is a connection on Q. To end the proof we have to show that $\nabla^{(i)}$ is a connection on E_i. Indeed, $\nabla^{(i)}_{fW} Z = fW(\eta_i(Z))\xi_i = f\nabla^{(i)}_W Z$ and

$$\nabla^{(i)}_W(fZ) = W(\eta_i(fZ))\xi_i = W(f)\eta_i(Z)\xi_i + fW(\eta_i(Z))\xi_i = W(f)Z + f\nabla^{(i)}_W Z. \quad \blacksquare$$

Now we can define a global connection on M by setting, for any $V, W \in \Gamma(TM)$,

$$\nabla_W V := \nabla^L_W V_L + \nabla^Q_W V_Q + \sum_{i=1}^r \nabla^{(i)}_W V_{E_i}.$$

It follows that, for all $W \in \Gamma(TM)$, $\nabla_W \xi_i = \nabla^{(i)}_W \xi_i = W(\eta_i(\xi_i))\xi_i = 0$, and

$$\nabla_{\xi_i} W = [\xi_i, W_L]_L + [\xi_i, W_Q]_Q + \sum_{j=1}^r \xi_i(\eta_j(W))\xi_j.$$

PROPOSITION 3.4. *The connection ∇ has the following properties*:

(i) $\nabla(\Gamma L) \subset \Gamma L$, $\nabla(\Gamma Q) \subset \Gamma Q$ *and* $\nabla(\Gamma E_i) \subset \Gamma E_i$ *for* $i \in \{1, \dots, r\}$;
(ii) $\nabla\eta_1 = \dots = \nabla\eta_r = 0$;
(iii) $\nabla\Phi = 0$.

Proof. (i) is a direct consequence of the definition of ∇. We prove (ii). For any $V, W \in \Gamma(TM)$ we have

$$\begin{aligned}(\nabla_W \eta_i)V &= W(\eta_i(V)) - \eta_i(\nabla_W V) = W(\eta_i(V)) - \sum_{j=1}^{r} \eta_i(\nabla_W^{(j)} V_{E_j}) \\ &= W(\eta_i(V)) - \sum_{j=1}^{r} \eta_i(W(\eta_j(V_{E_j}))\xi_j) \\ &= W(\eta_i(V)) - W(\eta_i(V_{E_i})) = 0,\end{aligned}$$

so $\nabla \eta_i = 0$ for each $i \in \{1, \dots, r\}$. It remains to prove that $(\nabla_Z \Phi)(V, W) = 0$ for all $V, W, Z \in \Gamma(TM)$. This clearly holds if $V, W \in \Gamma L$ or $V, W \in \Gamma Q$, since

$$(\nabla_Z \Phi)(V, W) = Z(\Phi(V, W)) - \Phi(\nabla_Z V, W) - \Phi(V, \nabla_Z W)$$

and each term of the right hand side vanishes (by (i)). Also the case $V \in \Gamma(TM)$, $W \in \Gamma E_i$ is obvious. Indeed, W can be written as $W = f\xi_i$ for some $f \in C^\infty(M)$ and we have

$$\begin{aligned}(\nabla_Z \Phi)(V, f\xi_i) &= Z(\Phi(V, f\xi_i)) - \Phi(\nabla_Z V, f\xi_i) - \Phi(V, \nabla_Z (f\xi_i)) \\ &= -\Phi(V, f\nabla_Z \xi_i) = 0.\end{aligned}$$

So we only have to prove that $(\nabla_Z \Phi)(X, Y) = 0$ for $X \in \Gamma L$ and $Y \in \Gamma Q$. It is sufficient to consider the two cases $Z \in \Gamma\mathcal{H}$ and $Z = \xi_i$. In the first case we have

$$\begin{aligned}(\nabla_Z \Phi)(X, Y) &= Z(\Phi(X, Y)) - \Phi(\nabla_Z^L X, Y) - \Phi(X, \nabla_Z^Q Y) \\ &= Z(\Phi(X, Y)) - \Phi(H(Z_L, X)_L + [Z_Q, X]_L, Y) \\ &\quad - \Phi(X, H(Z_Q, Y)_Q + [Z_L, Y]_Q) \\ &= Z(\Phi(X, Y)) - \Phi(H(Z_L, X), Y) - \Phi([Z_Q, X], Y) \\ &\quad + \Phi(H(Z_Q, Y), X) + \Phi([Z_L, Y], X) \\ &= Z(\Phi(X, Y)) - Z_L(\Phi(X, Y)) - Z_Q(\Phi(X, Y)) = 0\end{aligned}$$

by the definition of H. Finally, by Lemma 2.1,

$$\begin{aligned}(\nabla_{\xi_i} \Phi)(X, Y) &= \xi_i(\Phi(X, Y)) - \Phi([\xi_i, X]_L, Y) - \Phi(X, [\xi_i, Y]_Q) \\ &= \xi_i(\Phi(X, Y)) - \Phi([\xi_i, X], Y) - \Phi(X, [\xi_i, Y]) \\ &= (\mathcal{L}_{\xi_i} \Phi)(X, Y) = 0. \ \blacksquare\end{aligned}$$

Now we compute the torsion of ∇.

PROPOSITION 3.5. *The torsion of* ∇ *is given by*

(i) $T(X, X') = -p_{L^\perp}([X, X'])$ *for* $X, X' \in \Gamma L$,
(ii) $T(Y, Y') = -p_{Q^\perp}([Y, Y'])$ *for* $Y, Y' \in \Gamma Q$,
(iii) $T(X, Y) = 2\Phi(X, Y)\overline{\xi}$ *for* $X \in \Gamma L$ *and* $Y \in \Gamma Q$,

(iv) $T(W,\xi_i) = [\xi_i, W_L]_Q + [\xi_i, W_Q]_L$ *for* $W \in \Gamma(TM)$.

In particular, by (iv), $T(\xi_i,\xi_j) = 0$.

Proof. First take $X \in \Gamma L$ and $Y \in \Gamma Q$. Then

$$\begin{aligned} T(X,Y) &= \nabla^Q_X Y - \nabla^L_Y X - [X,Y] = [X,Y]_Q - [Y,X]_L - [X,Y] \\ &= -\sum_{i=1}^r \eta_i([X,Y])\xi_i = \sum_{i=1}^r 2d\eta_i(X,Y)\xi_i = 2\Phi(X,Y)\overline{\xi}. \end{aligned}$$

Moreover, for any $W \in \Gamma(TM)$, we have

$$\begin{aligned} T(W,\xi_i) = &\ -[\xi_i, W_L]_L - [\xi_i, W_Q]_Q - \sum_{j=1}^r \xi_i(\eta_j(W))\xi_j - [W,\xi_i] \\ = &\ -[\xi_i, W_L]_L - [\xi_i, W_Q]_Q - \sum_{j=1}^r \xi_i(\eta_j(W))\xi_j + [\xi_i, W_L] + [\xi_i, W_Q] \\ &+ \sum_{j=1}^r [\xi_i, \eta_j(W)\xi_j] \\ = &\ [\xi_i, W_L]_Q + [\xi_i, W_Q]_L. \end{aligned}$$

It remains to prove the statement for $X, X' \in \Gamma L$ and $Y, Y' \in \Gamma Q$. Indeed,

$$\begin{aligned} T(X,X') &= H(X,X')_L - H(X',X)_L - [X,X'] \\ &= (H(X,X') - H(X',X) - [X,X'])_L - [X,X']_{L^\perp}, \end{aligned}$$

so it is sufficient to prove that $H(X,X') - H(X',X) = [X,X']_{\mathcal{H}}$. Indeed, from the definition of H we have, for every $Z \in \Gamma\mathcal{H}$,

$$\begin{aligned} \Phi(H(X,X'),Z) &= X(\Phi(X',Z)) - \Phi(X',[X,Z]), \\ \Phi(H(X',X),Z) &= X'(\Phi(X,Z)) - \Phi(X,[X',Z]). \end{aligned}$$

Subtracting the last two equations we get

$$\begin{aligned} &\Phi(H(X,X') - H(X',X),Z) \\ &\quad = X(\Phi(X',Z)) - X'(\Phi(X,Z)) - \Phi(X',[X,Z]) + \Phi(X,[Y,Z]) \\ &\quad = 3d\Phi(X,X',Z) + \Phi([X,X'],Z) = \Phi([X,X'],Z), \end{aligned}$$

from which, since Φ is closed and non-degenerate on $\mathcal{H}$, we conclude that $H(X,X') - H(X',X) = [X,X']_{\mathcal{H}}$. In the same way one can prove that $T(Y,Y') = -p_{Q^\perp}([Y,Y'])$. ∎

Corollary 3.6. *If L and Q are involutive then ∇ is torsion free along the leaves of the Legendrian foliations $\mathcal{F}$ and $\mathcal{G}$.*

Corollary 3.7. *If L and Q are strongly flat then $T(V,\xi_i) = 0$ for every $V \in \Gamma(TM)$.*

Now we can prove that the connection ∇ is uniquely determined by the properties stated in Proposition 3.4 and 3.5:

THEOREM 3.8. *Let L and Q be two complementary Legendrian distributions on the almost $\mathcal{S}$-manifold $(M^{2n+r}, \phi, \eta_i, \xi_i, g)$. There exists a unique connection ∇ on M with the following properties*:

(i) $\nabla\Phi = 0$;
(ii) $\nabla(\Gamma L) \subset \Gamma L$, $\nabla(\Gamma Q) \subset \Gamma Q$ *and* $\nabla(\Gamma E_i) \subset \Gamma E_i$ *for* $i \in \{1, \dots, r\}$;
(iii) $T(X, Y) = 2\Phi(X, Y)\overline{\xi}$ *for all* $X \in \Gamma L$ *and* $Y \in \Gamma Q$,
$T(V, \xi_i) = [\xi_i, V_L]_Q + [\xi_i, V_Q]_L$ *for all* $V \in \Gamma(TM)$ *and* $i \in \{1, \dots, r\}$,

where T denotes the torsion tensor of ∇.

Proof. We only have to prove the uniqueness. Let ∇' be any connection on M^{2n+r} satisfying (i)–(iii). First we show that our hypotheses yield $\nabla'_W \xi_i = 0$ for all $W \in \Gamma(TM)$. Indeed, it is sufficient to consider the two cases $W \in \Gamma\mathcal{H}$ and $W = \xi_j$. In the first case we have

$$\begin{aligned}\nabla'_W \xi_i &= \nabla'_{\xi_i} W + [W, \xi_i] + T'(W, \xi_i)\\ &= \nabla'_{\xi_i} W + [W, \xi_i] + [\xi_i, W_Q]_L + [\xi_i, W_L]_Q\\ &= \nabla'_{\xi_i} W - [\xi_i, W_L]_L - [\xi_i, W_Q]_Q \in \Gamma\mathcal{H}.\end{aligned}$$

On the other hand, $\nabla'_W \xi_i \in \Gamma E_i$, so necessarily $\nabla'_W \xi_i = 0$, from which we also deduce

$$\nabla'_{\xi_i} W = [\xi_i, W_L]_L + [\xi_i, W_Q]_Q. \tag{3}$$

In the case $W = \xi_j$ we have

$$\nabla'_{\xi_j} \xi_i = \nabla'_{\xi_i} \xi_j + [\xi_j, \xi_i] + T'(\xi_j, \xi_i) = \nabla'_{\xi_i} \xi_j \in \Gamma E_j,$$

so $\nabla'_{\xi_j} \xi_i = 0$. Thus, for all $Z \in \Gamma E_i$ and $W \in \Gamma(TM)$ we have

$$\nabla'_W Z = \nabla'_W(\eta_i(Z)\xi_i) = \eta_i(Z)\nabla'_W \xi_i + W(\eta_i(Z))\xi_i = W(\eta_i(Z))\xi_i = \nabla^{(i)}_W Z.$$

Now take $X, X' \in \Gamma L$ and $Y \in \Gamma Q$. Then, as Φ is parallel with respect to ∇', we get

$$\Phi(\nabla'_{X'} X, Y) = X'(\Phi(X, Y)) - \Phi(X, \nabla'_{X'} Y).$$

On the other hand, the conditions on the torsion yield

$$\nabla'_{X'} Y = \nabla'_Y X' + [X', Y] + T(X', Y) = \nabla'_Y X' + [X', Y] + 2\Phi(X', Y)\overline{\xi},$$

so

$$\begin{aligned}\Phi(\nabla'_{X'} X, Y) &= X'(\Phi(X, Y)) - \Phi(X, \nabla'_Y X') - \Phi(X, [X', Y])\\ &\quad - 2\Phi(X', Y')\Phi(X, \overline{\xi})\\ &= X'(\Phi(X, Y)) - \Phi(X, [X', Y]),\end{aligned}$$

from which we deduce $\nabla'_{X'}X = H(X,X')_L = \nabla^L_{X'}X$. In a similar way one can show that $\nabla'_{Y'}Y = H(Y,Y')_Q = \nabla^Q_{Y'}Y$ for any $Y,Y' \in \Gamma Q$. Moreover, if $X \in \Gamma L$ and $Y,Y' \in \Gamma Q$, we have

$$\begin{aligned}\Phi(\nabla'_Y X, Y') &= Y(\Phi(X,Y')) + \Phi(\nabla'_Y Y', X)\\ &= Y(\Phi(X,Y')) + \Phi(H(Y,Y')_Q, X)\\ &= Y(\Phi(X,Y')) + \Phi(H(Y,Y'), X)\\ &= Y(\Phi(X,Y')) + Y(\Phi(Y',X)) - \Phi(Y',[Y,X])\\ &= \Phi([Y,X],Y')\end{aligned}$$

from which we get $\nabla'_Y X = [Y,X]_Q = \nabla^L_Y X$. Finally, if Z is any section of E_i, then, by (3),

$$\begin{aligned}\nabla'_Z X &= \nabla'_{\eta_i(Z)\xi_i}X = \eta_i(Z)\nabla'_{\xi_i}X = \eta_i(Z)[\xi_i,X]_L = [\eta_i(Z)\xi_i,X]_L\\ &= [Z,X]_L = \nabla^L_Z X.\end{aligned}$$

Therefore, for any $W \in \Gamma(TM)$ and $X \in \Gamma L$,

$$\begin{aligned}\nabla'_W X &= \nabla'_{W_L}X + \nabla'_{W_Q}X + \nabla'_{W_E}X\\ &= H(W_L,X)_L + [W_Q,X]_L + [W_E,X]_L = \nabla^L_W X.\end{aligned}$$

In a similar way one can show that $\nabla'_W Y = \nabla^Q_W Y$ for all $W \in \Gamma(TM)$ and $Y \in \Gamma Q$. ∎

The connection of the previous theorem is called the *bi-Legendrian connection* associated to the pair (L,Q) of complementary Legendrian distributions.

Proposition 3.9. *Let $(\mathcal{F},\mathcal{G})$ a strongly flat bi-Legendrian structure on M. Then the bi-Legendrian connection ∇ associated to $(\mathcal{F},\mathcal{G})$ is flat along the leaves of the foliations $\mathcal{F}$ and $\mathcal{G}$.*

Proof. As usual, let L and Q denote the tangent bundles of the foliations $\mathcal{F}$ and $\mathcal{G}$, respectively. Let $X,X' \in \Gamma L$. We have to prove that $R(X,X')Z = 0$ for any $Z \in \Gamma(TM)$. Clearly this is true for $Z = \xi_i$, $i \in \{1,\ldots,r\}$, so it remains to check it for $Z \in \Gamma L$ and $Z \in \Gamma Q$. In the first case we have

$$\begin{aligned}R(X,X')Z &= \nabla^L_X(H(X',Z)_L) - \nabla^L_{X'}(H(X,Z)_L) - H([X,X'],Z)_L\\ &= H(X,H(X',Z)_L)_L - H(X',H(X,Z)_L)_L - H([X,X'],Z)_L.\end{aligned}$$

We examine separately the three terms of the last formula. For any $Y \in \Gamma Q$,

$$\begin{aligned}\Phi(H(X,H(X',Z)_L)_L,Y) &= \Phi(H(X,H(X',Z)_L),Y)\\ &= X(\Phi(H(X',Z)_L,Y)) - \Phi(H(X',Z)_L,[X,Y])\\ &= X(\Phi(H(X',Z),Y)) - \Phi(H(X',Z),[X,Y]_Q)\end{aligned}$$

$$
\begin{aligned}
&= X(X'(\Phi(Z,Y))) - X(\Phi(Z,[X',Y])) \\
&\quad - X'(\Phi(Z,[X,Y])) + \Phi(Z,[X',[X,Y]]) \\
&\quad - \sum_{i=1}^{r} \eta_i([X,Y])\Phi(Z,[X',\xi_i]) \\
&= X(X'(\Phi(Z,Y))) - X(\Phi(Z,[X',Y])) \\
&\quad - X'(\Phi(Z,[X,Y])) + \Phi(Z,[X',[X,Y]]),
\end{aligned}
$$

$$
\begin{aligned}
\Phi(H(X',H(X,Z)_L)_L,Y) &= \Phi(H(X',H(X,Z)_L),Y) \\
&= X'(\Phi(H(X,Z)_L,Y)) - \Phi(H(X,Z)_L,[X',Y]) \\
&= X'(\Phi(H(X,Z),Y)) - \Phi(H(X,Z),[X',Y]_Q) \\
&= X'(X(\Phi(Z,Y))) - X'(\Phi(Z,[X,Y])) \\
&\quad - X(\Phi(Z,[X',Y])) + \Phi(Z,[X,[X',Y]]) \\
&\quad - \sum_{i=1}^{r} \eta_i([X,Y])\Phi(Z,[X,\xi_i]) \\
&= X'(X(\Phi(Z,Y))) - X'(\Phi(Z,[X,Y])) \\
&\quad - X(\Phi(Z,[X',Y])) + \Phi(Z,[X,[X',Y]])
\end{aligned}
$$

and

$$
\Phi(H([X,X'],Z)_L,Y) = [X,X'](\Phi(Z,Y)) - \Phi(Z,[[X,X'],Y]),
$$

where in the first two equations we have used the strong flatness of L. Thus

$$
\begin{aligned}
&\Phi(H(X,H(X',Z)),Y) - \Phi(H(X',H(X,Z)),Y) - \Phi(H([X,X'],Z),Y) \\
&\qquad = [X,X'](\Phi(Z,Y)) + \Phi(Z,[X',[X,Y]]) - \Phi(Z,[X,[X',Y]]) \\
&\qquad\quad - [X,X'](\Phi(Z,Y)) + \Phi(Z,[[X,X'],Y]) \\
&\qquad = \Phi(Z,[[X,X'],Y] + [[X',Y],X] + [[Y,X],X']) = 0
\end{aligned}
$$

as a consequence of the Jacobi identity. Thus, $R(X,X')Z = 0$ if $Z \in \Gamma L$. Now we prove the same in the case $Z \in \Gamma Q$. We have

$$
\begin{aligned}
R(X,X')Z &= \nabla^Q_X([X',Z]_Q) - \nabla^Q_{X'}([X,Z]_Q) - [[X,X'],Z]_Q \\
&= [X,[X',Z]_Q]_Q - [X',[X,Z]_Q]_Q - [[X,X'],Z]_Q \\
&= -\sum_{i=1}^{r} \eta_i([X',Z])[X,\xi_i]_Q + [X,[X',Z]]_Q \\
&\quad - [X,[X',Z]_L]_Q - [X',[X,Z]]_Q \\
&\quad + [X',[X,Z]_L]_Q - [[X,X'],Z]_Q + \sum_{i=1}^{r}[X,Z][X',\xi_i]_Q \\
&= p_Q([X,[X',Z]] + [X',[Z,X]] + [Z,[X,X']]) = 0,
\end{aligned}
$$

since L is strongly flat and hence $[X,\xi_i]_Q = [X',\xi_i]_Q = 0$. This shows that $R(X,X') = 0$. Similarly one can prove the flatness along Q. ■

PROPOSITION 3.10. *Let $(\mathcal{F},\mathcal{G})$ be a strongly flat bi-Legendrian structure on M^{2n+r}. Then $R(V,\xi_i) = 0$ for all $V \in \Gamma(TM)$ and $i \in \{1,\dots,r\}$.*

Proof. Indeed, by a straightforward computation,

$$\begin{aligned}\Phi(R(X,\xi_i)X',Y) &= (\mathcal{L}_{[X,\xi_i]_Q}\Phi)(X',Y) - \Phi([\xi_i,X']_Q,[X,Y]),\\ \Phi(R(X,\xi_i)Y,X') &= (\mathcal{L}_{[X,\xi_i]_Q}\Phi)(X',Y) - \Phi([[X,Y]_L,\xi_i],X'),\end{aligned}$$

for all $X,X' \in \Gamma L$ and $Y \in \Gamma Q$, from which we deduce that if $(\mathcal{F},\mathcal{G})$ is strongly flat then $R(X,\xi_i) = 0$ for all $X \in \Gamma L$ and, in the same way, $R(Y,\xi_i) = 0$ for all $Y \in \Gamma Q$. Moreover it is easy to see that

$$\begin{aligned}R(\xi_i,\xi_j)X &= p_L([\xi_j,[\xi_i,X]_Q] - [\xi_i,[\xi_j,X]_Q]),\\ R(\xi_i,\xi_j)Y &= p_Q([\xi_j,[\xi_i,Y]_L] - [\xi_i,[\xi_j,Y]_L]),\end{aligned}$$

from which we have $R(\xi_i,\xi_j) = 0$. ■

REMARK 3.11. It is easy to see that the last proposition is also true when L and Q are not integrable.

4. Examples and further remarks. Now we can give a basic example of bi-Legendrian structure with its relative bi-Legendrian structure.

EXAMPLE 4.1. Consider $\mathbb{R}^{2n+r}$ with coordinates $x_1,\dots,x_n,y_1,\dots,y_n$, $z_1,\dots,z_r$ and its standard $f\cdot pk$-metric structure (ϕ,η_i,ξ_i,g), $i \in \{1,\dots,r\}$, where

$$\begin{aligned}&\eta_i = dz_i - \sum_{j=1}^{n} y_j dx_j, \quad \xi_i = \frac{\partial}{\partial z_i},\\ &g = \sum_{i=1}^{r} \eta_i \otimes \eta_i + \frac{1}{2}\sum_{j=1}^{n}((dx_j)^2 + (dy_j)^2)\end{aligned} \tag{4}$$

and ϕ is given, with respect the frame $(\partial/\partial x_1,\dots,\partial/\partial x_n,\partial/\partial y_1,\dots,\partial/\partial y_n$, $\partial/\partial z_1,\dots,\partial/\partial z_r)$, by the $(2n+r)\times(2n+r)$-matrix

$$\begin{pmatrix} 0 & I_n & 0\\ -I_n & 0 & 0\\ 0 & Y & 0\end{pmatrix} \tag{5}$$

where Y is the $r \times n$-matrix given by

$$\begin{pmatrix} y_1 & \cdots & y_n\\ \vdots & \ddots & \vdots\\ y_1 & \cdots & y_n\end{pmatrix}.$$

Note that from (4) it follows that $\Phi = d\eta_1 = \dots = d\eta_r = \sum_{i=1}^{n} dx_i \wedge dy_i$. For all $k \in \{1, \dots, n\}$, let

$$X_k := \frac{\partial}{\partial y_k} \quad \text{and} \quad Y_k := \frac{\partial}{\partial x_k} + y_k \frac{\partial}{\partial z_1} + \dots + y_k \frac{\partial}{\partial z_r}$$

and set $L = \operatorname{span}\{X_1, \dots, X_n\}$, $Q = \operatorname{span}\{Y_1, \dots, Y_n\}$. It is easy to check that L, Q are Legendrian distributions on $\mathbb{R}^{2n+r}$, $\phi(L) = Q$ (more precisely, $\phi(X_k) = Y_k$ for each $k \in \{1, \dots, r\}$) and L, Q are integrable. Thus $(\mathcal{F}, \mathcal{G})$ is a bi-Legendrian structure on $\mathbb{R}^{2n+r}$, where, as usual, we have denoted by $\mathcal{F}$ and $\mathcal{G}$ the integral foliations of L and Q, respectively. Since $[X_k, \xi_\alpha] = 0$ and $[Y_k, \xi_\alpha] = 0$ for each $k \in \{1, \dots, n\}$ and $\alpha \in \{1, \dots, r\}$, $\mathcal{F}$ and $\mathcal{G}$ are strongly flat. Consider the bi-Legendrian connection ∇ associated to $(\mathcal{F}, \mathcal{G})$. We show that the curvature tensor of ∇ vanishes identically. First of all it is easy to check that $H(\partial/\partial x_i, \partial/\partial x_j) = H(\partial/\partial y_i, \partial/\partial y_j) = H(\partial/\partial x_i, \partial/\partial y_j) = H(\partial/\partial y_j, \partial/\partial x_i) = 0$ for all $i, j \in \{1, \dots, n\}$. Then ∇ is flat. For example we compute $R(\partial/\partial x_i, \partial/\partial x_j)\partial/\partial x_k$, the other cases being similar. Indeed, by a direct computation we obtain

$$\begin{aligned}\nabla_{\partial/\partial x_j} \frac{\partial}{\partial x_k} &= \sum_\beta \left(y_k H\left(\frac{\partial}{\partial x_j}, \xi_\beta\right) + \left(\frac{\partial y_k}{\partial x_j}\, \xi_\beta\right)_{\mathcal{H}} \right)_Q + \sum_\alpha y_j H\left(\xi_\alpha, \frac{\partial}{\partial x_k}\right)_Q \\ &\quad + \sum_\alpha \sum_\beta \left(y_j y_k \left[\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial z_\beta}\right] + y_j \frac{\partial y_k}{\partial z_\alpha} \frac{\partial}{\partial z_\beta} \right)_Q = 0,\end{aligned}$$

and $R(\partial/\partial x_i, \partial/\partial x_j)\partial/\partial x_k = 0$.

The relevance of this example lies in the fact that, locally, the converse holds, as stated in the following

THEOREM 4.2. *Let $(\mathcal{F}, \mathcal{G})$ be a strongly flat bi-Legendrian structure on the almost $\mathcal{S}$-manifold $(M^{2n+r}, \phi, \eta_i, \xi_i, g)$ and suppose that the corresponding bi-Legendrian connection ∇ is flat. Then the bi-Legendrian structure $(\mathcal{F}, \mathcal{G})$ is locally equivalent to the standard bi-Legendrian structure on* $\mathbb{R}^{2n+r}$.

Proof. Let $p \in M$ be a point and $U \subset M$ a chart containing p. Since Φ_p is a symplectic form on the subspace $\mathcal{H}_p \subset T_pM$, it follows that there exists a g-orthogonal basis $\{e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}, \xi_{1p}, \dots, \xi_{rp}\}$ of T_pM such that $\{e_1, \dots, e_n\}$ is a basis of L_p, $\{e_{n+1}, \dots, e_{2n}\}$ is a basis of Q_p, $e_{n+i} = \phi(e_i)$ and

$$\Phi(e_i, e_j) = \Phi(e_{n+i}, e_{n+j}) = 0, \qquad \Phi(e_i, e_{n+j}) = -\frac{1}{2}\,\delta_{ij} \tag{6}$$

for all $i, j \in \{1, \dots, n\}$. For each $k \in \{1, \dots, 2n\}$ we define a vector field E_k on U by the ∇-parallel transport along curves. More precisely, for any $q \in U$ we consider a curve $\gamma : [0, 1] \to U$ such that $\gamma(0) = p$, $\gamma(1) = q$ and we define $E_k(q) := \tau_\gamma(e_k)$, $\tau_\gamma : T_pM \to T_qM$ being the parallel transport along γ. Note that $E_k(q)$ does not depend on the curve joining p and q, since $R=0$. So we obtain $2n$ vector fields $E_1,\dots,E_{2n}$ on U such that, for each $i\in\{1,\dots,n\}$, $E_i\in\Gamma L$ and $E_{n+i}\in\Gamma Q$, since the bi-Legendrian connection ∇ preserves the foliations $\mathcal{F}$ and $\mathcal{G}$. Moreover, (6) holds at any point of U, that is, for any $q\in U$ and $i,j\in\{1,\dots,n\}$,

$$\varPhi(E_i(q),E_j(q))=\varPhi(E_{n+i}(q),E_{n+j}(q))=0, \tag{7}$$

$$\varPhi(E_i(q),E_{n+j}(q))=-\frac{1}{2}\delta_{ij}. \tag{8}$$

Indeed, since $\varPhi$ is parallel with respect to ∇, for all $h,k\in\{1,\dots,2n\}$,

$$\frac{d}{dt}\varPhi_{\gamma(t)}(E_h(\gamma(t)),E_k(\gamma(t)))=\varPhi_{\gamma(t)}(\nabla_{\gamma'}E_h,E_k)+\varPhi_{\gamma(t)}(E_h,\nabla_{\gamma'}E_k)=0$$

so that $\varPhi_p(e_h,e_k)=\varPhi_q(E_h(q),E_k(q))$ for all $q\in U$. Note that, by construction, we have $\nabla_{E_h}E_k=0$ and $\nabla_{\xi_\alpha}E_k=0$ for all $h,k\in\{1,\dots,2n\}$ and $\alpha\in\{1,\dots,r\}$. From this, Proposition 3.5 and Corollary 3.7, we get

$$[E_i,E_j]=0, \tag{9}$$

$$[E_{n+i},E_{n+j}]=0, \tag{10}$$

$$[E_k,\xi_\alpha]=0, \tag{11}$$

$$[E_i,E_{n+j}]=-T(E_i,E_{n+j})=-2\varPhi(E_i,E_{n+j})\overline{\xi}=\delta_{ij}\sum_{\alpha=1}^{r}\xi_\alpha, \tag{12}$$

for all $i,j\in\{1,\dots,n\}$, $k\in\{1,\dots,2n\}$ and $\alpha\in\{1,\dots,r\}$, and (9)–(12) imply that there exist coordinates $\{x_1,\dots,x_n,y_1,\dots,y_n,z_1,\dots,z_r\}$ such that $E_i=\partial/\partial y_i$, $E_{n+j}=\partial/\partial x_j+y_j\sum_{\alpha=1}^{r}\partial/\partial z_\alpha$, $\xi_\alpha=\partial/\partial z_\alpha$ for any $i\in\{1,\dots,n\}$ and $\alpha\in\{1,\dots,r\}$. Note that from (7) it follows that, in these coordinates, $\varPhi=\sum_{k=1}^{n}dx_k\wedge dy_k$, from which we have, for each $i\in\{1,\dots,r\}$, $d(\eta_i+\sum_{k=1}^{n}y_kdx_k)=0$ and $\eta_i=df_i-\sum_{k=1}^{n}y_kdx_k$ for some $f_i\in C^\infty(U)$. But $\eta_i(E_j)=0$, $\eta_i(E_{n+j})=0$ and $\eta_i(\xi_l)=\delta_{il}$ imply $\partial f_i/\partial y_j=0$, $\partial f_i/\partial x_j=0$ and $\partial f_i/\partial z_l=\delta_{il}$, respectively. So $df_i=dz_i$ and, in this coordinate system,

(i) L is spanned by $\partial/\partial y_h$, $h=1,\dots,n$,
(ii) Q is spanned by $\partial/\partial x_h+y_h\sum_{\alpha=1}^{r}\partial/\partial z_\alpha$, $h=1,\dots,n$,
(iii) the 1-forms η_i, $i\in\{1,\dots,r\}$, are given by $\eta_i=dz_i-\sum_{k=1}^{n}y_kdx_k$.

Finally, from (7) we deduce that $E_{n+i}=\phi(E_i)$ and so ϕ is represented, in the local frame $(\partial/\partial x_1,\dots,\partial/\partial x_n,\partial/\partial y_1,\dots,\partial/\partial y_n,\partial/\partial z_1,\dots,\partial/\partial z_r)$, by the matrix (5). Hence this coordinate system gives the local equivalence between $(\mathcal{F},\mathcal{G})$ and the standard bi-Legendrian structure on $\mathbb{R}^{2n+r}$. ∎

We conclude with another example, showing the relation between the bi-Legendrian connection and the Bott connection. Consider an almost $\mathcal{S}$-manifold $(M^{2n+r},\phi,\eta_i,\xi_i,g)$ such that each ξ_i is a Killing vector field and there exists a strongly flat Legendrian foliation $\mathcal{F}$ on M such that the conjugate Legendrian foliation exists, i.e. $Q = \phi(L)$ is involutive. Then, by Lemma 2.1, also $\mathcal{G}$ is a strongly flat Legendrian foliation, where, as usual, $\mathcal{G}$ denotes the integral foliation of Q. In this situation, as shown in [3], we can define a connection $\overline{\nabla}$ on M in the following way. First of all we consider the Bott connection on $L^{\perp} = Q \oplus E_1 \oplus \cdots \oplus E_r$ given by

$$\nabla_X^{L^\perp} Y := p_{L^\perp}([X,Y])$$

for all $X \in \Gamma L$ and $Y \in \Gamma L^\perp$, where $p_{L^\perp}$ denotes the projection onto $L^\perp$. Then $\nabla^{L^\perp}$ defines a Bott partial connection $\nabla^{L^{\perp^*}}$ in the dual bundle $L^{\perp^*}$ by

$$(\nabla_X^{L^{\perp^*}} v)Y = X(v(Y)) - v([X,Y]) = 2dv(X,Y)$$

for $X \in \Gamma L$, $Y \in \Gamma L^\perp$ and $v \in \Gamma L^{\perp^*}$, which induces a partial connection ∇^{Q^*} defined by

$$\nabla_X^{Q^*} v := p_{Q^*}(\nabla_X^{L^{\perp^*}} v)$$

for $X \in \Gamma L$ and $v \in \Gamma Q^*$. Now, we consider the isomorphism $\Psi : L \to Q^*$ given by $\Psi(X) = \frac{1}{2} i_X \Phi$ and define a partial connection along L by setting

$$\widetilde{\nabla}_X^L X' := \Psi^{-1}(\nabla_X^{Q^*} \Psi(X')).$$

This connection was introduced for the case $r = 1$ by Pang (cf. [9]) who proved that $\widetilde{\nabla}^L$ is torsion free and its curvature vanishes if, as in our case, the Legendrian foliation $\mathcal{F}$ is flat. These results are still valid in the general case (see [3]). The Bott connection $\nabla^{L^\perp}$ also induces a connection ∇^Q on Q given by the formula

$$\nabla_X^Q Y := p_Q([X,Y]).$$

It can be proved that the hypothesis of strong flatness of $\mathcal{F}$ implies that the curvature tensor of ∇^Q vanishes identically. Now, let $\overline{\nabla}'$ be the partial connection along L defined by

$$\overline{\nabla}'_X V := \widetilde{\nabla}_X^L V_L + \nabla_X^Q V_Q + p_{L^\perp}([X, V_E])$$

for all $X \in \Gamma L$ and $V \in \Gamma(TM)$. Then $\overline{\nabla}'$ is a flat connection along L, that is, $R'(X,X') = 0$ for all $X, X' \in \Gamma L$, since both $\widetilde{\nabla}^L$ and ∇^Q are flat connections along L. The same construction can be repeated for Q, as also $\mathcal{G}$ is a strongly flat Legendrian foliation, so we have a partial connection ∇'' along Q given by

$$\overline{\nabla}''_Y V := \widetilde{\nabla}_Y^Q V_Q + \nabla_Y^L V_L + p_{Q^\perp}([Y, V_E])$$

for all $Y \in \Gamma Q$ and $V \in \Gamma(TM)$, which, as before, is flat along Q. Finally, for each $i \in \{1, \dots, r\}$, we set, for all $Z \in \Gamma E_i$ and $V \in \Gamma(TM)$,

$$\overline{\nabla}^{(i)}_Z V := p_L([Z, V_L]) + p_Q([Z, V_Q]) + \sum_{j=1}^{r} Z(\eta_j(V))\xi_j,$$

thus obtaining a connection along the bundle E_i. Using these connections we can define a global connection $\overline{\nabla}$ on M by setting

$$\overline{\nabla}_W V := \overline{\nabla}'_{W_L} V + \overline{\nabla}''_{W_Q} V + \sum_{i=1}^{r} \overline{\nabla}^{(i)}_{W_{E_i}} V$$

for all $V, W \in \Gamma(TM)$. It is not difficult to check that $\overline{\nabla}$ is a connection and, as a consequence of the flatness of $\overline{\nabla}'$ and $\overline{\nabla}''$, it is flat along the leaves of the foliations $\mathcal{F}$ and $\mathcal{G}$. Moreover, for all $i \in \{1, \dots, r\}$,

$$\begin{aligned}\overline{\nabla}_W \xi_i &= \overline{\nabla}'_{W_L} \xi_i + \overline{\nabla}''_{W_Q} \xi_i + \sum_{j=1}^{r} \nabla^{(j)}_{W_{E_j}} \xi_i \\ &= p_Q([W_L, \xi_i]) - p_L([W_Q, \xi_i]) - \sum_{j=1}^{r}\sum_{k=1}^{r} W_{E_j}(\delta_{ki}) = 0,\end{aligned}$$

since both L and Q are strongly flat. It can be easily showed that the torsion $\overline{T}$ of $\overline{\nabla}$ vanishes along L and Q as a consequence of the symmetry of $\widetilde{\nabla}^L$ and $\widetilde{\nabla}^Q$, and, for any $X \in \Gamma L$ and $Y \in \Gamma Q$,

$$\begin{aligned}\overline{T}(X, Y) &= \nabla^Q_X Y - \nabla^L_Y X - [X, Y] = [X, Y]_Q - [Y, X]_L - [X, Y] \\ &= -\sum_{i=1}^{r} \eta_i([X, Y])\xi_i = 2\Phi(X, Y)\overline{\xi}.\end{aligned}$$

THEOREM 4.3. *$\overline{\nabla}$ coincides with the bi-Legendrian connection ∇ associated to the bi-Legendrian structure $(\mathcal{F}, \mathcal{G})$.*

Proof. By the uniqueness of the bi-Legendrian connection associated to $(\mathcal{F}, \mathcal{G})$, it is enough to verify that $\overline{\nabla}$ has all the properties stated in Theorem 3.8. First, directly by our definitions, we see that $\overline{\nabla}$ preserves the foliations $\mathcal{F}$, $\mathcal{G}$ and E_i. Moreover, for all $W \in \Gamma(TM)$,

$$\begin{aligned}\overline{T}(W, \xi_i) &= \overline{\nabla}_W \xi_i - \overline{\nabla}_{\xi_i} W - [W, \xi_i] = -[\xi_i, W_L]_L - [\xi_i, W_Q]_Q \\ &\quad - \sum_{j=1}^{r} \xi_i(\eta_j(W))\xi_j - [W_L, \xi_i] - [W_Q, \xi_i] - \sum_{j=1}^{r} [\eta_j(W)\xi_j, \xi_i] \\ &= [\xi_i, W_L]_Q + [\xi_i, W_Q]_L - \sum_{j=1}^{r} \xi_i(\eta_j(W))\xi_j + \sum_{j=1}^{r} \xi_i(\eta_j(W))\xi_j \\ &= [\xi_i, W_L]_Q + [\xi_i, W_Q]_L.\end{aligned}$$

Since $\overline{\nabla}_{\xi_i} Z = \overline{\nabla}_Z \xi_i = 0$ for all $Z \in \Gamma(TM)$, conditions (i) and (ii) of Theorem 3.8 are satisfied. Finally, as $\overline{\nabla}$ preserves the foliations, we see directly that $(\overline{\nabla}_Z \Phi)(V,W) = Z(\Phi(V,W)) - \Phi(\overline{\nabla}_Z V, W) - \Phi(V, \overline{\nabla}_Z W) = 0$ for $V, W \in \Gamma L$ or $V, W \in \Gamma Q$, so it remains to check that $(\overline{\nabla}_Z \Phi)(X,Y) = 0$ for all $Z \in \Gamma(TM)$, $X \in \Gamma L$ and $Y \in \Gamma Q$. We consider the two cases $Z = \xi_i$ and $Z \in \Gamma\mathcal{H}$. We have

$$\begin{aligned}(\overline{\nabla}_{\xi_i}\Phi)(X,Y) &= \xi_i(\Phi(X,Y)) - \Phi([\xi_i, X]_L, Y) - \Phi(X, [\xi_i, Y]_Q) \\ &= \xi_i(\Phi(X,Y)) - \Phi([\xi_i, X], Y) - \Phi(X, [\xi_i, Y]) \\ &= (\mathcal{L}_{\xi_i}\Phi)(X,Y) = 0,\end{aligned}$$

and, if $Z \in \Gamma\mathcal{H}$,

$$\begin{aligned}(\overline{\nabla}_Z\Phi)(X,Y) &= Z(\Phi(X,Y)) - \Phi(\widetilde{\nabla}^L_{Z_L}X + [Z_Q, X]_L, Y) \\ &\qquad - \Phi(X, [Z_L, Y]_Q + \widetilde{\nabla}^Q_{Z_Q}Y) \\ &= Z(\Phi(X,Y)) - \Phi(\widetilde{\nabla}^L_{Z_L}X, Y) \\ &\qquad - \Phi([Z_Q, X], Y) - \Phi(X, [Z_L, Y]) - \Phi(X, \widetilde{\nabla}^Q_{Z_Q}Y) \\ &= Z(\Phi(X,Y)) - \Phi(\Psi^{-1}(\nabla^{Q^*}_{Z_L}\Psi(X)), Y) - \Phi([Z_Q, X], Y) \\ &\qquad - \Phi(X, [Z_L, Y]) - \Phi(X, \Psi^{-1}(\nabla^{L^*}_{Z_Q}\Psi(Y))) \\ &= Z(\Phi(X,Y)) - Z_L(\Psi(X)(Y)) + \Psi(X)([Z_L, Y]) \\ &\qquad - \Phi([Z_Q, X], Y) - \Phi(X, [Z_L, Y]) + Z_Q(\Psi(Y)(X)) - \Psi(Y)([Z_Q, X]) \\ &= Z(\Phi(X,Y)) - Z_L(\Phi(X,Y)) + \Phi(X, [Z_L, Y]) - \Phi([Z_Q, X], Y) \\ &\qquad - \Phi(X, [Z_L, Y]) + Z_Q(\Phi(Y,X)) - \Phi(Y, [Z_Q, X]) = 0.\end{aligned}$$

Therefore $\overline{\nabla}$ has all the properties which characterize the bi-Legendrian connection ∇ associated to the bi-Legendrian structure $(\mathcal{F}, \mathcal{G})$, hence, by the uniqueness of this connection, $\overline{\nabla} = \nabla$. ∎

In particular, from Theorem 4.3 and Proposition 3.10 it follows that, for the connection $\overline{\nabla}$ associated to a strongly flat bi-Legendrian structure $(\mathcal{F}, \mathcal{G})$,

$$\overline{R}(V, \xi_i) = 0 \tag{13}$$

for every $V \in \Gamma(TM)$ and $i \in \{1, \ldots, r\}$. Note that (13) is rather difficult to check directly.

Remark 4.4. We emphasize that for $r = 0$ the theory of bi-Legendrian connections reduces to the theory of bi-Lagrangian connections in symplectic geometry. In particular Theorem 4.2 is a generalization of the well known theorem of Hess which states that if the curvature of the bi-Lagrangian connection associated to a bi-Lagrangian structure on a symplectic manifold (M^{2n}, ω) vanishes identically, then the bi-Lagrangian structure is locally isomorphic to the standard structure $(\mathcal{F}, \mathcal{G})$ on $\mathbb{R}^{2n}$ given by $\mathcal{F} = \{x_1 = \text{const}, \ldots, x_n = \text{const}\}$ and $\mathcal{G} = \{y_1 = \text{const}, \ldots, y_n = \text{const}\}$.

For $r = 1$ we obtain the theory of bi-Legendrian connections on contact manifolds, which was the initial motivation for this work. We note that in this case the notions of flatness and strong flatness of a Legendrian foliation are equivalent.

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