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On the Euler characteristic of the link of a weighted homogeneous mapping

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Abstract. The paper is concerned with an effective formula for the Euler characteristic of the link of a weighted homogeneous mapping $F : \mathbb{R}^n \to \mathbb{R}^k$ with an isolated singularity. The formula is based on Szafraniec's method for calculating the Euler characteristic of a real algebraic manifold (as the signature of an appropriate bilinear form). It is shown by examples that in the case of a weighted homogeneous mapping it is possible to make the computer calculations of the Euler characteristics much more effective.

Let $F : \mathbb{R}^n \to \mathbb{R}^k$, where $n > k + 1$, be a polynomial mapping and let $L = \{x \in S^{n-1} \mid F(x) = 0\}$. Let $\chi(L)$ denote the Euler characteristic of L. There is an algebraic formula, due to Szafraniec [7], which expresses $\chi(L)$ in terms of the signature of an appropriate bilinear form (provided that L is smooth).

The aim of this paper is to show that in the weighted homogeneous case the general formula for the Euler characteristic of L can be replaced by a simpler one. It makes practical (computer) calculations of the Euler characteristic more effective. Actually, in some cases the general method is ineffective because of huge computations.

The basic idea is to consider the intersection of $F^{-1}(0)$ with the hyperplanes $\{x_n = 0\}$ and $\{x_n = 1\}$. We prove that the Euler characteristic of L is determined by the behavior of F on these spaces, so the formula for $\chi(L)$ becomes simpler (we consider only the case when the manifold L is of even dimension, for otherwise its Euler characteristic is zero).

We also consider the simpler invariant $\chi(L)/2 \pmod 2$; we give a simple formula which requires only calculating the dimension of an appropriate algebra (instead of computing a signature).

There is a computer program by Andrzej Łęcki to compute the dimensions of quotient algebras and signatures of bilinear forms on these algebras;

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however, reducing the dimension of the domain of F makes the use of the program considerably more effective.

Let $d_1, \ldots, d_k, w_1, \ldots, w_n$ be positive integers. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we write $\lambda.x = (\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n)$ and $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, so $x = (x', x_n) \in \mathbb{R}^n$.

Let $F : \mathbb{R}^n \to \mathbb{R}^k$ be a polynomial mapping such that every F_i is a weighted homogeneous polynomial of degree d_i with respect to the weights $\mathrm{wt}(x_j) = w_j$, i.e. $F_i(\lambda.x) = \lambda^{d_i} F_i(x)$ for every $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

Renumbering the variables if necessary, we may assume that w_n is odd (there is at least one odd weight). Let c be the smallest positive integer such that each w_j divides c. Let $a_j = c/w_j$ and $\omega(x) = \sum_{j=1}^n x_j^{2a_j}/(2a_j)$. Put $\Sigma_r = \{x \in \mathbb{R} \mid \omega(x) = r^{2c}\}$, $\Gamma_r = \{x \in \mathbb{R} \mid \omega(x) \le r^{2c}\}$ and $L_r = F^{-1}(0) \cap \Sigma_r$. Note that if $x \in L_r$ then $\lambda.x \in L_{\lambda r}$ for every $\lambda \neq 0$.

Write $G(x') = F(x', 1)$ and $H(x') = F(x', 0)$ for every $x' \in \mathbb{R}^{n-1}$. Assume that

$$\operatorname{rank}[DG(x')] = k \quad \text{for every } x' \in G^{-1}(0), \tag{1}$$

$$\operatorname{rank}[DH(x')] = k \quad \text{for every } x' \in H^{-1}(0) - 0. \tag{2}$$

As a consequence of (1) and (2) we obtain

Lemma 1. *$\operatorname{rank}[DF(x)] = k$ for every $x \in F^{-1}(0) - 0$ and $F^{-1}(0)$ meets each Σ_r transversally.*

Proof. Suppose that $\operatorname{rank}[DF(x)] < k$ for some $x \in F^{-1}(0)$, $x \neq 0$. Clearly, $D_j H(x') = D_j F(x', 0)$ for $j = 1, \ldots, n-1$, hence if $x_n = 0$ then $\operatorname{rank}[DH(x')] < k$.

Assume that $x_n \neq 0$. Each F_i is a weighted homogeneous polynomial, hence it satisfies the Euler formula:

$$\sum_{j=1}^n w_j x_j D_j F_i(x) = d_i F_i(x)$$

for $i = 1, \ldots, k$. In particular, if $x_n \neq 0$, then the last column of the matrix $[DF(x)]$ is a linear combination of the $n-1$ preceding columns. Again by weighted homogeneity of each F_i we have

$$\lambda^{w_j} D_j F_i(\lambda.x) = \lambda^{d_i} D_j F_i(x),$$

and thus

$$D_j F_i(\lambda.x) = \lambda^{d_i - w_j} D_j F_i(x).$$

Therefore, $\operatorname{rank}[DF(\lambda.x)] = \operatorname{rank}[DF(x)]$ for $\lambda \neq 0$. Then, letting $\lambda = x_n^{-1/w_n}$ we have $\lambda.x \in F^{-1}(0) \cap \{x_n = 1\} = G^{-1}(0)$ and $\operatorname{rank}[DG(x')] < k$, where $\lambda.x = (x', 1)$, which proves the first claim.

In order to prove the second assertion of the lemma, we shall show that the rank of the matrix

$$M = \begin{bmatrix} D_1\omega(x) & \dots & D_n\omega(x) \\ D_1F_1(x) & \dots & D_nF_1(x) \\ & \dots & \\ D_1F_k(x) & \dots & D_nF_k(x) \end{bmatrix}$$

is $k+1$ for every $x \in F^{-1}(0) \cap \Sigma_r$. We may assume that $x_1 \neq 0$. Let us multiply the first column of M by w_1x_1 and then add to it a linear combination of the form $\sum_{j=2}^n w_jx_jA_j$, where A_j denotes the jth column of M. Then the first column becomes

$$\begin{bmatrix} \sum_{j=1}^n w_jx_jD_j\omega(x) \\ \sum_{j=1}^n w_jx_jD_jF_1(x) \\ \dots \\ \sum_{j=1}^n w_jx_jD_jF_k(x) \end{bmatrix}.$$

Again, by the Euler formula we have

$$\sum_{j=1}^n w_jx_jD_jF_i(x) = d_iF_i(x)$$

for $i = 1, \dots, k$ and

$$\sum_{j=1}^n w_jx_jD_j\omega(x) = 2c\omega(x),$$

so the first column has the form

$$\begin{bmatrix} 2cr^{2c} \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}.$$

Thus the first row of M cannot be written as a linear combination of the remaining rows. So the first part of the lemma yields the assertion. ∎

Therefore each L_r is a smooth $(n-k-1)$-dimensional compact manifold and $\chi(L_r) = \chi(L)$ (the independence of the link from the function ω has been proven in [3]; in our case it may be observed that L and L_r are homeomorphic by flowing along a vector field tangent to $F^{-1}(0)$). The Euler characteristic of every compact odd-dimensional manifold is zero, so from now on we shall assume that $n-k$ is odd.

Note that H is also weighted homogeneous, hence the same reasoning shows that $H^{-1}(0)$ meets each $\widetilde{\Sigma}_r = \Sigma_r \cap \{x_n = 0\}$ transversally and

$$\chi(H^{-1}(0) \cap \widetilde{\Sigma}_r) = 0. \tag{3}$$

We shall denote by π the projection on the nth coordinate in $\mathbb{R}^n$.

LEMMA 2. *For sufficiently large r,*

$$\chi(L_r \cap \{-1 \le x_n \le 1\}) = 0.$$

Proof. It is easy to check (by the same method as in the proof of Lemma 1) that the rank of the matrix

$$\begin{bmatrix} D_1\pi(x) & \dots & D_{n-1}\pi(x) & D_n\pi(x) \\ D_1\omega(x) & \dots & D_{n-1}\omega(x) & D_n\omega(x) \\ D_1F_1(x) & \dots & D_{n-1}F_1(x) & D_nF_1(x) \\ & \dots & & \\ D_1F_k(x) & \dots & D_{n-1}F_k(x) & D_nF_k(x) \end{bmatrix}$$

is $k+2$ for every $x \in L_1 \cap \{x_n = 0\}$. This means that $\pi_{|L_1}$ has no critical points in the hyperplane $\{x_n = 0\}$ (see [4]). Hence for small $\varepsilon > 0$, $\pi_{|L_1}$ has no critical points in the set $L_1 \cap \{-\varepsilon \le x_n \le \varepsilon\}$.

So $L_1 \cap \{x_n = 0\}$ is a deformation retract of $L_1 \cap \{-\varepsilon \le x_n \le \varepsilon\}$, and then by (3), $\chi(L_1 \cap \{-1 \le x_n \le 1\}) = 0$. The proof is completed by observing that if $r = \varepsilon^{-1/w_n}$, then $x \mapsto r.x$ maps homeomorphically the set $L_1 \cap \{-\varepsilon \le x_n \le \varepsilon\}$ onto $L_r \cap \{-1 \le x_n \le 1\}$. ∎

THEOREM 3. $\chi(L) = 2\chi(F^{-1}(0) \cap \{x_n = 1\})$.

Proof. Clearly,

$$\begin{aligned} \chi(L) &= \chi(L_r) \\ &= \chi(L_r \cap \{-1 \le x_n \le 1\}) + \chi(L_r \cap \{x_n \ge 1\}) \\ &\quad + \chi(L_r \cap \{x_n \le -1\}) - \chi(L_r \cap \{x_n = \pm 1\}). \end{aligned}$$

By Lemma 2, $\chi(L_r \cap \{-1 \le x_n \le 1\}) = 0$ for large r. Using similar arguments to those above we can prove that for large r, L_r meets $\{x_n = \pm 1\}$ transversally and then $\chi(L_r \cap \{x_n = \pm 1\}) = 0$.

If $x \in L_r \cap \{x_n \ge 1\}$ then set $\gamma(x) = x_n^{-1/w_n}$. Then $x \mapsto \gamma(x).x$ maps $L_r \cap \{x_n \ge 1\}$ homeomorphically onto the set $F^{-1}(0) \cap \{x_n = 1\} \cap \Gamma_r$, which for large r is a deformation retract of $F^{-1}(0) \cap \{x_n = 1\} = G^{-1}(0)$. Moreover, the map $x \mapsto (-1).x$ is a homeomorphism between $L_r \cap \{x_n = 1\}$ and $L_r \cap \{x_n = -1\}$ (recall that w_n is odd), so the proof is complete. ∎

Let $I(H) \subset \mathbb{R}[x'] = \mathbb{R}[x_1, \dots, x_{n-1}]$ be the ideal generated by $H_1, \dots, H_k$ and all $(k+1) \times (k+1)$-minors $\partial(\omega, H_1, \dots, H_k)/\partial(x_{j_1}, \dots, x_{j_{k+1}})$.

If $\dim \mathbb{R}[x']/I(H) < \infty$ then there are a finite number of singular points of $H^{-1}(0)$ and of critical points of ω restricted to the smooth part of $H^{-1}(0)$. But the map H is weighted homogeneous, hence $H^{-1}(0)$ is transversal to each $\widetilde{\Sigma}_r$, so there are no critical points of ω on the smooth part of $H^{-1}(0)$. It follows by the same method as in the proof of Lemma 1 that if $(x_1, \dots, x_{n-1}) \in H^{-1}(0)$ is a singular point of $H^{-1}(0)$, then for every

$\lambda \neq 0$, $(\lambda^{w_1} x_1, \ldots, \lambda^{w_{n-1}} x_{n-1})$ is also a singular point of $H^{-1}(0)$. Thus we have proved

LEMMA 4. *If* $\dim \mathbb{R}[x']/I(H) < \infty$ *then* $\operatorname{rank}[DH(x')] = k$ *for every* $x' \in H^{-1}(0) - \{0\}$. ∎

There is an algebraic formula, due to Szafraniec [7], which expresses the Euler characteristic of an algebraic manifold as the signature of an appropriate bilinear form. There is also an algebraic method for verifying assumption (1).

Assume that $\dim \mathbb{R}[x']/I(G) = d < \infty$. According to [7], there is a certain (explicitly defined) linear functional $\phi : \mathbb{R}[x']/I(G) \to \mathbb{R}$. Set $M = \partial(G_1, \ldots, G_k)/\partial(x_1, \ldots, x_k)$ and define a bilinear form $\Phi_M : \mathbb{R}[x']/I(G) \times \mathbb{R}[x']/I(G) \to \mathbb{R}$ by $\Phi_M(f, g) = \phi(Mfg)$.

THEOREM 5. *Assume that* Φ_M *is non-degenerate. Then* $G^{-1}(0)$ *is a smooth manifold and* $\chi(G^{-1}(0)) = \operatorname{signature} \Phi_M$.

For the proof see [7].

Therefore we have

THEOREM 6. *Assume that* $\dim \mathbb{R}[x']/I(H) < \infty$ *and* $\dim \mathbb{R}[x']/I(G) < \infty$ *and that* Φ_M *is non-degenerate. Then* $\chi(L) = 2 \cdot \operatorname{signature} \Phi_M$. ∎

Since Φ_M is non-degenerate, $\operatorname{signature} \Phi_M \equiv \dim \mathbb{R}[x']/I(G) \pmod 2$, hence we obtain a simpler formula for the simpler invariant $\chi(L)/2 \pmod 2$:

COROLLARY 7. *Under the assumptions of Theorem* 6,

$$\chi(L)/2 \equiv \dim \mathbb{R}[x']/I(G) \pmod 2.$$

In [2] another formula is given which expresses $\chi(L)/2 \pmod 2$ in terms of the dimensions of appropriate local algebras in the case of $F : \mathbb{R}^n \to \mathbb{R}$ with a non-isolated singularity.

EXAMPLE 1. Let $F(x, y, z, t) = x^2zt + t^2 + x^5yz + y^2z^2 + x^3y + x^2y^2 + xy^3 + y^5$. The polynomial F is weighted homogeneous of degree 10 with respect to the weights $w_1 = 1$, $w_2 = 2$, $w_3 = 3$, $w_4 = 5$. Then $H(x, y, z) = x^5yz + y^2z^2 + x^3y + x^2y^2 + xy^3 + y^5$ and $G(x, y, z) = x^2z + 1 + x^5yz + y^2z^2 + x^3y + x^2y^2 + xy^3 + y^5$. It is easily seen that $G^{-1}(0)$ is non-empty. A computer calculation shows that $\dim \mathbb{R}[x']/I(H) = 140$ and $\dim \mathbb{R}[x']/I(G) = 142$ and that the form Φ_M is non-degenerate and $\operatorname{signature} \Phi_M = \chi(L) = 0$.

EXAMPLE 2. Let $F_1(x, y, z, t, u) = xyz + xu + z^2 + yt$, $F_2(x, y, z, t, u) = u + yz + xt + x^2z$. The polynomials are weighted homogeneous of degrees 6 and 5 respectively, with respect to the weights $w_1 = 1$, $w_2 = 2$, $w_3 = 3$, $w_4 = 4$, $w_5 = 5$. Then $H_1(x, y, z, t) = xyz + z^2 + yt$, $H_2(x, y, z, t) = yz + xt + x^2z$, $G_1(x, y, z, t) = xyz + x + z^2 + yt$, $G_2(x, y, z, t) = 1 + yz + xt + x^2z$. A computer calculation shows that $\dim \mathbb{R}[x']/I(H) = 57$ and $\dim \mathbb{R}[x']/I(G) = 33$, the form Φ_M is non-degenerate and $\operatorname{signature} \Phi_M = 1$ so $\chi(L) = 2$.

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