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On the Łojasiewicz exponent near the fibre of polynomial mappings

by Ha Huy Vui and Nguyen Hong Duc (Hanoi)

Abstract. We give the formula expressing the Łojasiewicz exponent near the fibre of polynomial mappings in two variables in terms of the Puiseux expansions at infinity of the fibre.

1. Introduction. Let M, N, L be finite-dimensional real vector spaces and let $g\colon X \to N$ and $f\colon X \to L$ be semialgebraic mappings, where $X \subset M$.

For a set $S \subset X$, put

$$\mathcal{L}_\infty(g|_S) := \sup\{\nu \in \mathbb{R}\colon \exists C, R > 0, \forall x \in S\ (\|x\| \geq R \Rightarrow \|g(x)\| \geq C\|x\|^\nu)\}.$$

For $\lambda \in L$, put

$$\mathcal{L}_{\infty,f\to\lambda}(g) := \sup\{\mathcal{L}_\infty(g|_{f^{-1}(U)})\colon U \subset L \text{ is a neighbourhood of } \lambda\}.$$

Motivated by results of [H], [C-K1], [C-K2], [P], [KOS], ... on bifurcation values at infinity of polynomial functions, the number $\mathcal{L}_{\infty,f\to\lambda}(g)$, called the *Łojasiewicz exponent at infinity of g near the fibre* $f^{-1}(\lambda)$, was introduced and studied in [Sk] and [R-S]. The authors of [R-S] proved that:

(i) $\mathcal{L}_{\infty,f\to\lambda}(g) \in \mathbb{Q} \cup \{\pm\infty\}$.
(ii) There is a semialgebraic stratification $L = S_1 \cup \cdots \cup S_j$ such that the function $\nu\colon L \ni \lambda \mapsto \mathcal{L}_{\infty,f\to\lambda}(g)$ is constant on each stratum S_i.

Our aim in this paper is to study $\mathcal{L}_{\infty,f\to\lambda}(g)$ in the case when f and g are polynomials in two real or complex variables. In this very restrictive setting we can give complete results about the Łojasiewicz exponent at infinity near the fibre in the complex case. In brief, our results are the following. Let $f(x,y)$ be a non-constant monic polynomial in x, i.e. $f(x,y) = x^d + a_1(y)x^{d-1} + \cdots + a_d(y)$, where $a_i \in \mathbb{C}[y]$ and $\deg a_i \leq i$. Then:

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(i) The set of $\lambda \in \mathbb{C}$ such that $\mathcal{L}_{\infty,f\to\lambda}(g) = -\infty$ coincides with a certain set $A(f,g)$ which is defined in terms of the Puiseux expansions at infinity of $g = 0$.

(ii) If $\mathcal{L}_{\infty,f\to\lambda}(g) \neq -\infty$ then
$$\mathcal{L}_{\infty,f\to\lambda}(g) = \min\{\deg g(x_i(y), y)\}$$
in the complex case, and
$$\mathcal{L}_{\infty,f\to\lambda}(g) = \min\{\deg g(x_i^{\mathbb{R}}(y), y)\}$$
in the real case, where $x_i(y)$ runs over the set of Puiseux expansions at infinity of the fibre $f^{-1}(\lambda)$, and $x_j^{\mathbb{R}}(y)$ is the real approximation of $x_j(y)$.

(iii) If f, g are complex polynomials in two variables, then the function
$$\nu\colon \mathbb{C} \setminus A(f,g) \to \mathbb{Q} \cup \{\pm\infty\}, \qquad \lambda \mapsto \mathcal{L}_{\infty,f\to\lambda}(g),$$
is constant.

The paper is organized as follows. In Section 2 we describe the process of sliding of [K-P] in the form which is most convenient for us. The main results are stated and proved in Section 3.

2. Sliding. In this section we prove some lemmas about the process of sliding in both complex and real cases.

If $\varphi(\tau)$ is a series of the form
$$\varphi(\tau) = a_0\tau^{\alpha} + \text{terms of lower degree} \quad \text{with } a_0 \neq 0,$$
then the number α is denoted by $\deg\varphi$.

Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be a polynomial. For a series
$$x = \varphi(y) = c_1 y^{n_1/N} + c_2 y^{n_2/N} + \cdots,$$
where $c_i \in \mathbb{C}, n_i \in \mathbb{Z}$ and $c_1 \neq 0, n_1 > n_2 > \cdots$, we put
$$M(X,Y) = f\left(X + \varphi\left(\frac{1}{Y}\right), \frac{1}{Y}\right) = \sum_{i,j} c_{ij} X^i Y^{j/N}.$$

For each $c_{ij} \neq 0$, let us plot a dot at $(i, j/N)$, called a *Newton dot*. The set of Newton dots is called the *Newton diagram*. The boundary of its convex hull is the *Newton polygon of f relative to φ*, to be denoted by $\mathbb{P}(f,\varphi)$ or $\mathbb{P}(M)$.

Assume that $x = \varphi(y)$ is not a Puiseux root at infinity of $f = 0$. Then the Y-axis contains at least one dot of M. Let $(0, h_M)$ be the lowest one. We see that $h_M = -\deg f(\varphi(y), y)$.

By the *highest Newton edge H_M* of M we mean the edge of $\mathbb{P}(M)$ with one extremity $(0, h_M)$ and such that all Newton dots of M lie on or above the line containing H_M. Let $\theta_M = \tan\varphi$, where φ is the angle between H_M and the X-axis. Note that if $(i, j/N)$ is a Newton dot of M then $\theta_M i + j/N \geq h_M$, and $(i, j/N) \in H_M$ if and only if $\theta_M i + j/N = h_M$. If $x = \varphi(y)$ is a Puiseux root at infinity of $f = 0$, we set $h_M = +\infty$ and $\theta_M = +\infty$.

We associate with H_M the polynomial $\varepsilon_M(x) := \varepsilon_M(x, 1)$, where

$$\varepsilon_M(X, Y) = \sum_{(i,j/N)\in H_M} c_{ij} X^i Y^{j/N}.$$

LEMMA 2.1. *Let* $\widetilde{M}(X, Y) = M(X + cY^\theta, Y)$. *We have*:

(a) *If* $\theta > \theta_M$, *then* $h_{\widetilde{M}} = h_M$ *and* $\theta_{\widetilde{M}} = \theta_M$.
(b) *If* $\theta = \theta_M$ *and* c *is a non-zero root of* $\varepsilon_M(x)$, *then* $h_{\widetilde{M}} > h_M$ *and* $\theta_{\widetilde{M}} > \theta_M$.
(c) *If* $\theta = \theta_M$ *and* $\varepsilon_M(c) \neq 0$, *then* $h_{\widetilde{M}} = h_M$ *and* $\theta_{\widetilde{M}} = \theta_M$.

Proof. (a) Let $(0, h_M)$ and $(i_0, j_0/N)$ be the extremities of H_M. It is clear that the coefficient $\widetilde{c}_{0h_M}$ of $X^0Y^{h_M}$ in $\widetilde{M}$ is $\sum_{i\geq 0} c_{ij}c^i$, $\theta i + j/N = h_M$. Since $\theta_M i + j/N \geq h_M$ and by the hypothesis $\theta > \theta_M$, we see that $\widetilde{c}_{0h_M} = c_{0h_M}$. Therefore $(0, h_M)$ is a Newton dot of $\widetilde{M}$.

Analogously, we can show that $(i_0, j_0/N)$ is also a Newton dot of $\widetilde{M}$ and all Newton dots of $\widetilde{M}$ lie on or above H_M. Thus $H_M \equiv H_{\widetilde{M}}$. Hence $h_{\widetilde{M}} = h_M$ and $\theta_{\widetilde{M}} = \theta_M$.

(b) Let $(0, \beta)$ be any Newton dot of $\widetilde{M}$ on the Y-axis. Since $\theta = \theta_M$ and the coefficient of X^0Y^β in $\widetilde{M}$ is $\sum_{\theta i + j/N = \beta} c_{ij}c^i$, it is clear that $\beta \geq h_M$ and therefore $h_{\widetilde{M}} \geq h_M$.

Since $\varepsilon_M(c) = 0$, $H_{\widetilde{M}}$ does not contain the dot $(0, h_M)$. Hence $h_{\widetilde{M}} > h_M$.

As in the proof of (a), $(i_0, j_0/N)$ is a Newton dot of $\widetilde{M}$. Therefore

$$h_M - \theta_M i_0 = j_0/N \geq h_{\widetilde{M}} - \theta_{\widetilde{M}} i_0 > h_M - \theta_{\widetilde{M}} i_0.$$

Thus $\theta_{\widetilde{M}} > \theta_M$.

(c) follows easily from the proof of (b). ■

If c is a non-zero root of $\varepsilon_M(x)$, the series $\varphi_1(y) = \varphi(y) + cy^{-\theta_M}$ will be called a *sliding of* $\varphi(y)$ *along* f. A recursive sliding $\varphi \to \varphi_1 \to \cdots$ produces a limit, φ_∞, where $\varphi_\infty(y) = \varphi_i(y)$ if $f(\varphi_i(y), y) = 0$. The series φ_∞ is a Puiseux root at infinity of $f = 0$ and will be called a *final result of sliding* φ *along* f.

LEMMA 2.2. *Let* $f, g\colon \mathbb{C}^2 \to \mathbb{C}$ *be two polynomials. For a series* $x = \varphi(y)$, *put*

$$M(X, Y) = f(X + \varphi(1/Y), 1/Y), \qquad N(X, Y) = g(X + \varphi(1/Y), 1/Y).$$

We have:

(a) *If* $\theta_M > \theta_N$, *then* $\deg g(\varphi_\infty(y), y) = \deg g(\varphi(y), y)$.
(b) *If* $\theta_M = \theta_N$, *then* $\deg g(\varphi_\infty(y), y) \leq \deg g(\varphi(y), y)$.

Here $x = \varphi_\infty(y)$ *is a final result of sliding* φ *along* f.

Proof. Assume that $\varphi(y) \to \varphi_1(y) \to \cdots \to \varphi_\infty(y)$ is a process of sliding of φ along f, where $\varphi_0(y) = \varphi(y)$ and $\varphi_{i+1}(y) = \varphi_i(y) + c_i y^{-\theta_i}$. Put

$$\begin{aligned} M_0 &= M, \quad M_{i+1}(X,Y) = M_i(X + c_i Y^{\theta_i}, Y), \\ N_0 &= N, \quad N_{i+1}(X,Y) = N_i(X + c_i Y^{\theta_i}, Y) \end{aligned}$$

We get

$$M_\infty(X,Y) = f(X + \varphi_\infty(1/Y), 1/Y), \quad N_\infty(X,Y) = g(X + \varphi_\infty(1/Y), 1/Y).$$

Since $\theta_i = \theta_{M_i}$ by the definition of sliding, Lemma 2.1(b) implies $\theta_i < \theta_{i+1}$.

(a) Since $\theta_M > \theta_N$, applying Lemma 2.1(a), we have $h_{N_1} = h_N$ and $\theta_{N_1} = \theta_N$. Therefore

$$\theta_1 > \theta_0 = \theta_M > \theta_N = \theta_{N_1}.$$

Again by Lemma 2.1(a), we get

$$h_{N_2} = h_{N_1} \quad \text{and} \quad \theta_{N_2} = \theta_{N_1}.$$

Applying Lemma 2.1(a) infinitely many times we finally obtain $h_{N_\infty} = h_N$, which means that $\deg g(\varphi_\infty(y), y) = \deg g(\varphi(y), y)$.

(b) Suppose that $\theta_M = \theta_N$. Assume that k is a natural number such that $\varepsilon_{N_i}(c_i) = 0$ and $\theta_{M_i} = \theta_{N_i}$ for $i = 0, 1, \ldots, k-1$, but either $\varepsilon_{N_k}(c_k) \neq 0$ or $\theta_{M_k} \neq \theta_{N_k}$. Since $\varepsilon_{N_i}(c_i) = 0$ and $\theta_{M_i} = \theta_{N_i}$ for $i = 0, 1, \ldots, k-1$, Lemma 2.1(b) gives $h_{N_{i+1}} \geq h_{N_i}$ for $i = 0, 1, \ldots, k$. Therefore $h_{N_k} \geq h_N$.

Claim. *$h_{N_{k+1}} \geq h_N$ and $\theta_{N_{k+1}} < \theta_{M_{k+1}}$.*

To see this, we have to consider several cases.

If $\theta_{M_k} > \theta_{N_k}$ then by Lemma 2.1(a),

$$h_{N_{k+1}} = h_{N_k} \geq h_N \quad \text{and} \quad \theta_{N_{k+1}} = \theta_{N_k} < \theta_{M_k} < \theta_{M_{k+1}}.$$

If $\theta_{M_k} = \theta_{N_k}$, then $\varepsilon_{N_k}(c_k)$ must be non-zero and Lemma 2.1(c) yields $h_{N_{k+1}} = h_{N_k}$ and $\theta_{N_{k+1}} = \theta_{N_k}$. As before, we see that $h_{N_{k+1}} \geq h_N$ and $\theta_{N_{k+1}} < \theta_{M_{k+1}}$.

If $\theta_{M_k} < \theta_{N_k}$, then as in the proof of Lemma 2.1(b), if $(i_0, j_0/N)$ is the other extremity of $H_{N_{k-1}}$, it is also the other extremity of H_{N_k}. Therefore

$$\begin{aligned} h_{N_{k+1}} &= \theta_{M_k} i_0 + j_0/N > \theta_{N_{k-1}} i_0 + j_0/N = h_{N_{k-1}} \geq h_N, \\ \theta_{N_{k+1}} &= \theta_{M_k} < \theta_{M_{k+1}}. \end{aligned}$$

Now, using the claim and by the same argument as in the proof of (a), we get

$$h_N \leq h_{N_{k+1}} = h_{N_\infty}.$$

Hence $\deg g(\varphi_\infty(y), y) \leq \deg g(\varphi(y), y)$. ∎

Let us consider a series $x = \lambda(y)$ of the form

$$x = \lambda(y) = a_1 y^{\alpha_1} + a_2 y^{\alpha_2} + \cdots,$$

where $\alpha_1 > \alpha_2$. If $a_1, \dots, a_{s-1} \in \mathbb{R}$ and $a_s \notin \mathbb{R}$, we put

$$\lambda^{\mathbb{R}}(y) := a_1 y^{\alpha_1} + \dots + a_{s-1} y^{\alpha_{s-1}} + c y^{\alpha_s},$$

where c is a generic real number. We call $\lambda^{\mathbb{R}}(y)$ the *real approximation of* $\lambda(y)$.

LEMMA 2.3. *Let $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ be polynomials. For a series $x = \varphi(y)$, put*

$$M(X,Y) = f(X + \varphi(1/Y), 1/Y), \qquad N(X,Y) = g(X + \varphi(1/Y), 1/Y).$$

Let $x = \varphi_\infty(y)$ be a final result of sliding φ along f and $\varphi_\infty^{\mathbb{R}}(y)$ be the real approximation of $\varphi_\infty(y)$. We have:

(a) *If $\theta_M > \theta_N$, then* $\deg g(\varphi_\infty^{\mathbb{R}}(y), y) = \deg g(\varphi(y), y)$.

(b) *If $\theta_M = \theta_N$, then* $\deg g(\varphi_\infty^{\mathbb{R}}(y), y) \le \deg g(\varphi(y), y)$.

In particular with $g = f$, we have $\deg f(\varphi_\infty^{\mathbb{R}}(y), y) \le \deg f(\varphi(y), y)$.

Proof. (a) If $\varphi_\infty(y) = \varphi(y)$ then $\varphi_\infty^{\mathbb{R}}(y) = \varphi(y)$ and then automatically $\deg g(\varphi(y), y) = \deg g(\varphi_\infty^{\mathbb{R}}(y), y)$. Otherwise, we write

$$\varphi_\infty(y) - \varphi(y) = b_0 y^{\beta_0} + \dots + b_{s-1} y^{\beta_{s-1}} + b_s y^{\beta_s} + \cdots,$$

where $\beta_0 = -\theta_M > \beta_1 > \cdots$, $b_0, b_1, \dots, b_{s-1} \in \mathbb{R}$ and $b_s \notin \mathbb{R}$ for some $s \ge 0$. We write $\varphi(y)$ as a sum $\varphi(y) = \psi(y) + \gamma(y)$ with

$$\psi(y) = \sum_{\alpha > \beta_s} a_\alpha y^\alpha, \qquad \gamma(y) = \sum_{\alpha \le \beta_s} a_\alpha y^\alpha.$$

Clearly

$$\varphi_\infty(y) = \psi(y) + (b_0 y^{\beta_0} + \dots + b_{s-1} y^{\beta_{s-1}} + b_s y^{\beta_s} + \cdots) + \gamma(y).$$

Therefore

$$\begin{aligned}\varphi_\infty^{\mathbb{R}}(y) &= \psi(y) + b_0 y^{\beta_0} + \dots + b_{s-1} y^{\beta_{s-1}} + c y^{\beta_s} \\ &= \varphi(y) + b_0 y^{\beta_0} + \dots + b_{s-1} y^{\beta_{s-1}} + c y^{\beta_s} - \gamma(y) \\ &= \varphi(y) + a_0 y^{\alpha_0} + a_1 y^{\alpha_1} + \cdots,\end{aligned}$$

where $\beta_0 = \alpha_0 > \alpha_1 > \cdots$. By putting

$$\begin{aligned} M_0 &= M, \qquad M_{i+1}(X,Y) = M_i(X + a_i Y^{-\alpha_i}, Y), \\ N_0 &= N, \qquad N_{i+1}(X,Y) = N_i(X + a_i Y^{-\alpha_i}, Y), \end{aligned}$$

we get

$$M_\infty(X,Y) = f(X + \varphi_\infty^{\mathbb{R}}(1/Y), 1/Y), \qquad N_\infty(X,Y) = g(X + \varphi_\infty^{\mathbb{R}}(1/Y), 1/Y).$$

Since, by the hypothesis, $\theta_M > \theta_N$ and $\theta_M = -\beta_0$ $(= -\alpha_0 < -\alpha_1 < \cdots)$, the same argument as in the proof of Lemma 2.2(a) gives $h_{N_\infty} = h_N$. Hence $\deg g(\varphi_\infty^{\mathbb{R}}(y), y) = \deg g(\varphi(y), y)$.

(b) Now assume that $\theta_M = \theta_N$. Let k be a natural number such that $\varepsilon_{N_i}(a_i) = 0$ and $\theta_{M_i} = \theta_{N_i}$ for $i = 0, 1, \dots, k-1$, but either $\varepsilon_{N_k}(a_k) \ne 0$ or

$\theta_{M_k} \neq \theta_{N_k}$. It is clear that $k \leq s$. Since $\theta_M = \theta_N$, by the same argument as in the proof of Lemma 2.2(b) we get $\deg g(\varphi^{\mathbb{R}}_{\infty}(y), y) \leq \deg g(\varphi(y), y)$. ■

3. Main results. Let $f, g\colon \mathbb{K}^2 \to \mathbb{K}$, with $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$, be polynomial functions and let $\lambda \in \mathbb{K}$. Put

$$\widetilde{\mathcal{L}}_{\infty, f\to\lambda}(g) = \inf_{\Phi} \frac{\deg g \circ \Phi}{\deg \Phi},$$

where Φ runs over the set of meromorphic function at infinity such that

$$\deg \Phi > 0, \qquad \deg(f-\lambda) \circ \Phi < 0.$$

If f is monic in x, then Φ can be written in the form $x = \varphi(y)$ with $\deg \varphi \leq 1$ and

$$\frac{\deg g \circ \Phi}{\deg \Phi} = \deg g(\varphi(y), y).$$

According to [Sk, Theorem 2.1], we know that

$$\widetilde{\mathcal{L}}_{\infty, f\to\lambda}(g) = \mathcal{L}_{\infty, f\to\lambda}(g).$$

Theorem 3.1. *Let f and g be polynomials in two complex variables (x, y). Assume that f is monic in x. Let $x = x_i(y)$, $i = 1, \dots, d$, (respectively, $x = \widetilde{x}_j(y)$, $j = 1, \dots, s$) be the Puiseux expansions at infinity of $f(x, y) - \lambda = 0$ (respectively, of $g(x, y) = 0$). Let*

$$A(f, g) := \{\lambda_j \in \mathbb{C}\colon \lambda_j = \lim_{y\to\infty} f(\widetilde{x}_j(y), y),\ j = 1, \dots, s\}.$$

Then:

(a) $\mathcal{L}_{\infty, f\to\lambda}(g) = -\infty$ *if and only if* $\lambda \in A(f, g)$.
(b) *If* $\mathcal{L}_{\infty, f\to\lambda}(g) \neq -\infty$ *then*

$$\mathcal{L}_{\infty, f\to\lambda}(g) = \min\{\deg g(x_i(y), y) : i = 1, \dots, d\}.$$

Proof. (a) Put

$$l = \min_i \deg g(x_i(y), y).$$

It is obvious that $\lambda \in A(f, g)$ if $l = -\infty$. Let $l > -\infty$. Since $\mathcal{L}_{\infty, f\to\lambda}(g) = -\infty$, there is a curve $x = \gamma(y)$ meromorphic at infinity such that

$$\deg(f(\gamma(y), y) - \lambda) < 0 \quad \text{and} \quad \deg g(\gamma(y), y) < l.$$

Put

$$M(X, Y) = f(X + \gamma(1/Y), 1/Y) - \lambda, \qquad N(X, Y) = g(X + \gamma(1/Y), 1/Y).$$

First, we see that $\theta_M < \theta_N$. Indeed, assume that this is not the case. Take a final result $\gamma_\infty(y)$ of sliding $\gamma(y)$ along $f - \lambda$. This series will be a Puiseux root at infinity of $f - \lambda$: $f(\gamma_\infty(y), y) - \lambda = 0$. Then Lemma 2.2 yields

$$\deg g(\gamma_\infty(y), y) \leq \deg g(\gamma(y), y) < l,$$

a contradiction. Now applying Lemma 2.2(a) with $\theta_M < \theta_N$ we get

$$\deg(f(\widetilde{\gamma}_\infty(y), y) - \lambda) \le \deg(f(\gamma(y), y) - \lambda) < 0,$$

where $\widetilde{\gamma}_\infty(y)$ is a final result of sliding $\gamma(y)$ along g. Hence $\lambda \in A(f, g)$.

(b) Let $x = \varphi(y)$ be a meromorphic curve at infinity which satisfies $\deg(f(\varphi(y), y) - \lambda) < 0$. Put

$$M(X, Y) = f(X + \varphi(1/Y)) - \lambda, \qquad N(X, Y) = g(X + \varphi(1/Y)).$$

Since $\mathcal{L}_{\infty, f \to \lambda}(g) \ne -\infty$, we can show as before that $\theta_M \ge \theta_N$. Therefore Lemma 2.2 yields $\deg g(\varphi_\infty(y)(y), y) \le \deg g(\varphi(y), y)$, where $\varphi_\infty(y)$ is a final result of sliding $\varphi(y)$ along $f - \lambda$. Thus

$$\mathcal{L}_{\infty, f \to \lambda}(g) = \inf_{\varphi} \deg g(\varphi(y), y) \ge \min_i \deg g(x_i(y), y).$$

Since the opposite inequality is always satisfied, the assertion follows. ∎

THEOREM 3.2. *With the notations of Theorem* 3.1, *the function* $\vartheta(\lambda) = \mathcal{L}_{\infty, f \to \lambda}(g)$ *is constant on* $\mathbb{C} \setminus A(f, g)$.

Proof. Suppose $0, \lambda \notin A(f, g)$. We only need to prove that $\vartheta(0) \ge \vartheta(\lambda)$. By Theorem 3.1 with $0 \in \mathbb{C} \setminus A(f, g)$, there is a Puiseux root at infinity $x = \varphi(y)$ of f such that $\vartheta(0) = \deg g(\varphi(y), y)$. Put

$$M_\lambda(X, Y) = f(X + \varphi(1/Y), 1/Y) - \lambda, \qquad N(X, Y) = g(X + \varphi(1/Y), 1/Y).$$

We shall show that $\theta_{M_\lambda} \ge \theta_N$. By contradiction suppose that $\theta_{M_\lambda} < \theta_N$. Let $\widetilde{\varphi}_\infty(y)$ be a final result of sliding $\varphi(y)$ along g. By Lemma 2.2(a),

$$\deg f(\widetilde{\varphi}_\infty(y), y) = \deg f(\varphi(y), y) < 0,$$

which is impossible, because 0 is not in $A(f, g)$.

Now, since $\theta_{M_\lambda} \ge \theta_N$, Lemma 2.2 shows that $\deg g(\varphi_\infty(y), y) \le \nu(0)$, where $\varphi_\infty(y)$ is a final result of sliding $\varphi(y)$ along $f - \lambda$. Thus $\vartheta(0) \ge \vartheta(\lambda)$. ∎

We denote by

$$J_\Phi := \{(u, v) \in \mathbb{C}^2 : \exists \{z_n\} \subset \mathbb{C}^2,\ z_n \to \infty,\ \Phi(z_n) \to (u, v)\}$$

the Jelonek set of $\Phi = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$. The following proposition is also a consequence of [C-K2, Theorem 1].

PROPOSITION 3.1. *Let* $\Phi = (f, g) : \mathbb{C}^2 \to \mathbb{C}^2$ *be a polynomial mapping with* f, g *monic in* x. *Let* $(u, v) \in \mathbb{C}^2$. *Then*:

(a) $(u, v) \in J_\Phi$ *if and only if either there exists a Puiseux expansion at infinity* $x = x(y)$ *of* $f(x, y) = u$ *such that* $\deg(g(x(y), y) - v) < 0$, *or there exists a Puiseux expansion at infinity* $x = \widetilde{x}(y)$ *of* $g(x, y) = v$ *such that* $\deg(f(\widetilde{x}(y), y) - u) < 0$.

(b) $(u, v) \in J_\Phi$ *if and only if either* $\mathcal{L}_{\infty, f \to u}(g - v)$ *or* $\mathcal{L}_{\infty, g \to v}(f - u)$ *is* $-\infty$.

Proof. (a) Let us proceed as in [Sk]. Suppose that $(u,v) \in J_\Phi$. For every $\delta > 0$ there is $z^0 \in \mathbb{C}^2$ such that

$$\|z^0\| > 1/\delta \quad \text{and} \quad \|(f(z^0) - u, g(z^0) - v)\| < \delta.$$

Let $B = \{z \in \mathbb{C}^2 \colon \|z\| < 1\}$. The mapping $H \colon B \ni z \mapsto z/(1 - \|z\|^2) \in \mathbb{C}^2$ is a rational homeomorphism. Hence, the set

$$X := \{(z,\delta) \in B \times (0,+\infty) \colon \|H(z)\| > 1/\delta \text{ and } \|(f(H(z)) - u, g(H(z)) - v)\| < \delta\}$$

is semialgebraic and there is a sequence of points $(\omega^k, \delta_k) \in X$ convergent to a point $(\omega^0, 0)$ such that $\omega^0 \in \partial B$. Therefore by the curve selection lemma, there exists a curve $\widetilde{\Psi} = (\widetilde{\varphi}, \psi) \colon (R, +\infty) \to X$, meromorphic at infinity, such that $\lim_{t\to\infty} \widetilde{\Psi} = (\omega^0, 0)$. By putting $\varphi = H \circ \widetilde{\varphi}$, we obtain the curve $\Psi = (\varphi, \psi)$ meromorphic at infinity such that $\deg \varphi \geq -\deg \psi > 0$, $\deg(f - u) \circ \varphi < 0$ and $\deg(g - v) \circ \varphi < 0$. We can take φ in the form $x = \varphi(y)$, so $\deg(f(\varphi(y), y) - u) < 0$ and $\deg(g(\varphi(y), y) - v) < 0$. Put

$$M(X,Y) = f(X + \varphi(1/Y), 1/Y) - u, \qquad N(X,Y) = g(X + \varphi(1/Y), 1/Y) - v.$$

Then $h_M > 0$ and $h_N > 0$.

If $\theta_M \geq \theta_N$, let $x = x(y)$ be a final result of sliding φ along $f - u$. By Lemma 2.2, $\deg(g(x(y), y) - v) < 0$.

If $\theta_M < \theta_N$ then Lemma 2.2(a) yields $\deg(f(\widetilde{x}(y), y) - u) < 0$, where $x = \widetilde{x}(y)$ is a final result of sliding φ along $g - v$.

(b) follows easily from (a) and Theorem 3.1. ∎

THEOREM 3.3. *Let $f, g \colon \mathbb{R}^2 \to \mathbb{R}$ be real polynomials monic in x. Let $\lambda \in \mathbb{R}$. Let $x = x_i(y)$, $i = 1, \ldots, d$, be the Puiseux expansions at infinity of $f(x,y) - \lambda = 0$ and $x_i^{\mathbb{R}}(y)$ be the real approximation of $x_i(y)$. Put*

$$V_{\mathbb{R}}(f) := \{x_i(y) \colon \deg(f(x_i^{\mathbb{R}}(y), y) - \lambda) < 0\}.$$

Let $x = \widetilde{x}_j(y)$, $j = 1, \ldots, s$, be the real Puiseux expansions at infinity of $g(x,y) = 0$. Put

$$A_{\mathbb{R}}(f,g) := \{\lambda \in \mathbb{R} \colon \lim_{y\to\infty} f(\widetilde{x}_j(y), y),\ j = 1, \ldots, s\}.$$

(If $g(x,y) = 0$ has no real Puiseux root at infinity, we put $A_{\mathbb{R}}(f,g) = \emptyset$.) Then:

(a) *If $\mathcal{L}_{\infty, f_{\mathbb{C}} \to \lambda}(g_{\mathbb{C}}) \neq -\infty$ then*

$$\mathcal{L}_{\infty, f\to\lambda}(g) = \min\{\deg g(x^{\mathbb{R}}(y), y) \colon x(y) \in V_{\mathbb{R}}(f)\}.$$

(b) $\mathcal{L}_{\infty, f\to\lambda}(g) = -\infty$ *if and only if* $\lambda \in A_{\mathbb{R}}(f,g)$.

(c) $\mathcal{L}_{\infty, f\to\lambda}(g) = +\infty$ *if and only if* $\lambda \notin J_f$.

Proof. (a) Let $x = \varphi(y)$ be a meromorphic real curve at infinity with $\deg(f(\varphi(y), y) - \lambda) < 0$. Put

$$M(X,Y) = f(X + \varphi(1/Y), 1/Y) - \lambda$$

and

$$N(X,Y) = g(X + \varphi(1/Y), 1/Y).$$

First, we show that $\theta_M > \theta_N$. In fact, otherwise let $\widetilde{\varphi}_\infty$ be a final result of sliding φ along g. By Lemma 2.2, $\deg(f(\widetilde{\varphi}(y), y) - \lambda) < 0$ and therefore $\mathcal{L}_{\infty, f_{\mathbb{C}} \to \lambda}(g_{\mathbb{C}}) = -\infty$, which is impossible.

Since $\theta_M > \theta_N$, Lemma 2.3 yields

$$\deg g(\varphi_\infty^{\mathbb{R}}(y), y) \le \deg g(\varphi(y), y),$$
$$\deg(f(\varphi_\infty^{\mathbb{R}}(y), y) - \lambda) \le \deg(f(\varphi(y), y) - \lambda) < 0,$$

where $\varphi_\infty(y)$ is a final result of sliding $\varphi(y)$ along $f - \lambda$. Thus

$$\mathcal{L}_{\infty, f \to \lambda}(g) \ge \min\{\deg g(x^{\mathbb{R}}(y), y) \colon x(y) \in V_{\mathbb{R}}(f)\}.$$

The opposite inequality always holds.

(b) Assume that $\mathcal{L}_{\infty, f \to \lambda}(g) = -\infty$. Clearly, if

$$k := \min\{\deg g(x^{\mathbb{R}}(y), y) \colon x(y) \in V_{\mathbb{R}}(f)\} = -\infty$$

or

$$l := \min\{\deg g(\widetilde{x}^{\mathbb{R}}(y), y) \colon g(\widetilde{x}(y), y) = 0, \deg(f(\widetilde{x}^{\mathbb{R}}(y), y) - \lambda) < 0\} = -\infty$$

then $\lambda \in A_{\mathbb{R}}(f, g)$. Assume that k and l are finite. Since $\mathcal{L}_{\infty, f \to \lambda}(g) = -\infty$, there exists a real curve $x = \varphi(y)$ such that

$$\deg(f(\varphi(y), y) - \lambda) < 0 \quad \text{and} \quad \deg g(\varphi(y), y) < \min\{k, l\}.$$

Put

$$M(X,Y) = f(X + \varphi(1/Y), 1/Y) - \lambda, \qquad N(X,Y) = g(X + \varphi(1/Y), 1/Y).$$

If $\theta_M > \theta_N$, then Lemma 2.3 yields $\deg(f(\varphi_\infty^{\mathbb{R}}(y), y) - \lambda) < 0$ and

$$\deg g(\varphi_\infty^{\mathbb{R}}(y), y) = \deg g(\varphi(y), y),$$

where $x = \varphi_\infty(y)$ is a final result of sliding φ along $f - \lambda$. This contradicts the fact that $\deg g(\varphi(y), y) < k$.

If $\theta_M \le \theta_N$, then by Lemma 2.3, if $x = \widetilde{\varphi}_\infty(y)$ is a final result of sliding φ along g then $\deg(f(\widetilde{\varphi}_\infty^{\mathbb{R}}(y), y) - \lambda) < 0$ and $\deg g(\widetilde{\varphi}_\infty^{\mathbb{R}}(y), y) \le \deg g(\varphi(y), y)$, which is impossible, since $\deg g(\varphi(y), y) < l$.

(c) Straightforward. ■

Remark 3.1. Parts (b) and (c) of Theorem 3.3 are known by [R-S, Remark 2.4] and [Sp, Theorem 3.5].

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Institute of Mathematics
18 Hoang Quoc Viet Road
Cau Giay District
10307, Hanoi, Vietnam
E-mail: hhvui@math.ac.vn
nhduc@math.ac.vn

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