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Weak solutions to the complex Monge–Ampère equation on hyperconvex domains

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Abstract. We show a very general existence theorem for a complex Monge–Ampère type equation on hyperconvex domains.

1. Introduction. Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ and F a nonnegative function defined on $\mathbb{R} \times \Omega$. In the present note, we shall consider the existence and uniqueness of a weak solution of the complex Monge–Ampère type equation

$$(dd^c u)^n = F(u, \cdot) d\mu \tag{1.1}$$

where u is plurisubharmonic on Ω and μ is a nonnegative measure. This problem has been studied extensively by various authors; see for example [2], [4], [9], [10], [12], [14], [15], [16], [19], [20], [21], [22] and references therein for further information about complex Monge–Ampère equations.

The problem was first considered by Bedford and Taylor [3]. In connection with the problem of finding complete Kähler–Einstein metrics on pseudoconvex domains, Cheng and Yau [16] treated the case $F(t, z) = e^{Kt} f(z)$. More recently, Czyż [17] treated the case where F is bounded by a function independent of the first variable and μ is the Monge–Ampère measure of a plurisubharmonic function v, generalizing some results of Cegrell [13], Kołodziej [21] and Cegrell and Kołodziej [14], [15].

In this paper we will consider a more general case. With notations introduced in the next section, our main result is stated as follows.

Main Theorem. *Let Ω be a bounded hyperconex domain and μ be a nonnegative measure which vanishes on all pluripolar subsets of Ω. Assume that $F : \mathbb{R} \times \Omega \to [0, \infty)$ is a measurable function such that:*

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(1) *For all* $z \in \Omega$, *the function* $t \mapsto F(t,z)$ *is continuous and nondecreasing;*
(2) *For all* $t \in \mathbb{R}$, *the function* $z \mapsto F(t,z)$ *belongs to* $L^1_{\mathrm{loc}}(\Omega,\mu)$;
(3) *There exists a function* $v_0 \in \mathcal{N}^a$ *which is a subsolution to* (1.1), *i.e.*

$$(dd^c v_0)^n \ge F(v_0,\cdot)d\mu.$$

Then for any maximal function $f \in \mathcal{E}$ *there exists a unique solution* u *in* $\mathcal{N}^a(f)$ *to the complex Monge–Ampère equation*

$$(dd^c u)^n = F(u,\cdot)d\mu.$$

Note that the solution, as we will see in the proof, is given by the following upper envelope of all subsolutions:

$$u = \sup\{v \in \mathcal{E}(\Omega) : v \le f \text{ and } (dd^c v)^n \ge F(v,\cdot)d\mu\}$$

where $\mathcal{E}(\Omega)$ is the set of nonpositive plurisubharmonic functions defined on Ω for which the complex Monge–Ampère operator is well defined as a nonnegative measure (a precise definition will be given shortly).

2. Background and definitions. Recall that $\Omega \Subset \mathbb{C}^n$, $n \ge 1$, is a *bounded hyperconvex domain* if it is a bounded, connected, and open set such that there exists a bounded plurisubharmonic function $\rho : \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega : \rho(z) < c\}$ is compact in Ω, for every $c \in (-\infty, 0)$. We denote by $\mathrm{PSH}(\Omega)$ the family of plurisubharmonic functions defined on Ω.

We say that a bounded plurisubharmonic function φ defined on Ω belongs to $\mathcal{E}_0$ if $\lim_{z\to\zeta} \varphi(z) = 0$ for every $\zeta \in \partial\Omega$, and $\int_\Omega (dd^c\varphi)^n < \infty$ (see [10] for details).

Let $\mathcal{E}(\Omega)$ be the set of plurisubharmonic functions u such that for all $z_0 \in \Omega$, there exists a neighborhood V_{z_0} of z_0 and a decreasing sequence $u_j \in \mathcal{E}_0(\Omega)$ which converges towards u in V_{z_0} and satisfies

$$\sup_j \int_\Omega (dd^c u_j)^n < \infty.$$

U. Cegrell [10] has shown that the operator $(dd^c\,\cdot)^n$ is well defined on $\mathcal{E}(\Omega)$, is continuous under decreasing limits, and the class $\mathcal{E}(\Omega)$ is stable under taking maximum, i.e. if $u \in \mathcal{E}(\Omega)$ and $v \in \mathrm{PSH}^-(\Omega)$ then $\max(u,v) \in \mathcal{E}(\Omega)$. $\mathcal{E}(\Omega)$ is the largest class with these properties [10, Theorem 4.5]. The class $\mathcal{E}(\Omega)$ has been further characterized by Z. Błocki [7], [8].

The class $\mathcal{F}(\Omega)$ is the "global version" of $\mathcal{E}(\Omega)$: a function u belongs to $\mathcal{F}(\Omega)$ iff there exists a decreasing sequence $u_j \in \mathcal{E}_0(\Omega)$ converging towards u *in all of* Ω, with $\sup_j \int_\Omega (dd^c u_j)^n < \infty$. Further characterizations are given in [5], [6].

Define $\mathcal{N}(\Omega)$ to be the family of all $u \in \mathcal{E}(\Omega)$ which satisfy: if $v \in \mathrm{PSH}(\Omega)$ is maximal and $u \le v$ then $v \ge 0$, i.e. the smallest maximal psh function above u is null. In fact, this class is the analogue of potentials for subharmonic functions (see [9] for more details).

The class $\mathcal{F}^a(\Omega)$ (resp. $\mathcal{N}^a(\Omega)$, $\mathcal{E}^a(\Omega), \ldots$) is the set of functions u in $\mathcal{F}(\Omega)$ (resp. $\mathcal{N}(\Omega)$, $\mathcal{E}(\Omega), \ldots$) whose Monge–Ampère measure $(dd^c u)^n$ is absolutely continuous with respect to capacity, i.e. it does not charge pluripolar sets.

Finally, for $f \in \mathcal{E}$, we denote by $\mathcal{N}(f)$ (resp. $\mathcal{F}(f)$) the family of those $u \in \mathrm{PSH}(\Omega)$ such that there exists $\varphi \in \mathcal{N}$ (resp. $\varphi \in \mathcal{F}$) satisfying

$$\varphi(z) + f(z) \le u(z) \le f(z), \qquad \forall z \in \Omega.$$

We shall use repeatedly the following well known comparison principle from [4] as well as its generalizations to the class $\mathcal{N}(f)$ (cf. [1], [9]).

THEOREM 2.1 ([1], [4], [9]). *Let $f \in \mathcal{E}(\Omega)$ be a maximal function and let $u, v \in \mathcal{N}(f)$ be such that $(dd^c u)^n$ vanishes on all pluripolar sets in Ω. Then*

$$\int_{(u<v)} (dd^c v)^n \le \int_{(u<v)} (dd^c u)^n.$$

Furthermore, if $(dd^c u)^n = (dd^c v)^n$ then $u = v$.

3. Proof of Main Theorem

LEMMA 3.1 (Stability). *Let μ be a finite nonnegative measure which vanishes on all pluripolar subsets of Ω and $f \in \mathcal{E}(\Omega)$ be a maximal function. Fix a function $v_0 \in \mathcal{E}(\Omega)$. Then for any $u_j, u \in \mathcal{N}^a(f)$ that satisfy*

$$(dd^c u_j)^n = h_j d\mu, \qquad (dd^c u)^n = h d\mu$$

and $0 \le h d\mu$, $h_j d\mu \le (dd^c v_0)^n$, and $h_j d\mu \to h d\mu$ as measures, the sequence u_j converges towards u weakly.

The statement of the lemma fails if no control on the complex Monge–Ampère measures is assumed (see [15]).

Proof. It follows from the comparison principle that $u_j \ge v_0$ for all $j \in \mathbb{N}$. Therefore by the general properties of psh functions $(u_j)_j$ is relatively compact in the L^1_{loc} topology. Let $\tilde{u} \in \mathcal{N}^a(f)$ be any cluster point of the sequence u_j. Assume that $u_j \to \tilde{u}$ pointwise $d\lambda$-almost everywhere, where $d\lambda$ denotes the Lebesgue measure. By [11, Lemma 2.1], after extracting a subsequence if necessary, we have $u_j \to \tilde{u}$ $d\mu$-almost everywhere. Then

$$\tilde{u} = \Big(\limsup_{j\to\infty} u_j\Big)^* = \lim_{j\to\infty} \Big(\sup_{k\ge j} u_k\Big)^*.$$

Now, consider the following auxiliary functions:

$$\tilde{u}_j = \Big(\sup_{k\geq j} u_k\Big)^* = \Big(\lim_{l\to\infty} \sup_{l\geq k\geq j} u_k\Big)^* = \Big(\lim_{l\to\infty} \tilde{u}_j^l\Big)^*.$$

Observe that

$$(dd^c \max(u_j, u_k))^n \geq \min(h_j, h_k) d\mu.$$

Therefore

$$(dd^c\tilde{u}_j)^n = \lim_{l\to\infty} (dd^c\tilde{u}_j^l)^n \geq \lim_{l\to\infty} \min_{l\geq k\geq j} h_k \, d\mu.$$

We let j converge to ∞ to get

$$(dd^c\tilde{u})^n \geq h d\mu.$$

Now, for the reverse inequality, pick a negative psh function $\varphi \in \mathcal{E}_0$. For any $j \geq 1$ and since $u_j \leq \tilde{u}_j$, by integration by parts, which is valid in $\mathcal{N}^a(f)$ (cf. [1]), we have

$$\int_\Omega -\varphi(dd^c u_j)^n \geq \int_\Omega -\varphi(dd^c\tilde{u}_j)^n.$$

Therefore

$$\lim_{j\to\infty} \int_\Omega \varphi h_j \, d\mu \leq \lim_{j\to\infty} \int_\Omega \varphi(dd^c\tilde{u}_j)^n = \int_\Omega \varphi(dd^c\tilde{u})^n.$$

Together with the first inequality, we get

$$\int_\Omega \varphi(dd^c\tilde{u})^n = \int_\Omega \varphi h \, d\mu, \qquad \forall\varphi \in \mathcal{E}_0.$$

We have $\mathcal{D}(\Omega) \subset \mathcal{E}_0 - \mathcal{E}_0$ (cf. [10, Lemma 2.1]), so the equality holds for any $\varphi \in \mathcal{D}(\Omega)$. Hence

$$(dd^c\tilde{u})^n = h d\mu = (dd^c u)^n.$$

Uniqueness in $\mathcal{N}^a(f)$ implies that $\tilde{u} = u$, which concludes the proof. ∎

Proof of Main Theorem. Assume first that $F(t, \cdot) \in L^1(d\mu)$. Then $F(f, \cdot) \in L^1(d\mu)$. It follows from [9] and [1] that the nonnegative measure $F(f, \cdot)d\mu$ is the Monge–Ampère measure of a function u_0 from the class $\mathcal{F}^a(f)$. Then

$$(dd^c u_0)^n = F(f, \cdot)d\mu \geq F(u_0, \cdot)d\mu.$$

We denote by $\mathcal{A}$ the set of all $u \in \mathcal{F}^a(f)$ such that $u \geq u_0$. The set $\mathcal{A}$ is convex and compact with respect to the $L^1(d\lambda)$ topology, where $d\lambda$ denotes the Lebesgue measure in $\mathbb{C}^n$. Once more, by [9] (see also [1]), for each $u \in \mathcal{A}$ there exists a unique $\hat{u} \in \mathcal{F}^a(f)$ such that

$$(dd^c\hat{u})^n = F(u, \cdot)d\mu.$$

Since $\hat{u} \leq f$ and F is nondecreasing in the first variable, we have

$$(dd^c\hat{u})^n = F(u, \cdot)d\mu \leq F(f, \cdot)d\mu = (dd^c u_0)^n.$$

The comparison principle yields $\hat{u} \geq u \geq u_0$, hence $\hat{u} \in \mathcal{A}$.

We define the map $T : \mathcal{A} \to \mathcal{A}$ by $u \mapsto \hat{u}$. By Schauder's fixed point theorem, we are done as soon as we show that the map T is continuous. Let $u_j \in \mathcal{A}$ converge towards $u \in \mathcal{A}$. By Lemma 3.1, it is enough to show that $F(u_j,\cdot)d\mu \to F(u,\cdot)d\mu$. After extracting a subsequence, we may assume that $u_j \to u$ $d\lambda$-a.e. Applying Lemma 2.1. in [11], we get $u_j \to u$ $d\mu$-a.e. By Lebesgue's convergence theorem we have $F(u_j,\cdot)d\mu \to F(u,\cdot)d\mu$.

We now complete the proof of the general case. Set

$$\mathcal{K} := \{\varphi \in \mathcal{N}^a(f) : (dd^c\varphi)^n \geq F(\varphi,\cdot)d\mu\}.$$

Claim 1. *$\mathcal{K}$ is not empty.*

Indeed, it follows from the monotonicity of F that

$$(dd^c v_0 + f)^n \geq (dd^c v_0)^n \geq F(v_0,)d\mu \geq F(v_0 + f,\cdot)d\mu,$$

so the function $\varphi_0 := v_0 + f$ belongs to $\mathcal{K}$.

Let

$$\mathcal{K}_0 := \{\varphi \in \mathcal{K} : \varphi \geq \varphi_0\}.$$

Claim 2. *$\mathcal{K}_0$ is stable under taking the maximum.*

Indeed, let $\varphi_1, \varphi_2 \in \mathcal{K}_0$. It is clear that $\max(u_1,u_2) \geq \varphi_0$. Since $\mathcal{N}^a(f)$ is stable under taking maximum, we have $\max(u_1,u_2) \in \mathcal{N}^a(f)$. On the other hand, from [18],

$$\begin{aligned}(dd^c \max(u_1,u_2))^n &\geq \mathbf{1}_{(u_1\geq u_2)}(dd^c u_1)^n + \mathbf{1}_{(u_1<u_2)}(dd^c u_2)^n \\ &\geq \mathbf{1}_{(u_1\geq u_2)}F(u_1,\cdot)d\mu + \mathbf{1}_{(u_1<u_2)}F(u_2,.)d\mu \\ &\geq F(\max(u_1,u_2),\cdot)d\mu.\end{aligned}$$

This implies that $\max(u_1,u_2) \in \mathcal{K}_0$.

Claim 3. *$\mathcal{K}_0$ is compact in $L^1_{\mathrm{loc}}(\Omega)$.*

It is enough to prove that $\mathcal{K}_0$ is closed. Let $\varphi_j \in \mathcal{K}_0$ be a sequence converging towards $\varphi \in \mathcal{N}^a(f)$. The limit function is given by $\varphi = (\limsup_{j\to\infty}\varphi_j)^*$. Then $\varphi_0 \leq \varphi \leq f$. The continuity of the complex Monge–Ampère operator and the properties of F yield

$$\begin{aligned}(dd^c\varphi)^n &= \lim_{j\to\infty}\Big(dd^c \sup_{k\geq j}\varphi_k\Big)^n = \lim_{j\to\infty}\lim_{l\to\infty}\Big(dd^c \max_{l\geq k\geq j}\varphi_k\Big)^n \\ &\geq \lim_{j\to\infty}\lim_{l\to\infty}F\Big(\max_{l\geq k\geq j}\varphi_k,\cdot\Big)d\mu.\end{aligned}$$

Therefore $\varphi \in \mathcal{K}_0$.

Consider the upper envelope

$$\phi(z) := \sup\{\varphi(z) : \varphi \in \mathcal{K}_0\}.$$

Notice that in order to get a psh function we should a priori replace ϕ by its upper semicontinuous regularization $\phi^*(z) := \limsup_{\zeta\to z}\phi(\zeta)$; but since $\phi^* \in \mathcal{K}_0$, also ϕ^* contributes to the envelope (i.e. $\phi^* \in \mathcal{K}_0$), and hence $\phi = \phi^*$.

Claim 4. *ϕ is a solution to the Monge–Ampère equation* (1.1).

It follows from Choquet's Lemma that there exists a sequence $\phi_j \in \mathcal{K}_0$ such that

$$\phi = \Big(\limsup_{j\to\infty}\phi_j\Big)^*.$$

Since $\mathcal{K}_0$ is stable under taking the maximum, we can assume that ϕ_j is nondecreasing. We use the classical balayage procedure to prove that ϕ is actually a solution of (1.1). Pick a ball $\mathbf{B} \Subset \Omega$ and define

$$\phi_j^B(z) := \sup\{v(z) : v^* \le \phi_j \text{ on } \partial\mathbf{B},\, v \in \mathrm{PSH}(\mathbf{B})\}, \qquad z \in \mathbf{B}.$$

By the first part of the proof, there exists $\tilde\phi_j \in \mathcal{F}^a(\phi_j^B, \mathbf{B})$ such that

$$(dd^c\tilde\phi_j)^n = \mathbf{1_B}F(\tilde\phi_j, \cdot)d\mu.$$

In fact, $\tilde\phi_j$ is the following upper envelope:

$$\tilde\phi_j = \sup\{w \in \mathcal{E}(\mathbf{B}) : w \le \phi_j^B \text{ and } (dd^cw)^n \ge F(w,\cdot)d\mu\}.$$

Indeed, if we denote by g the right hand side function, then $\tilde\phi_j \le g \le \phi_j^B$. Hence $g \in \mathcal{F}^a(\phi_j^B, \mathbf{B})$. It follows from [1, Lemma 3.3] that

$$\int_\Omega \chi(dd^c\tilde\phi_j)^n \le \int_\Omega \chi(dd^cg)^n, \qquad \forall\chi \in \mathcal{E}_0. \tag{3.1}$$

On the other hand, as before, we have $g = (\lim g_k)^*$ where $g_k \in \mathcal{E}(\mathbf{B})$ is a nondecreasing sequence satisfying $\phi_j^B \ge g_k \ge \phi_j$ and $(dd^cg_k)^n \ge F(g_k,\cdot)d\mu$. Therefore $(dd^cg)^n \ge F(g,\cdot)d\mu$. Thus

$$(dd^c\tilde\phi_j)^n = F(\tilde\phi_j,\cdot)d\mu \le F(g,\cdot)d\mu \le (dd^cg)^n. \tag{3.2}$$

Combining (3.1) and (3.2), we get

$$(dd^c\tilde\phi_j)^n = (dd^cg)^n,$$

therefore, by the comparison principle, $\tilde\phi_j = g$.

Now, for $j \in \mathbb{N}$, consider the function ψ_j defined on Ω by

$$\psi_j(z) = \begin{cases} \tilde\phi_j(z) & \text{if } z \in \mathbf{B},\\ \phi_j(z) & \text{if } z \notin \mathbf{B}.\end{cases}$$

On $\mathbf{B}$ we have $\phi_j \le \tilde\phi_j \le \phi_j^B \le f$ and on $\Omega\setminus\mathbf{B}$ we have $\tilde\phi_j = \phi_j \le f$. Hence $\psi_j \in \mathcal{N}^a(f)$. From the definition of ψ_j, we deduce that $(dd^c\psi_j)^n \ge F(\psi_j,\cdot)d\mu$. Therefore $\psi_j \in \mathcal{K}_0$ and

$$\phi = \Big(\lim_{j\to\infty}\psi_j\Big)^*.$$

Since the complex Monge–Ampère operator is continuous under monotonic sequences and $\mathbf{B}$ is arbitrary, to conclude the proof of the claim it is enough to observe that the sequence ψ_j is nondecreasing.

Uniqueness follows in a classical way from the comparison principle and the monotonicity of F. Indeed, assume that there exist two solutions φ_1 and φ_2 in $\mathcal{N}^a(f)$ such that

$$(dd^c\varphi_i)^n = F(\varphi_i,\cdot)d\mu, \quad i = 1,2.$$

Since F is nondecreasing in the first variable, we have

$$F(\varphi_1,\cdot)d\mu \le F(\varphi_2,\cdot)d\mu \quad \text{on } (\varphi_1 < \varphi_2).$$

On the other hand, by the comparison principle,

$$\begin{aligned}\int\limits_{(\varphi_1<\varphi_2)} F(\varphi_2,\cdot)\,d\mu &= \int\limits_{(\varphi_1<\varphi_2)} (dd^c\varphi_2)^n \le \int\limits_{(\varphi_1<\varphi_2)} (dd^c\varphi_1)^n \\ &= \int\limits_{(\varphi_1<\varphi_2)} F(\varphi_2,\cdot)\,d\mu.\end{aligned}$$

Therefore

$$F(\varphi_1,\cdot)d\mu = F(\varphi_2,\cdot)d\mu \quad \text{on } (\varphi_1 < \varphi_2).$$

In the same way, we get the equality on $(\varphi_1 > \varphi_2)$ and so on Ω. Hence $(dd^c\varphi_1)^n = (dd^c\varphi_2)^n$ on Ω. Therefore uniqueness in the class $\mathcal{N}^a(f)$ yields $\varphi_1 = \varphi_2$, and the proof is complete. ■

REMARKS. 1. We have no precise knowledge when a subsolution of (1.1) exists. However, if there exists a negative function $\psi \in \mathrm{PSH}(\Omega)$ such that

$$\int\limits_{\Omega} -\psi F(0,\cdot)\,d\mu < \infty,$$

then (1.1) admits a subsolution $v \in \mathcal{N}^a$. This is an immediate consequence of [9, Proposition 5.2].

2. Condition (2) in Main Theorem is necessary.

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