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Embedding polydisk algebras into the disk algebra and an application to stable ranks

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Abstract. It is shown how to embed the polydisk algebras (finite and infinite ones) into the disk algebra $A(\overline{\mathbb{D}})$. As a consequence, one obtains uniform closed subalgebras of $A(\overline{\mathbb{D}})$ which have arbitrarily prescribed stable ranks.

Introduction. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk, $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ its closure, and $A(\overline{\mathbb{D}})$ the *disk algebra*, that is, the space of all functions continuous on $\overline{\mathbb{D}}$ and holomorphic on $\mathbb{D}$. In this note I am interested in the question whether there are subalgebras of $A(\overline{\mathbb{D}})$ that do not have Bass stable rank one (see below for the definitions). As is well known, Jones, Marshall and Wolff showed that the stable rank of $A(\overline{\mathbb{D}})$ is one. Whereas in [7] I found for any $n \in \mathbb{N} \cup \{\infty\}$ subalgebras of H^∞ on the disk which have stable rank n, the problem whether these algebras could be chosen to be subalgebras of $A(\overline{\mathbb{D}})$ remained open. The examples given in [7] always meet $H^\infty(\mathbb{D}) \setminus A(\overline{\mathbb{D}})$. It is a quite recent result developed together with Rudolf Rupp (see Corollary 1.3) that any subalgebra B of $A(\overline{\mathbb{D}})$ containing the polynomials and satisfying Royden's property (α_0) has Bass stable rank one (note that B is not assumed to be closed in $A(\overline{\mathbb{D}})$). On the other hand, it is easy to construct a subalgebra of $A(\overline{\mathbb{D}})$ that has stable rank two: just take the restriction $\mathbb{C}[z]\,|_{\overline{\mathbb{D}}}$ of the polynomials to $\overline{\mathbb{D}}$. In an oral communication Amol Sasane gave a first example of a non-closed subalgebra of $A(\overline{\mathbb{D}})$ with stable rank infinity: if φ is a conformal map of the disk $\{|z| < 2\}$ onto the upper half-plane H^+, then the algebra

$$A = \{f \circ \varphi|_{\overline{\mathbb{D}}} : f \in \mathrm{AP}^+\}$$

of pull-backs of almost periodic functions that are analytic on H^+ is isomorphic to AP^+ and therefore has stable rank infinity (see [6] and [8]).

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It is the aim of this paper to prove, given $n \in \mathbb{N} \cup \{\infty\}$, the existence of *uniformly closed* subalgebras of $A(\overline{\mathbb{D}})$ that have Bass stable rank n. The proof is based on embedding the polydisk algebras $A(\overline{\mathbb{D}}^n)$ and $A(\mathbf{D}^\infty)$ isomorphically into $A(\overline{\mathbb{D}})$ (see below for the definitions). This will be done by using the Rudin–Carleson interpolation theorem for disk algebra functions, and the topological fact (known by the name of the Aleksandrov–Hausdorff theorem) that every compact metric space is a continuous image of the Cantor set (see for example [11]).

1. Background

DEFINITION 1.1. Let A be a commutative unital algebra (real or complex) with identity element denoted by 1.

(1) An n-tuple $(f_1, \dots, f_n) \in A^n$ is said to be *invertible* (or *unimodular*) if there exists $(x_1, \dots, x_n) \in A^n$ such that the Bézout equation $\sum_{j=1}^n x_j f_j = 1$ is satisfied. The set of all invertible n-tuples is denoted by $U_n(A)$. Note that $U_1(A) = A^{-1}$. An $(n+1)$-tuple $(f_1, \dots, f_n, g) \in U_{n+1}(A)$ is called *reducible* if there exists $(a_1, \dots, a_n) \in A^n$ such that $(f_1 + a_1 g, \dots, f_n + a_n g) \in U_n(A)$.

(2) The *Bass stable rank* of A, denoted by $\operatorname{bsr} A$, is the smallest integer n such that every element in $U_{n+1}(A)$ is reducible. If no such n exists, then $\operatorname{bsr} A = \infty$.

It is obvious that if A and B are two commutative unital algebras such that A is isomorphic to B, then $\operatorname{bsr} A = \operatorname{bsr} B$, because any isomorphism ι between A and B induces a bijection between $U_n(A)$ and $U_n(B)$. The following two observations stem from joint work with R. Rupp [9]. Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

PROPOSITION 1.2. *Let X be a topological space and B a subalgebra of $C_b(X, \mathbb{K})$ with $\mathbb{K} \subseteq B$. Suppose that B has Royden's property (α_0), that is,*

(α_0) *for every $f \in B$, if $\|1 - f\|_\infty < 1$, then $f \in B^{-1}$.*

Then $\operatorname{bsr} B \le \operatorname{bsr} \overline{B}^{\|\cdot\|_\infty}$, *where $\overline{B}^{\|\cdot\|_\infty}$ is the uniform closure of B.*

Proof. Let $A := \overline{B}^{\|\cdot\|_\infty}$. We show that $U_n(B) = U_n(A) \cap B^n$. Since $U_n(B) \subseteq U_n(A) \cap B^n$, it only remains to show the reverse inclusion. So let $(b_1, \dots, b_n) \in U_n(A) \cap B^n$. Then there is $(a_1, \dots, a_n) \in A^n$ such that $1 = \sum_{j=1}^n a_j b_j$. Uniformly approximating a_j by elements $x_j \in B$ yields $\|\sum_{j=1}^n x_j b_j - 1\|_\infty < 1/2$. By assumption (α_0), $f := \sum_{j=1}^n x_j b_j \in B^{-1}$. Hence $(b_1, \dots, b_n) \in U_n(B)$. It is now a standard observation that $\operatorname{bsr} B \le \operatorname{bsr} A$ (see [2] or [6]). ∎

COROLLARY 1.3. *Let B be a subalgebra of the disk algebra $A(\overline{\mathbb{D}})$ such that*

($*$) *B contains the polynomials (that is, $\mathbb{C}[z]\,|_{\overline{\mathbb{D}}} \subseteq B$)*

and (α_0) holds. Then $\operatorname{bsr} B = 1$.

Proof. By ($*$), B is uniformly dense in $A(\overline{\mathbb{D}})$. In view of (α_0), we may apply Proposition 1.2 to conclude that $\operatorname{bsr} B \leq \operatorname{bsr} A(\overline{\mathbb{D}})$. Since by the Jones–Marshall–Wolff theorem $\operatorname{bsr} A(\overline{\mathbb{D}}) = 1$ (see [5]), we are done. ■

2. An embedding theorem. Recall that $\overline{\mathbb{D}}^n$ is the closed polydisk and $\mathbf{D}^\infty := \prod_{n\in\mathbb{N}} \overline{\mathbb{D}}$ the infinite polydisk. By Tikhonov's theorem, $\mathbf{D}^\infty$ is a compact metric space when endowed with the product topology. Moreover, each $\overline{\mathbb{D}}^n$ and $\mathbf{D}^\infty$ are separable. The *polydisk algebra* $A(\overline{\mathbb{D}}^n)$ is the set of functions continuous on $\overline{\mathbb{D}}^n$ and holomorphic on the open polydisk $\mathbb{D}^n$. In the same spirit, one defines the infinite polydisk algebra $A(\mathbf{D}^\infty)$ as the smallest uniformly closed subalgebra of $C(\mathbf{D}^\infty, \mathbb{C})$ containing all the coordinate functions $z_1, z_2, \ldots$. Let $\mathbb{C}[z_1, z_2, \ldots]$ denote the set of polynomials

$$\sum_{\boldsymbol{j}\in\mathbb{N}^n} a_{\boldsymbol{j}} z_1^{j_1} \ldots z_n^{j_n}, \qquad n \in \mathbb{N},$$

over $\mathbb{C}$, where $\boldsymbol{j} = (j_1, \ldots, j_n) \in \mathbb{N}^n$. Hence

$$\mathbb{C}[z_1, z_2, \ldots]\,|_{\mathbf{D}^\infty} \subseteq A(\mathbf{D}^\infty).$$

THEOREM 2.1. *There are uniformly closed subalgebras A_n, respectively A_∞, of $A(\overline{\mathbb{D}})$ that are algebraically isomorphic to $A(\overline{\mathbb{D}}^n)$, respectively $A(\mathbf{D}^\infty)$.*

Proof. Let $C \subseteq \mathbb{T}$ be the homeomorphic image of the usual ternary Cantor set on $[0,1]$ via the map $e^{i\pi x}$. By the Aleksandrov–Hausdorff theorem [11], there is a continuous surjective map

$$M_n = (\phi_1, \ldots, \phi_n) : C \to \overline{\mathbb{D}}^n,$$

respectively

$$M_\infty = (\phi_1, \phi_2, \ldots) : C \to \mathbf{D}^\infty.$$

Since C has one-dimensional Lebesgue measure zero, the Rudin–Carleson interpolation theorem [4, p. 58] implies that there are functions $f_j \in A(\overline{\mathbb{D}})$ such that $f_j|_C = \phi_j$ and $\|f_j\| = 1$. Define $F_n : \overline{\mathbb{D}} \to \overline{\mathbb{D}}^n$ by

$$F_n(\xi) = (f_1(\xi), \ldots, f_n(\xi)),$$

and $F_\infty : \overline{\mathbb{D}} \to \mathbf{D}^\infty$ by

$$F_\infty(\xi) = (f_1(\xi), f_2(\xi), \ldots).$$

By construction, the range of F_n on $\overline{\mathbb{D}}$ is $\overline{\mathbb{D}}^n$ and the range of F_∞ on $\overline{\mathbb{D}}$ is $\mathbf{D}^\infty$. Moreover, since $f_j(\mathbb{D}) \subseteq \mathbb{D}$, the functions $f \circ F_n$ and $f \circ F_\infty$ are holomorphic on $\mathbb{D}$ for any $f \in A(\overline{\mathbb{D}}^n)$ and $f \in A(\mathbf{D}^\infty)$, respectively. Hence

$$\Psi_n : A(\overline{\mathbb{D}}^n) \to A(\overline{\mathbb{D}}), \quad f \mapsto f \circ F_n,$$

and

$$\Psi_\infty : A(\mathbf{D}^\infty) \to A(\overline{\mathbb{D}}), \quad f \mapsto f \circ F_\infty,$$

are isometric isomorphisms of $A(\overline{\mathbb{D}}^n)$ and $A(\mathbf{D}^\infty)$, respectively, onto a uniformly closed subalgebra of $A(\overline{\mathbb{D}})$. ■

Corollary 2.1. *For every $n \in \mathbb{N} \cup \{\infty\}$ there is a uniformly closed subalgebra A_n of $A(\overline{\mathbb{D}})$ with* $\operatorname{bsr} A_n = n$.

Proof. Let $N \in \mathbb{N}$ be chosen so that $\lfloor N/2 \rfloor + 1 = n$. By Theorem 2.1, $A(\overline{\mathbb{D}}^N)$ is isomorphic to a uniformly closed subalgebra A_N of $A(\overline{\mathbb{D}})$. Hence

$$\operatorname{bsr} A_N = \operatorname{bsr} A(\overline{\mathbb{D}}^N) = \lfloor N/2 \rfloor + 1 = n,$$

where the penultimate equality is due to Corach and Suárez [2]. Moreover, by [7], $\operatorname{bsr} A(\mathbf{D}^\infty) = \infty$. Since by Theorem 2.1, $A(\mathbf{D}^\infty)$ is isomorphic to a uniformly closed subalgebra A_∞ of $A(\overline{\mathbb{D}})$, we deduce that

$$\operatorname{bsr} A_\infty = \operatorname{bsr} A(\mathbf{D}^\infty) = \infty. \ ■$$

3. The topological stable rank. Associated with the Bass stable rank is the notion of *topological stable rank* introduced by Rieffel [10].

Definition 3.1. Let A be a commutative unital complex Banach algebra. The *topological stable rank*, $\operatorname{tsr} A$, of A is the least integer n for which $U_n(A)$ is dense in A^n, or infinity if no such n exists.

It is straightforward to see (and well known) that $\operatorname{tsr} A(\overline{\mathbb{D}}) = 2$. Corach and Suárez [1] showed that $\operatorname{tsr} A(\overline{\mathbb{D}}^n) = n + 1$ for $n \in \mathbb{N}$. Because $\operatorname{bsr} A \leq \operatorname{tsr} A$ is always true, $\operatorname{tsr} A(\mathbf{D}^\infty) = \infty$. Since the topological stable rank is invariant under isometric isomorphisms, we obtain from Corollary 2.2 the following theorem.

Corollary 3.2. *For every $n \in \mathbb{N} \cup \{\infty\}$ there is a uniformly closed subalgebra A_n of $A(\overline{\mathbb{D}})$ with* $\operatorname{tsr} A_n = n + 1$.

Proof. Just take $A_n := \Psi_n(A(\overline{\mathbb{D}}^n))$, respectively $A_\infty := \Psi_\infty(A(\mathbf{D}^\infty))$. ■

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Added in proofs (September 2014). Meanwhile Joel Feinstein has communicated to me that an unpublished result of Brian Cole (2002) states that actually every non-trivial uniform algebra A contains a chain $B_1 \subseteq B_2 \subseteq \cdots$ of subalgebras B_n of A such that B_n is isomorphic to $A(\overline{\mathbb{D}}^n)$ (referenced in [3, p. 2834]).

References

[1] G. Corach and F. D. Suárez, *Extension problems and stable rank in commutative Banach algebras*, Topology Appl. 21 (1985), 1–8.

[2] G. Corach and F. D. Suárez, *Dense morphisms in commutative Banach algebras*, Trans. Amer. Math. Soc. 304 (1987), 537–547.

[3] T. W. Dawson and J. F. Feinstein, *On the denseness of the invertible group in Banach algebras*, Proc. Amer. Math. Soc. 131 (2003), 2831–2839.

[4] T. W. Gamelin, *Uniform Algebras*, Chelsea, New York, 1984.

[5] P. W. Jones, D. Marshall and T. Wolff, *Stable rank of the disc algebra*, Proc. Amer. Math. Soc. 96 (1986), 603–604.

[6] K. Mikkola and A. Sasane, *Bass and topological stable ranks of complex and real algebras of measures, functions and sequences*, Complex Anal. Oper. Theory 4 (2010), 401–448.

[7] R. Mortini, *An example of a subalgebra of H^∞ on the unit disk whose stable rank is not finite*, Studia Math. 103 (1992), 275–281.

[8] R. Mortini and R. Rupp, *The Bass and topological stable ranks for algebras of almost periodic functions on the real line*, Trans. Amer. Math. Soc., to appear.

[9] R. Mortini and R. Rupp, *An Introduction to Extension Problems, Bézout Equations and Stable Ranks in Classical Function Algebras*, a monograph accompanied by introductory chapters on point-set topology and function theory, in preparation, ca. 1000 pages.

[10] M. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. 46 (1983), 301–333.

[11] I. Rosenholtz, *Another proof that any compact metric space is the continuous image of the Cantor set*, Amer. Math. Monthly 83 (1976), 646–647.

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