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Interpolating sequences, Carleson measures and Wirtinger inequality

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Abstract. Let S be a sequence of points in the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ which is separated for the hyperbolic distance and contained in the zero set of a Nevanlinna function. We prove that the associated measure $\mu_S := \sum_{a\in S}(1-|a|^2)^n\delta_a$ is bounded, by use of the Wirtinger inequality. Conversely, if X is an analytic subset of $\mathbb{B}$ such that any δ-separated sequence S has its associated measure μ_S bounded by C/δ^n, then X is the zero set of a function in the Nevanlinna class of $\mathbb{B}$.

As an easy consequence, we prove that if S is a dual bounded sequence in $H^p(\mathbb{B})$, then μ_S is a Carleson measure, which gives a short proof in one variable of a theorem of L. Carleson and in several variables of a theorem of P. Thomas.

1. Introduction. Let $\mathbb{B}$ be the unit ball of $\mathbb{C}^n$ and σ the Lebesgue measure on $\partial\mathbb{B}$. As usual we define the Hardy spaces $H^p(\mathbb{B})$ as the closure in $L^p(\partial\mathbb{B})$ of the holomorphic polynomials, and $H^\infty(\mathbb{B})$ as the algebra of all bounded holomorphic functions in $\mathbb{B}$.

The *Nevanlinna class*, $\mathcal{N}(\mathbb{B})$, is the set of holomorphic functions f in $\mathbb{B}$ such that

$$\|f\|_* := \sup_{r<1} \int_{\partial\mathbb{B}} \ln^+ |f(r\zeta)|\, d\sigma(\zeta) < \infty.$$

The *hyperbolic distance* between $a, b \in \mathbb{B}$ is

$d_{\mathrm{h}}(a,b) := |\varPhi_a(b)|$ for any automorphism $\varPhi_a$ of $\mathbb{B}$ exchanging 0 and a.

Definition 1.1. Let S be a sequence of points in $\mathbb{B}$ and $\delta > 0$. We shall say that S is *δ-separated* if $\delta \le \inf_{a,b\in S,\, a\ne b} d_{\mathrm{h}}(a,b)$.

We shall need stronger notions.

Definition 1.2. We say that the sequence $S \subset \mathbb{B}$ is *dual bounded* in $H^p(\mathbb{B})$ if there is a bounded sequence $\{\varrho_a\}_{a\in S} \subset H^p(\mathbb{B})$ such that

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$$\forall a, b \in S, \quad \varrho_a(b) = \delta_{a,b}(1-|a|^2)^{-n/p}.$$

This coincides with the *uniform minimality* introduced by N. Nikolskii ([5, p. 131]) to study Carleson's interpolation theorem.

DEFINITION 1.3. We say that a sequence $S \subset \mathbb{B}$ is $H^p(\mathbb{B})$-*interpolating for* $1 \leq p < \infty$, $S \in IH^p(\mathbb{B})$ for short, if

$$\forall \lambda \in \ell^p, \exists f \in H^p(\mathbb{B}), \forall a \in S, \quad f(a) = \lambda_a(1-|a|^2)^{-n/p}.$$

We say that $S \subset \mathbb{B}$ is $H^\infty(\mathbb{B})$-*interpolating*, $S \in IH^\infty(\mathbb{B})$, if

$$\forall \lambda \in \ell^\infty, \exists f \in H^\infty(\mathbb{B}), \forall a \in S, \quad f(a) = \lambda_a.$$

Clearly if S is $H^p(\mathbb{B})$-interpolating, then S is dual bounded in $H^p(\mathbb{B})$.

In one variable, L. Carleson [1] proved that if S is dual bounded in $H^\infty(\mathbb{D})$ then the measure $\mu_S := \sum_{a\in S}(1-|a|^2)\delta_a$ is a Carleson measure, which was the main step in his characterization of interpolating sequences in the unit disc. Here we reprove this in a very simple way.

With the stronger hypothesis that S is $H^\infty(\mathbb{B})$-interpolating, N. Varopoulos [10] proved that μ_S is a Carleson measure, and P. Thomas [8] improved it: if the sequence S is $H^p(\mathbb{B})$-interpolating for a $p \geq 1$, then μ_S is a Carleson measure.

Our main result is the following

THEOREM 1.4. *Let X be an analytic subvariety of pure codimension* 1 *in the unit ball $\mathbb{B} \subset \mathbb{C}^n$. The variety X is the zero set of a function in the Nevanlinna class of $\mathbb{B}$ if and only if there is a constant C such that for any δ-separated sequence $S \subset X$,*

$$\delta^n \sum_{a\in S}(1-|a|^2)^n \leq C.$$

REMARK 1.5. In the unit disc $\mathbb{D}$ of the complex plane, this is just the well known Blaschke characterization of the zero sets of functions in the Nevanlinna class.

As a corollary of the direct part of Theorem 1.4 we get (an improvement of) P. Thomas' theorem:

THEOREM 1.6. *Let S be a sequence in the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ which is dual bounded in $H^p(\mathbb{B})$ for some $p \geq 1$. Then $\mu_S := \sum_{a\in S}(1-|a|^2)^n\delta_a$ is a Carleson measure.*

REMARK 1.7. This proof is simpler than those of L. Carleson [1], J. Garnett [3] and P. Thomas [8], but in fact they proved more: their theorems are also valid for harmonic interpolation.

I thank the referee for his pertinent questions and remarks.

2. Proof of the main result. We shall argue in the ball of $\mathbb{C}^2$, the general case being more combinatorial but completely analogous.

We shall use the following lemma ([2, p. 40]):

LEMMA 2.1. *Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^2$ and X an analytic subvariety of $\mathbb{B}$. Denote by $P_N(X)$ the projection of X on $N := \{z := (z_1, z_2) : z_2 = 0\}$, counting multiplicity, and $P_T(X)$ the projection of X on $T := \{z := (z_1, z_2) : z_1 = 0\}$, still counting multiplicity. Then*

(i) $\mathrm{Area}(X) = \mathrm{Area}(P_N(X)) + \mathrm{Area}(P_T(X))$.
(ii) $\mathrm{Area}(X) \geq \pi$ (*Wirtinger inequality*).

Let $a \in \mathbb{B}$ and define

$$\varPhi_a(z) := \frac{a - P_a z - s_a Q_a z}{1 - \overline{a} \cdot z},$$

with W. Rudin's notations ([6, Theorem 2.2.2]): P_a is the orthogonal projection of $\mathbb{C}^n$ on the subspace $[a]$ generated by a and $Q_a = I - P_a$ is the projection on the orthogonal complement of $[a]$. Precisely,

$$P_a z = \frac{\overline{a} \cdot z}{|a|^2} a \quad \text{for } a \neq 0 \quad \text{and} \quad s_a := \sqrt{1 - |a|^2}.$$

Let

$$Q(a, \delta) := \varPhi_a(B(0, \delta)),$$

the hyperbolic ball "centered" at a of radius δ.

Let X be an analytic subvariety of $\mathbb{B}$ and $a \in X$. Denote by P_N the orthogonal projection on the complex normal at a to $\partial\mathbb{B}$, counting multiplicity, and by P_T the orthogonal projection on the complex tangent at a to $\partial\mathbb{B}$, still counting multiplicity.

Let $X_a := X \cap Q(a, \delta)$ and $Y_a := \varPhi_a^{-1}(X_a) \subset B(0, \delta)$; we have

LEMMA 2.2.

(i) $\mathrm{Area}(P_N(X_a))$ *is comparable to* $(1 - |a|^2)^2 \mathrm{Area}(P_N(Y_a))$.
(ii) $\mathrm{Area}(P_T(X_a))$ *is comparable to* $(1 - |a|^2)\mathrm{Area}(P_T(Y_a))$.
(iii) $\mathrm{Area}(Y_a) = \mathrm{Area}(P_N(Y_a)) + \mathrm{Area}(P_T(Y_a)) \geq \delta^2 \pi$.

Proof. By rotation we can suppose that $a = (a_1, 0)$. Let $X_1 := P_N(X_a)$, $X_2 := P_T(X_a)$, and similarly $Y_1 := P_N(Y_a)$, $Y_2 := P_T(Y_a)$. Because $a = (a_1, 0)$, we have $\varPhi_a(z) = (Z_1(z), Z_2(z))$ with

$$Z_1(z) = \frac{a_1 - z_1}{1 - \overline{a}_1 z_1}, \qquad Z_2(z) = \frac{z_2 \sqrt{1 - |a_1|^2}}{1 - \overline{a}_1 z_1}. \tag{2.1}$$

Hence $X_1 = Z_1(Y_1)$ and Z_1 is an automorphism of the unit disc. Its jacobian is equivalent to $(1-|a|^2)^2$ on the disc $D(0, \delta)$. The change of variables formula gives

$$\mathrm{Area}(X_1) \simeq (1 - |a|^2)^2 P_N(Y_a).$$

For X_2 we have

$$Z_2 \in X_2 \Leftrightarrow \exists (z_1, z_2) \in Y_a,\ Z_2(z) = \frac{z_2\sqrt{1-|a_1|^2}}{1-\bar{a}_1 z_1};$$

we also have

$$Z_2 \in \Phi_a(Y_2)\ (\subset \{Z_1 = a_1\}) \Leftrightarrow \exists (z_1, z_2) \in Y_a,\ Z_2(z) = z_2\sqrt{1-|a_1|^2}.$$

Hence $Z_2 \in X_2 \Leftrightarrow Z_2(1-a_1 z_1) \in \Phi_a(Y_2)$, for all $(z_1, z_2) \in Y_a$. Because $z_1 \in D(0,\delta)$, we get

$$\frac{\mathrm{Area}(\Phi_a(Y_2))}{(1+\delta)^2} \le \mathrm{Area}(X_2) \le \frac{\mathrm{Area}(\Phi_a(Y_2))}{(1-\delta)^2}.$$

On Y_2 we have $\Phi_a(z) = z_2\sqrt{1-|a|^2}$ because $z_1 = 0$, and its jacobian is $1-|a|^2$, so we get

$$\mathrm{Area}(X_2) \simeq (1-|a|^2)P_T(Y_a).$$

This gives (i) and (ii) of the lemma. Item (iii) is just the Wirtinger inequality applied to $Y_a \subset B(0,\delta)$. ■

2.1. *Proof of the direct part of Theorem 1.4.* Let X be the zero set of a function u in the Nevanlinna class containing S; S separated implies the existence of $\delta > 0$ such that the hyperbolic balls $\{Q(a,\delta) : a \in S\}$ are disjoint. Then the sets $X_a := Q(a,\delta) \cap X,\ a \in S$, are still disjoint.

Let $\Theta := \partial\bar{\partial} \ln |u|$, the current of integration on X. By [7], with $\varrho := |z|^2 - 1$ we get

$$A_T := \int_X (-\varrho)\Theta < \infty \qquad \text{(Blaschke condition)},$$

$$A_N := \int_X \Theta \wedge \partial\varrho \wedge \bar{\partial}\varrho < \infty \qquad \text{(Malliavin condition)}.$$

Let $a \in X$. Lemma 2.2 gives

$$\mathrm{Area}(P_N(X_a)) = (1-|a|^2)^2 \mathrm{Area}(P_N(Y_a)),$$
$$\mathrm{Area}(P_T(X_a)) = (1-|a|^2) \mathrm{Area}(P_T(Y_a)).$$

Hence

$$(1-|a|^2)\mathrm{Area}(P_T(X_a)) = (1-|a|^2)^2 \mathrm{Area}(P_T(Y_a)),$$

so

$$(1-|a|^2)^2[\mathrm{Area}(P_T(Y_a)) + \mathrm{Area}(P_N(Y_a))] = (1-|a|^2)\mathrm{Area}(P_T(X_a)) + \mathrm{Area}(P_N(X_a)).$$

By Lemma 2.1(iii),

$$\delta^2(1-|a|^2)^2\pi \le (1-|a|^2)^2 \mathrm{Area}(Y_a) = (1-|a|^2)\mathrm{Area}(P_T(X_a)) + \mathrm{Area}(P_N(X_a)). \tag{2.2}$$

We have

$$\int_{X_a} (-\varrho)\Theta \geq (1-|a|^2) \int_{X_a} \Theta \geq (1-|a|^2)\mathrm{Area}(P_T(X_a)),$$

because on X_a, $-\varrho \simeq 1-|a|^2$ and $\mathrm{Area}(X_a) \geq \mathrm{Area}(P_T(X_a))$.

Now we want to estimate $\mathrm{Area}(P_N(X_a))$. We have

$$\mathrm{Area}(P_N(X_a)) = \int_{X_a} \Theta \wedge \partial\varrho(a) \wedge \overline{\partial}\varrho(a) \tag{2.3}$$

with $\partial\varrho(z) = \overline{z}_1\, dz_1 + \overline{z}_2\, dz_2$ and $\partial\varrho(a) = \overline{a}_1\, dz_1 + \overline{a}_2\, dz_2$, because $\partial\varrho(z) \wedge \overline{\partial}\varrho(z)$ is the area element on the complex normal to the ball at z. The Taylor formula, with $\varrho(z) := |z|^2 - 1$, gives $\partial\varrho(a) = \partial\varrho(z) + (a-z)\cdot dz$, so

$$\begin{aligned}\partial\varrho(a) \wedge \overline{\partial}\varrho(a) = \partial\varrho(z) \wedge \overline{\partial}\varrho(z) &+ |a_1 - z_1|^2\, dz_1 \wedge d\overline{z}_1 + |a_2 - z_2|^2\, dz_2 \wedge d\overline{z}_2 \\ &+ (a_1 - z_1)(\overline{a}_2 - \overline{z}_2)\, dz_2 \wedge d\overline{z}_1 \\ &+ (a_2 - z_2)(\overline{a}_1 - \overline{z}_1)\, dz_1 \wedge d\overline{z}_2.\end{aligned}$$

But for $z \in Q(a,\delta)$ we have

$$|(a_i - z_i)(\overline{a}_k - \overline{z}_k)| \lesssim \delta^2(1-|z|^2) = \delta^2(-\varrho(z)), \qquad i,j = 1,2;$$

this can be easily seen for $a = (a_1, 0)$, by (2.1), hence is always true by rotation.

Putting this in (2.3) we get

$$\mathrm{Area}(P_N(X_a)) \leq \int_{X_a} \Theta \wedge \partial\varrho(z) \wedge \overline{\partial}\varrho(z) + \delta^2 \int_{X_a} (-\varrho(z))\Theta(z).$$

By (2.2) we then have

$$\delta^2(1-|a|^2)^2\pi \leq (1+\delta^2) \int_{X_a} (-\varrho)\Theta + \int_{X_a} \Theta \wedge \partial\varrho \wedge \overline{\partial}\varrho.$$

Summing over $a \in S$ and using the Blaschke and Malliavin conditions, we get

$$\begin{aligned}\pi\delta^2 \sum_{a\in S} (1-|a|^2)^2 &\leq (1+\delta^2) \sum_{a\in S} \int_{X\cap Q(a,\delta)} (-\varrho)\Theta + \sum_{a\in S} \int_{X\cap Q(a,\delta)} \Theta \wedge \partial\varrho \wedge \overline{\partial}\varrho \\ &\leq (1+\delta^2)A_T + A_N < \infty.\end{aligned}$$

Hence

$$\sum_{a\in S} (1-|a|^2)^2 \leq \frac{(1+\delta^2)A_T + A_N}{\pi\delta^2} = \frac{C}{\delta^2}.$$

2.2. *Proof of the converse part of Theorem 1.4.* We still give the proof in two variables to simplify notation.

Let X be an analytic variety of pure codimension 1 in the ball $\mathbb{B}$ of $\mathbb{C}^2$ and let σ_X be the area measure [4] on X.

Let $r < 1$. Denote by $\Sigma(r)$ the singular set of $X_r := X \cap B(0,r)$; it has a finite number $n(r)$ of points (we are in $\mathbb{C}^2$), and each singularity has a finite number of branches, $b(r)$ at most.

At a singular point of X, all the branches are regularly situated [9], hence there is a number $m = m(r)$ such that outside $R := \bigcup_{s\in\Sigma(r)} B(s,\delta^{1/m})$ one can find a δ-separated sequence S covering $X_r \backslash R$ and such that $X \cap Q(a,\delta)$ is a manifold for each $a \in S$.

The σ_X-area of $R \cap X_r$ then goes to 0 as $\delta \to 0$, r being fixed; by hypothesis we have

$$\delta^2 \sum_{a\in S} (1-|a|^2)^2 \le C,$$

so there is a $\delta_0 = \delta_0(r) > 0$ such that

$$\forall \delta < \delta_0, \quad \sigma_X(X_r \cap R) \le C. \tag{2.4}$$

Moreover, for $r > 0$ fixed, there is a $\delta_1 = \delta_1(r) > 0$ so small that the pseudo-ball $Q(a,\delta)$ for $\delta < \delta_1$ contains only the sheet of X passing through a, which is a manifold, and $X \cap Q(a,\delta)$ is as near as we wish to $T_a(X) \cap Q(a,\delta)$, where $T_a(X)$ is the tangent space to X at a. Using this and the geometry of the pseudo-balls, we get

$$\forall \delta < \delta_1, \quad \sigma_X(X_r \cap Q(a,\delta)) \le 2\delta^2(1-|a|^2).$$

On the other hand,

$$\int_{X_r\backslash R} \varrho\, d\sigma_X \le \sum_{a\in S} (1-|a|^2)\sigma_X(X_r \cap Q(a,\delta)),$$

hence

$$\forall \delta < \delta_1, \quad \int_{X_r\backslash R} \varrho\, d\sigma_X \le 2\delta^2 \sum_{a\in S} (1-|a|^2)^2 \le 2C.$$

Now using (2.4) we get

$$\forall \delta < \min(\delta_0,\delta_1), \quad \int_{X_r} \varrho\, d\sigma_X = \int_{X_r\backslash R} \varrho\, d\sigma_X + \int_{X_r\cap R} \varrho\, d\sigma_X \le 2C + C = 3C.$$

This is true for any $r < 1$, so finally

$$\int_X \varrho\, d\sigma_X \le 3C$$

and X satisfies the Blaschke condition, hence by the Henkin or Skoda theorem, X is the zero set of a function in the Nevanlinna class of $\mathbb{B}$. ∎

3. Proof of Theorem 1.6

LEMMA 3.1. *If S is a dual bounded sequence in $H^p(\mathbb{B})$ then $\phi(S)$ is dual bounded in $H^p(\mathbb{B})$ for any automorphism ϕ of $\mathbb{B}$, with a constant independent of ϕ.*

Proof. Let $\phi \in \mathrm{Aut}(\mathbb{B})$, $\alpha := \phi(0)$, $p \in [1, \infty[$, and set

$$T_\phi f(z) := \frac{(1-|\alpha|^2)^{n/p}}{(1-\overline{\alpha}\cdot z)^{2n/p}}\, f(\phi^{-1}(z)).$$

Then T_ϕ is a surjective isometry on $H^p(\mathbb{B})$ (as proved in [6, p. 155]). Because S is dual bounded, there is a dual sequence $\{\varrho_a\}_{a\in S}$ such that (Definition 1.2)

$$\exists C>0, \forall a \in S, \qquad \|\varrho_a\|_p \le C,$$
$$\forall a, b \in S, \qquad \varrho_a(b) = \delta_{a,b}(1-|a|^2)^{-n/p}.$$

To have a dual sequence for $\phi(S)$, just set

$$\widetilde{\varrho}_{\phi(a)} := T_\phi \varrho_a.$$

By isometry we already have $\|\widetilde{\varrho}_{\phi(a)}\|_p = \|\varrho_a\|_p \le C$; now let us compute

$$\widetilde{\varrho}_{\phi(a)}(\phi(b)) = T_\phi \varrho_a = \frac{(1-|\alpha|^2)^{n/p}}{(1-\overline{\alpha}\cdot\phi(b))^{2n/p}}\, \varrho_a(\phi^{-1}(\phi(b)))$$
$$= \frac{(1-|\alpha|^2)^{n/p}}{(1-\overline{\alpha}\cdot\phi(b))^{2n/p}}\, \varrho_a(b).$$

But $\varrho_a(b) = \delta_{a,b}(1-|a|^2)^{-n/p}$, hence

$$\widetilde{\varrho}_{\phi(a)}(\phi(b)) = \delta_{ab}\, \frac{(1-|\alpha|^2)^{n/p}}{(1-\overline{\alpha}\cdot\phi(b))^{2n/p}}\, (1-|a|^2)^{-n/p}.$$

If $a \ne b$, then $\widetilde{\varrho}_{\phi(a)}(\phi(b)) = 0$, which is the right value, so it remains to compute for $b = a$:

$$\widetilde{\varrho}_{\phi(a)}(\phi(a)) = \frac{(1-|\alpha|^2)^{n/p}}{(1-\overline{\alpha}\cdot\phi(a))^{2n/p}}\, (1-|a|^2)^{-n/p}. \tag{3.1}$$

A simple computation gives ([6, Theorem 2.2.2])

$$1-|\phi(a)|^2 = \frac{(1-|\alpha|^2)(1-|a|^2)}{|1-\overline{\alpha}\cdot a|^2}, \tag{3.2}$$

hence, putting this in (3.1), we get

$$\widetilde{\varrho}_{\phi(a)}(\phi(a) = (1-|\phi(a)|^2)^{-n/p},$$

and this is again the right value, proving the lemma. ∎

Lemma 3.2. *If S is dual bounded in $H^p(\mathbb{B})$, then*

$$\exists C>0,\ \forall \phi \in \mathrm{Aut}(\mathbb{B}), \qquad \sum_{a\in S} (1-|\phi(a)|^2)^n < C.$$

Proof. Let $\phi \in \mathrm{Aut}(\mathbb{B})$. We have just seen that $\phi(S)$ is still a dual bounded sequence with the same constant. An $H^p(\mathbb{B})$ dual bounded sequence

S' is always contained in the zero set of a nonzero $H^p(\mathbb{B})$ function, namely choose any $a \in S'$ and set $f(z) := (z_1 - a_1)\varrho_a(z) \in H^p(\mathbb{B}) \subset \mathcal{N}(\mathbb{B})$.

Hence S' is contained in a zero set of a Nevanlinna function. Because the separating constant is also controlled by the dual constant, using Theorem 1.4 we get

$$\sum_{a\in S} (1 - |\phi(a)|^2)^n < C,$$

and C being independent of $\phi \in \mathrm{Aut}(\mathbb{B})$, we get the assertion of the lemma. ∎

Lemma 3.3. *If*

$$\exists C > 0, \forall \phi \in \mathrm{Aut}(\mathbb{B}), \qquad \sum_{a\in S} (1 - |\phi(a)|^2)^n < C,$$

then $\mu_S := \sum_{a\in S} (1 - |a|^2)^n \delta_a$ *is a Carleson measure.*

To prove this, we use a lemma by Garnett ([3, p. 239]) which generalizes straightforwardly to the ball of $\mathbb{C}^n$:

Lemma 3.4 (J. Garnett). *A positive measure* μ *in the unit ball of* $\mathbb{C}^n$ *is Carleson if and only if*

$$\sup_{z\in\mathbb{B}} \int_{\mathbb{B}} P(z,\zeta)\, d\mu(\zeta) = M < \infty,$$

where $P(z,\zeta) = (1 - |z|^2)^n / |1 - \overline{z} \cdot \zeta|^{2n}$ *is the Poisson–Szegö kernel of the ball.*

Proof of Lemma 3.3. Let ϕ_α be an automorphism of $\mathbb{B}$ which exchanges α and 0:

$$\phi_\alpha(\zeta) := \frac{\alpha - P_\alpha\zeta - s_\alpha Q_\alpha \zeta}{1 - \overline{\alpha} \cdot \zeta}.$$

Then $\sum_{a\in S} (1 - |\phi_\alpha(a)|^2)^n \le C$. By (3.2),

$$1 - |\phi_\alpha(a)|^2 = \frac{(1 - |\alpha|^2)(1 - |a|^2)}{|1 - \overline{\alpha} \cdot a|^2},$$

hence,

$$\sum_{a\in S} (1 - |\phi_\alpha(a)|^2)^n = \sum_{a\in S} \left(\frac{(1 - |\alpha|^2)(1 - |a|^2)}{|1 - \overline{\alpha} \cdot a|^2} \right)^n \le C. \tag{3.3}$$

Let $d\mu := \sum_{a\in S} (1 - |a|^2)^n \delta_a$ be the measure associated to S. Then the inequality (3.3) says

$$\int_{\mathbb{B}} P(\alpha,\zeta)\, d\mu(\zeta) \le C,$$

hence the measure μ is Carleson by Garnett's lemma. ∎

Now combining Lemma 3.1 with Lemma 3.3 we get the assertion of Theorem 1.6. ■

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