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Triangularization properties of power linear maps and the Structural Conjecture

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Abstract. We discuss several additional properties a power linear Keller map may have. The Structural Conjecture of Drużkowski (1983) asserts that certain two such properties are equivalent, but we show that one of them is stronger than the other. We even show that the property of linear triangularizability is strictly in between. Furthermore, we give some positive results for small dimensions and small Jacobian ranks.

1. Introduction. Throughout this paper, we will write K for any field of characteristic zero, $\bar{K}$ for its algebraic closure, and $K[x] = K[x_1, \ldots, x_n]$ for the polynomial algebra over K with n indeterminates $x = x_1, \ldots, x_n$. Let $F = (F_1, \ldots, F_n) : K^n \to K^n$ be a polynomial map, that is, $F_i \in K[x]$ for all $1 \le i \le n$, or briefly $F \in K[x]^n$. We view F and x as column matrices, and write $\partial = \partial_1, \ldots, \partial_n$, where $\partial_i = \partial/\partial x_i$. Just like $F = (F_1, \ldots, F_n)$, we view any other tuple whose elements are separated by commas as a column vector. Let M^{t} be the transpose of a matrix M and write

$$\mathcal{J}F = (\partial(F^{\mathrm{t}}))^{\mathrm{t}} = \begin{pmatrix} \partial_1 F_1 & \partial_2 F_1 & \cdots & \partial_n F_1 \\ \partial_1 F_2 & \partial_2 F_2 & \cdots & \partial_n F_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 F_n & \partial_2 F_n & \cdots & \partial_n F_n \end{pmatrix}.$$

We say that a polynomial map F is a *Keller map* if $\det \mathcal{J}F \in K^*$. The well-known Jacobian Conjecture, raised by O.-H. Keller in 1939 in [Kel], states that a polynomial map $F : K^n \to K^n$ is invertible if it is a Keller map. This conjecture is still open for all $n \ge 2$. In [Dru, Th. 3], Ludwik Drużkowski showed that it suffices to consider polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ of the form $F = x + (Ax)^{*3}$, where $A \in \mathrm{Mat}_n(\mathbb{C})$ and M^{*d} is the dth Hadamard power (repeated Hadamard product with itself) of a matrix M.

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In the same paper, Drużkowski also formulated the following Structural Conjecture. Write $M|_{x=G}$ for the substitution of x by G in a matrix M.

STRUCTURAL CONJECTURE. *If* $F = x + (Ax)^{*3}$ *and* $\det \mathcal{J}F = 1$, *then the following conditions are equivalent:*

(JC) $\det((\mathcal{J}F)|_{x=v_1} + (\mathcal{J}F)|_{x=v_2}) \neq 0$ *for all* $v_1, v_2 \in \mathbb{C}^n$.

$(**)$ *There exist* $b_i \in \mathbb{C}^n$ *and* $c_j \in \mathbb{C}^n$ *such that* $c_j^{\mathrm{t}} b_i = 0$ *for every* $i \geq j \geq 1$, *and* F *has the form* $x + \sum_{i=1}^{n-1} (c_i^{\mathrm{t}} x)^3 b_i$.

Actually, Drużkowski writes $F = x + \sum_{j=1}^{n} (a_j^{\mathrm{t}} x)^3 e_j$ instead of $F = x + (Ax)^{*3}$, where e_j is the jth standard unit vector. Hence a_j^{t} corresponds to the jth row A_j of A. Since the vectors c_j and b_i are viewed as column matrices, the matrix product $c_j^{\mathrm{t}} b_i$ has only one entry, which we regard as an element of $\mathbb{C}$.

We call a polynomial map F over K *linearly triangularizable* if there exists a $T \in \mathrm{GL}_n(K)$ such that the Jacobian of $T^{-1}F(Tx)$ is a triangular matrix. For Keller maps of the form $F = x + H$ with H homogeneous of degree $d \geq 2$, the existence of such a T automatically means that the diagonal of

$$\mathcal{J}(T^{-1}F(Tx)) = T^{-1}(\mathcal{J}H)|_{x=Tx}T$$

is zero, because $\mathcal{J}H$ has to be nilpotent due to the Keller condition.

We embed the Structural Conjecture in a more general context, where F has the form $x + H$ such that $\mathcal{J}H$ is nilpotent, and compare its conditions with linear triangularizability and other properties. We give positive results in special cases and counterexamples in general. When we give counterexamples, we produce one of the form $F = x + H$ with H homogeneous of degree d and one of the form $F = x + (Ax)^{*d}$, for every $d \geq 3$ and possibly also for $d = 2$.

2. Triangularization.

In the following proposition, the conditions (JC) and $(**)$ of the Structural Conjecture are included in a chain of six properties. Furthermore, we consider maps $x + H$ such that H has no constant terms instead of being homogeneous.

PROPOSITION 2.1. *Let* $F = x + H$ *be any polynomial map of degree* d *over* K. *Consider the following conditions:*

(JC^-) *F is invertible;*

(JC) $\det\Big(\sum_{i=1}^{d-1} \mathcal{J}F|_{x=v_i}\Big) \in \bar{K}^*$ *for all* $v_i \in \bar{K}^n$;

(JC^+) $\det\Big(\sum_{i=1}^{n} \mathcal{J}F|_{x=v_i}\Big) \in \bar{K}^*$ *for all* $v_i \in \bar{K}^n$;

and the existence of $b_i \in K^n$, $c_j \in K^n$ *and* $d_i \in \{1,\dots,d\}$ *such that* $c_j^{\mathrm{t}} b_i = 0$ *for every* $i \geq j \geq 1$, *and one of the following holds:*

$$(*) \qquad H = \sum_{i=1}^{N} (c_i^{\mathrm{t}} x)^{d_i} b_i \qquad \text{for some } N \in \mathbb{N};$$

$$(**) \qquad H = \sum_{i=1}^{n-1} (c_i^{\mathrm{t}} x)^{d_i} b_i;$$

$$(***) \qquad H = \sum_{i=1}^{n-1} (c_i^{\mathrm{t}} x)^{d_i} b_i \quad \text{and} \quad b_1,\dots,b_{n-1} \text{ are linearly independent.}$$

Then $(\mathrm{JC}^-) \Leftarrow (\mathrm{JC}) \Leftarrow (\mathrm{JC}^+) \Leftarrow (*) \Leftarrow (**) \Leftarrow (***)$.

Furthermore, (JC^-), (JC) *and* (JC^+) *are satisfied when* $d \leq 2$.

Proof. Notice that the last two implications are trivial. The first two follow from [GBDS, Cor. 2.3] and [GBDS, Th. 3.5] respectively. The last claim follows from the first two implications and [GBDS, Prop. 3.1].

To show the third implication, assume $(*)$ holds and take $v_1,\dots,v_n \in \bar{K}$ arbitrary. Then

$$S := \sum_{k=1}^{n} (\mathcal{J}H)|_{x=v_k} = \sum_{k=1}^{n}\sum_{i=1}^{N} b_i \cdot d_i (c_i^{\mathrm{t}} v_k)^{d_i-1} \cdot c_i^{\mathrm{t}} = \sum_{i=1}^{N} b_i \Big(d_i \sum_{k=1}^{n} (c_i^{\mathrm{t}} v_k)^{d_i-1} \Big) c_i^{\mathrm{t}}.$$

It follows that in the expansion of S^{N+1}, each term will have a factor $c_j^{\mathrm{t}} \cdot b_i$ such that $i \geq j \geq 1$, which is zero by assumption. Hence $S^{N+1} = 0$. Thus S is nilpotent and $\det(\sum_{i=1}^{n} \mathcal{J}F|_{x=v_i}) = \det(nI_n + S) = n^n \in \bar{K}^*$. ∎

In the last section, we will show that $(\mathrm{JC}^-) \not\Rightarrow (\mathrm{JC})$ and $(\mathrm{JC}^+) \not\Rightarrow (*) \not\Rightarrow (**) \not\Rightarrow (***)$, even in the case where H is *homogeneous power linear*, i.e. $H = \sum_{i=1}^{n} (c_i^{\mathrm{t}} x)^d e_i$ for some $c_i \in K^n$ and a $d \geq 1$. But first, we formulate a lemma and a theorem about the starred equations. We call H *nonhomogeneous power linear* if $H = \sum_{i=1}^{n} (c_i^{\mathrm{t}} x)^{d_i} e_i$ for some $c_i \in K^n$ and some $d_i \geq 1$.

LEMMA 2.2. *Let* $N \in \mathbb{N}$ *and suppose that there exist* $b_i, c_j \in K^n$ *and* $d_i \in \mathbb{N}$ *such that*

$$H = \sum_{i=1}^{N} (c_i^{\mathrm{t}} x)^{d_i} b_i.$$

Then the following statements are equivalent:

- (i) *There exists a* $\sigma \in S_N$ *such that* $c_{\sigma(j)}^{\mathrm{t}} b_{\sigma(i)} = 0$ *for every* $i \geq j \geq 1$.
- (ii) *There exists a* $T \in \mathrm{GL}_n(K)$ *such that the Jacobian of* $T^{-1}(c_i^{\mathrm{t}} T x)^{d_i} b_i$ *is lower triangular with zeroes on the diagonal for all* $i \leq N$.

Furthermore, if $b_1, \ldots, b_N$ *are linearly independent and* σ *is as in* (i), *then in* (ii) *we can choose* $T \in \mathrm{GL}_n(K)$ *such that*

$$b_{\sigma(i)} = Te_{n-N+i} \quad \textit{for each } i. \tag{1}$$

Proof. We prove that (i) and (ii) are equivalent, showing the last claim along the way.

(ii)⇒(i). Suppose that (ii) holds. For each j, let m_j be the number of trailing zero coordinates of $T^{\mathrm{t}}c_j$. By reordering the terms of H, we can obtain $m_j \geq m_i$ for $i \geq j \geq 1$. By (ii),

$$\mathcal{J}(T^{-1}(c_i^{\mathrm{t}}Tx)^d b_i) = T^{-1}b_i \cdot d_i(c_i^{\mathrm{t}}Tx)^{d_i-1} \cdot c_i^{\mathrm{t}}T \tag{2}$$

is lower triangular with zeroes on the diagonal. Hence the number of leading zero coordinates of $T^{-1}b_i$ is at least $n - m_i \geq n - m_j$ for $i \geq j \geq 1$. Comparing the numbers of leading and trailing zero coordinates, we get $c_j^{\mathrm{t}}b_i = c_j^{\mathrm{t}}T \cdot T^{-1}b_i = 0$ for $i \geq j \geq 1$, which is (i) with $\sigma = 1$. So we can take $\sigma = 1$ when $m_j \geq m_{j+1}$ for each j already before reordering the terms of H.

(i)⇒(ii). Suppose that (i) holds. Again by reordering the terms of H, we can obtain $\sigma = 1$. Suppose that the vector space spanned by the column vectors $b_1, \ldots, b_N$ has dimension r. Then there are $\tau(1) < \cdots < \tau(r)$ such that $b_{\tau(1)}, \ldots, b_{\tau(r)}$ is a basis of this vector space. Now choose $\tau(1)+\cdots+\tau(r)$ as large as possible. Then b_i is linearly dependent on $b_{\tau(k)}, b_{\tau(k+1)}, \ldots, b_{\tau(r)}$ for all k and all $i > \tau(k-1)$, where $\tau(0) = 0$ and where zero vectors are linearly dependent on the empty set. Furthermore, $c_i^{\mathrm{t}}b_{\tau(k)} = c_i^{\mathrm{t}}b_{\tau(k+1)} = \cdots = c_i^{\mathrm{t}}b_{\tau(r)} = 0$ for all k and all $i \leq \tau(k)$ on account of (i) with $\sigma = 1$.

Take $T \in \mathrm{GL}_n(K)$ such that the last r columns of T are $b_{\tau(1)}, \ldots, b_{\tau(r)}$, in that order. Then we have (1) with $\sigma = 1$ if $b_1, \ldots, b_N$ are linearly independent. Take $i \leq N$ arbitrary. It suffices to show that (2) is lower triangular with zeroes on the diagonal. This is trivial when $b_i = 0$, so assume that $b_i \neq 0$. Then by definition of r and τ, there exists a $k \geq 1$ such that $\tau(k) \geq i > \tau(k-1)$. As we have seen above, b_i is linearly dependent on $b_{\tau(k)}, b_{\tau(k+1)}, \ldots, b_{\tau(r)}$ and $c_i^{\mathrm{t}}b_{\tau(k)} = c_i^{\mathrm{t}}b_{\tau(k+1)} = \cdots = c_i^{\mathrm{t}}b_{\tau(r)} = 0$.

Hence $T^{-1}b_i$ is linearly dependent on $e_{n-r+k}, e_{n-r+k+1}, \ldots, e_n$ and $c_i^{\mathrm{t}}T \cdot e_{n-r+k} = c_i^{\mathrm{t}}Te_{n-r+k+1} = \cdots = c_i^{\mathrm{t}}Te_n = 0$ by definition of T. Consequently, all nonzero entries of (2) are within the submatrix consisting of rows $n-r+k, \ldots, n$ and columns $1, \ldots, n-r+k-1$. Since i was arbitrary, we obtain (ii). ∎

THEOREM 2.3. *Let* $x + H$ *be any map of degree* $d \geq 1$ *over* K *such that* $H(0) = 0$. *Then:*

(i) H *is of the form* $(*)$ *if and only if there exists a* $T \in \mathrm{GL}_n(K)$ *such that the Jacobian of* $T^{-1}H(Tx)$ *is lower triangular with zeroes on the diagonal, i.e.* H *is linearly triangularizable and* $\mathcal{J}H$ *is nilpotent.*

(ii) *H is of the form $(**)$ if and only if there exist $b_i, c_j \in K^n$ and $T \in \mathrm{GL}_n(K)$ such that $H = \sum_{i=1}^{n-1}(c_i^{\mathrm{t}}x)^{d_i}b_i$ and the Jacobian of $T^{-1}(c_i^{\mathrm{t}}Tx)^{d_i}b_i$ is lower triangular with zeroes on the diagonal for all $i \leq n-1$.*

(iii) *H is of the form $(***)$ if and only if there exists a $T \in \mathrm{GL}_n(K)$ such that each component of $T^{-1}H(Tx)$ is a power of a linear form and the Jacobian of $T^{-1}H(Tx)$ is lower triangular with zeroes on the diagonal.*

Proof. Since the three results have similarities, we structure the proof as follows.

Only-if-parts. All only-if-parts follow immediately from (i)⇒(ii) of Lemma 2.2, except the claim in (iii) that each component of $T^{-1}H(Tx)$ is a power of a linear form. So assume that H is of the form $(***)$. By $(***)$ and (1), we have

$$T^{-1}H(Tx) = \sum_{i=1}^{n-1} T^{-1}(c_{\sigma(i)}^{\mathrm{t}}Tx)^{d_i}b_{\sigma(i)} = \sum_{i=1}^{n-1}(c_{\sigma(i)}^{\mathrm{t}}Tx)^{d_i}e_{i+1}$$

for some $\sigma \in S_{n-1}$. So $T^{-1}H(Tx)$ is of the desired form.

If-parts. The if-part of (ii) follows immediately from (ii)⇒(i) of Lemma 2.2. To prove the if-part of (i), suppose that $T^{-1}H(Tx)$ has a lower triangular Jacobian with zeroes on the diagonal. Then there exists an $r \in \mathbb{N}$ such that we can write the $(i+1)$th component of $T^{-1}H(Tx)$ as a linear combination of r powers of linear forms $c_j^{\mathrm{t}}Tx$ in $K[x_1, \ldots, x_i]$, for each $i \geq 1$. Furthermore, the first component of $T^{-1}H(Tx)$ is zero on account of $H(0) = 0$. Hence

$$\begin{aligned} T^{-1}H(Tx) &= \sum_{i=1}^{n-1}\sum_{j=1}^{r}(c_{r(i-1)+j}^{\mathrm{t}}Tx)^{d_{r(i-1)+j}}e_{i+1} \\ &= T^{-1}\sum_{i=1}^{n-1}\sum_{j=1}^{r}(c_{r(i-1)+j}^{\mathrm{t}}Tx)^{d_{r(i-1)+j}}Te_{i+1}. \end{aligned}$$

Taking $b_{r(i-1)+j} = Te_{i+1}$ for all i and all j with $1 \leq j \leq r$, we have

$$H = \sum_{i=1}^{n-1}\sum_{j=1}^{r}(c_{r(i-1)+j}^{\mathrm{t}}x)^{d_{r(i-1)+j}}b_{r(i-1)+j} = \sum_{i=1}^{r(n-1)}(c_i^{\mathrm{t}}x)^{d_i}b_i.$$

Furthermore, for each j, the Jacobian of $(c_j^{\mathrm{t}}Tx)^{d_j}T^{-1}b_j$ only has nonzero entries in the submatrix consisting of row $i+1$ and columns $1, \ldots, i$, by definition of c_j and b_j, where $i = \lceil j/r \rceil$. Hence the Jacobian of $(c_j^{\mathrm{t}}Tx)^{d_j}T^{-1}b_j$ is lower triangular with zeroes on the diagonal for all j. Now the if-part of (i) follows from (ii)⇒(i) of Lemma 2.2. The if-part of (iii) follows as well, because we can take $r = 1$ in that case, so that $r(n-1) = n-1$ and the b_i's are linearly independent. ∎

3. Positive results. First, we formulate a theorem about maps $x + H$ with H homogeneous and $\mathcal{J}H$ nilpotent.

THEOREM 3.1. *Assume that $H \in K[x]^n$ is homogeneous of degree $d \geq 1$, and $\mathcal{J}H$ is nilpotent. Then we have* $(*{*}*)$ (*and hence five* $\Rightarrow$*'s*) *if $n \leq 2$, and* $(*)$ (*and hence three* $\Rightarrow$*'s*) *if $n = 3$ or $n = 4 = d+2$. Furthermore, the implication chain* (JC) $\Rightarrow$ (JC$^+$) $\Rightarrow$ $(*)$ *holds when $n = 4 = d+1$.*

If H is in addition power linear, then the above claims even hold with the estimates on n replaced by estimates on $\operatorname{rk} \mathcal{J}H$.

Proof. We show the equivalent properties in (i) and (iii) of Theorem 2.3 respectively instead of $(*)$ and $(*{*}*)$. We start with the cases where H is only homogeneous.

The case $n \leq 2$ follows from [Che, Lem. 3], and the case $n = 3$ follows from [BE, Th. 1.1]. The case $n = 4 = d + 2$ follows from a corresponding strong nilpotence result in [MO], and the equivalence of strong nilpotence and the property in (i) of Theorem 2.3, which is proved in [EH]. The case $n = 4 = d + 1$ follows from [B1, Th. 4.6.5] and the fact that $F = x + H$ with H as in [B1, Th. 4.6.5] does not satisfy (JC), because the rightmost two columns of $(\mathcal{J}F)|_{x=(1,\mathrm{i},0,0)} + (\mathcal{J}F)|_{x=(1,-\mathrm{i},0,0)}$ are equal, where $H = \big(0, \lambda x_1^3, x_2(x_1x_3 - x_2x_4) + p(x_1, x_2), x_1(x_1x_3 - x_2x_4) + q(x_1, x_2)\big)$.

Assume from now on that H is in addition power linear. The case $\operatorname{rk} \mathcal{J}H \leq 2$ follows from [TB, Th. 4.7], because $K = \mathbb{C}$ is not used in its proof, or from Theorem 3.2 below. The cases $\operatorname{rk} \mathcal{J}H = 3$ and $\operatorname{rk} \mathcal{J}H = 4 = d + 2$ follow via [Che, Th. 2] from the cases $n = 3$ and $n = 4 = d + 2$ respectively. The case $\operatorname{rk} \mathcal{J}H = 4 = d + 1$ follows from the case $n = 4 = d + 1$ by using a variant of [Che, Th. 2], namely with $(*)$ replaced by (JC) $\Rightarrow$ $(*)$. To prove this variant, one can follow the proof of [Che, Th. 2] to see that in that proof it suffices to show that $F_1 = T^{-1} \circ F \circ T$ satisfies (JC) if F does. ∎

In Theorem 3.2 below, which is the nonhomogeneous variant of Theorem 3.1, we must replace the estimates on n and $\operatorname{rk} \mathcal{J}H$ of Theorem 3.1 by estimates on $n + 1$ and $\operatorname{rk} \mathcal{J}H + 1$ respectively, except the estimate $n \leq 2$ for $(**) \Rightarrow (*{*}*)$, and the estimate $\operatorname{rk} \mathcal{J}H \leq 2$ for $(*{*}*)$, which can be maintained.

THEOREM 3.2. *Assume that $H \in K[x]^n$ has degree d, $H(0) = 0$ and $\mathcal{J}H$ is nilpotent. Then we have* $(*{*}*)$ (*and hence five* $\Rightarrow$*'s*) *if $n \leq 1$, both* $(*)$ *and* $(**) \Rightarrow (*{*}*)$ (*and hence four* $\Rightarrow$*'s*) *if $n = 2$, and* $(*)$ (*and hence three* $\Rightarrow$*'s*) *if $n = 3 = d + 1$. Furthermore, the implication chain* (JC) $\Rightarrow$ (JC$^+$) $\Rightarrow$ $(*)$ *holds when $n = 3 = d$.*

If H is in addition power linear, then the above claims even hold when we replace the estimates on n by estimates on $\operatorname{rk} \mathcal{J}H$*, and additionally* $(***)$ (*and hence five* $\Rightarrow$*'s) holds when* $\operatorname{rk} \mathcal{J}H = 2$.

Furthermore, if we replace $(*)$ *and* $(***)$ *by their equivalents in* (i) *and* (iii) *of Theorem* 2.3*, then the condition* $H(0) = 0$ *is no longer necessary.*

Proof. We show the equivalent properties in (i) and (iii) of Theorem 2.3 respectively instead of $(*)$ and $(***)$. We start with the cases where H only has a nilpotent Jacobian.

The case $n = 1$ is trivial, because $H = 0$ in that case. Notice that in the cases $n = 2$ and $n = 3 = d + 1$, the homogeneization $x_{n+1}^d H(x_{n+1}^{-1}x, 0)$ of H has a strongly nilpotent Jacobian on account of Theorem 3.1. By substituting $x_{n+1} = 1$, we see that the Jacobian of H itself is strongly nilpotent as well. By the equivalence of strong nilpotence and the property in (i) of Theorem 2.3, which is proved in [EH], we have the latter property, and hence also $(*)$, when $n = 2$ or $n = 3 = d + 1$. This gives the case $n = 3 = d + 1$, and also the case $n = 2$, because $(**)$ and $(***)$ are trivially equivalent when $n = 2$.

In order to prove the case $n = 3 = d$, assume that H does not have the property in (i) of Theorem 2.3. By [B1, Cor. 4.6.6], we may assume that the first component of $T^{-1}H(Tx)$ equals $\lambda \in K$ for some $T \in \mathrm{GL}_3(K)$. Following the proof of [B1, Th. 4.6.5], we see that $T^{-1}H(Tx) = \big(\lambda, x_1(x_2 - x_1x_3) + p(x_1), (x_2 - x_1x_3) + q(x_1)\big)$ for some $T \in \mathrm{GL}_3(K)$. Since the rightmost two columns of $(\mathcal{J}F)|_{x=(\mathrm{i},0,0)} + (\mathcal{J}F)|_{x=(-\mathrm{i},0,0)}$ are equal, we see that (JC) does not hold, as desired.

Assume from now on that H is in addition (nonhomogeneous) power linear. The cases $\operatorname{rk} \mathcal{J}H = 3 = d$ and $\operatorname{rk} \mathcal{J}H = 3 = d + 1$ follow in a similar manner to the cases $\operatorname{rk} \mathcal{J}H = 4 = d + 1$ and $\operatorname{rk} \mathcal{J}H = 4 = d + 2$ respectively in Theorem 3.1. So assume that $\operatorname{rk} \mathcal{J}H \le 2$. Take λ and μ as in Lemma 3.3 below. If $\mu^{\mathrm{t}}H$ is a power of a linear form, then we take $T \in \mathrm{GL}_n(K)$ such that λ^{t} and μ^{t} are the first two rows of T^{-1}, in that order, and for the remaining rows of T^{-1} we transpose standard unit vectors. Since λ^{t} and μ^{t} generate the row space of $\mathcal{J}H$, we see that $\lambda^{\mathrm{t}}T = e_1$ and $\mu^{\mathrm{t}}T = e_2$ generate the row space of $\mathcal{J}(T^{-1}H(Tx))$. Using additionally that $\mu^{\mathrm{t}}H \in K[\lambda^{\mathrm{t}}x]$, we find that $T^{-1}H(Tx) \in K \times K[x_1] \times K[x_1, x_2]^{n-2}$ has a lower triangular Jacobian with zeroes on the diagonal. So we have (i) of Theorem 2.3 and hence also $(*)$.

Now assume that $\mu^{\mathrm{t}}H$ is not a power of a linear form. By Lemma 3.3 below,

$$\mu^{\mathrm{t}}H = \nu_1(\lambda^{\mathrm{t}}x)^{d_1} + \cdots + \nu_r(\lambda^{\mathrm{t}}x)^{d_r} \tag{3}$$

where $r \ge 2$, $\nu \in (K \setminus \{0\})^r$ and $\{0, 1\} \ni \lambda^{\mathrm{t}}H \le d_1 < \cdots < d_r$. Take

$T \in \mathrm{GL}_n(K)$ such that λ^{t} is the first row of T^{-1}. Let $V \in \mathrm{Mat}_{r,n}(\{0,1\})$ with $V_{ij} = 1$ if and only if $\deg H_j = d_i$. Without worrying about linear independence of rows at this stage, take for each i with $2 \le i \le r$ the $(i+1)$th row of T^{-1} equal to $T^{-1}_{i+1} = \nu_i^{-1}\mu^{\mathrm{t}} * V_i$, where $*$ is the Hadamard product and V_i is the ith row of V. Then the $(i+1)$th component $T^{-1}_{i+1}H$ of $T^{-1}H$ is ν_i^{-1} times the homogeneous part of degree d_i of (3), which is $\nu_i^{-1}\nu_i(\lambda^{\mathrm{t}}x)^{d_i} = (\lambda^{\mathrm{t}}x)^{d_i}$, for each i with $2 \le i \le r$.

Still without worrying about linear independence of rows, take the second row of T^{-1} equal to

$$T_2^{-1} = \nu_1^{-1}\big(\mu^{\mathrm{t}} - (\mu^{\mathrm{t}} * (V_2 + \cdots + V_r))\big).$$

Since $\mu^{\mathrm{t}} * V_i = \nu_i T^{-1}_{i+1}$ for each i with $2 \le i \le r$ by definition of T^{-1}, we have

$$T_2^{-1} = \nu_1^{-1}\big(\mu^{\mathrm{t}} - (\nu_2 T_3^{-1} + \cdots + \nu_r T^{-1}_{r+1})\big) \tag{4}$$

and the second component of $T^{-1}H$ equals

$$\begin{aligned} T_2^{-1}H &= \nu_1^{-1}\big(\mu^{\mathrm{t}}H - (\nu_2 T_3^{-1}H + \cdots + \nu_r T^{-1}_{r+1}H)\big) \\ &= \nu_1^{-1}\big(\mu^{\mathrm{t}}H - (\nu_2(\lambda^{\mathrm{t}}x)^{d_2} + \cdots + \nu_r(\lambda^{\mathrm{t}}x)^{d_r})\big), \end{aligned}$$

which by (3) is equal to $\nu_1^{-1}\nu_1(\lambda^{\mathrm{t}}x)^{d_1} = (\lambda^{\mathrm{t}}x)^{d_1}$. Thus for each $i \in \{1,\dots,r\}$, the $(i+1)$th component of $T^{-1}H$ is equal to $(\lambda^{\mathrm{t}}x)^{d_i}$.

Since $\{0,1\} \ni \lambda^{\mathrm{t}}H \le d_1 < \cdots < d_r$, we have $\deg T_1^{-1}H = \deg \lambda^{\mathrm{t}}H < \lambda^{\mathrm{t}}H$ and the degrees of the first $r+1$ components of $T^{-1}H$ are strictly increasing. Hence as $T_1^{-1} = \lambda^{\mathrm{t}} \ne 0$, the first $r+1$ rows of T^{-1} are indeed linearly independent. Take transposed standard unit vectors for the remaining rows of T^{-1}. As $\lambda^{\mathrm{t}}T = e_1^{\mathrm{t}}$, the first $r+1$ components of $T^{-1}H(Tx)$ are $\lambda^{\mathrm{t}}H, x_1^{d_1},\dots,x_1^{d_r}$, so $T^{-1}H(Tx) \in K \times K[x_1]^r \times K[x]^{n-r-1}$. Furthermore, we see that $T^{-1}H(Tx)$ is power linear.

By (4), we have $\sum_{i=1}^r \nu_i T^{-1}_{i+1} = \mu^{\mathrm{t}}$, so μ^{t} is a linear combination of the first $r+1$ rows of T^{-1}. Since λ^{t} and μ^{t} generate the row space of $\mathcal{J}H$ and are linear combinations of the first $r+1$ rows of T^{-1}, we see that $\lambda^{\mathrm{t}}T$ and $\mu^{\mathrm{t}}T$ generate the row space of $(\mathcal{J}H)\cdot T$ and are linear combinations of $e_1^{\mathrm{t}},\dots,e_{r+1}^{\mathrm{t}}$. From the fact that H is (nonhomogeneous) power linear, we deduce that the row space of $(\mathcal{J}H)\cdot T$ is the same as that of $\mathcal{J}(T^{-1}H(Tx))$. Hence $T^{-1}H(Tx) \in K[x_1,\dots,x_{r+1}]^n$. Since we have shown above that $T^{-1}H(Tx) \in K \times K[x_1]^r \times K[x]^{n-r-1}$ as well, we can deduce that $T^{-1}H(Tx) \in K \times K[x_1]^r \times K[x_1,\dots,x_{r+1}]^{n-r-1}$. Hence $T^{-1}H(Tx)$ has a lower triangular Jacobian with zeroes on the diagonal. So we have (i) of Theorem 2.3 and hence also $(*)$. ∎

Lemma 3.3. *Assume that $H \in K[A_1x,\dots,A_nx]^n$, where A_j is the jth row of a matrix $A \in \mathrm{Mat}_n(K)$ such that* $\mathrm{rk}\,A \le 2$ *and $\mathcal{J}H$ is nilpotent. Then there exist linearly independent $\lambda,\mu \in K^n$ such that $\mu^{\mathrm{t}}H \in K[\lambda^{\mathrm{t}}x]$*

has no terms of degree less than $\lambda^{\mathrm{t}} H \in \{0,1\}$, *and* λ^{t} *and* μ^{t} *generate the row space of* A.

Proof. Using the case $n = 2$ of Theorem 3.2 (instead of the case $n = 3$ of Theorem 3.1), by similar techniques to the proof of the case $\operatorname{rk} \mathcal{J}H = 3$ of Theorem 3.1 we deduce that there exists a $T \in \mathrm{GL}_n(K)$ such that $ATx \in K[x_1, x_2]^n$ and the Jacobian of $T^{-1}H(Tx) \in K[x_1, x_2]^n$ is lower triangular with zeroes on the diagonal. By a subsequent linear conjugation on the first two coordinates, we can even have in addition the first component of $T^{-1}H(Tx)$ contained in $\{0,1\}$, and the second component without constant term if the first component already has one.

Now take for λ^{t} the first row of T^{-1} and for μ^{t} the second row of T^{-1}. Then $\lambda^{\mathrm{t}} H(Tx) \in \{0,1\}$ and $ATx \in K[x_1, x_2]^n$. Furthermore, $\mu^{\mathrm{t}} H(Tx) \in K[x_1]$ only has terms of degree greater than $\deg \lambda^{\mathrm{t}} H(Tx)$, and hence no terms of degree less than $\lambda^{\mathrm{t}} H(Tx)$ ($\in \{0,1\}$) itself. Thus substituting $x = T^{-1}x$ gives the desired results. ∎

Notice that in the case where H is power linear and $\operatorname{rk} \mathcal{J}H = 1$ in Theorem 3.2, we can even get $T^{-1}H(Tx) \in k[x_1]^n$ in (iii) of Theorem 2.3, namely by taking λ^{t} in the row space of $\mathcal{J}H$. This is similar to the case where H is power linear and $\operatorname{rk} \mathcal{J}H = 2$ in Theorem 3.1, in the proof of which T is taken such that $T^{-1}H(Tx) \in k[x_1, x_2]^n$ in (iii) of Theorem 2.3. It is however not always possible to take T such that $T^{-1}H(Tx) \in k[x_1, x_2]^n$ in (iii) of Theorem 2.3 when H is power linear and $\operatorname{rk} \mathcal{J}H = 2$ in Theorem 3.2: consider e.g. $H = (0, x_1^d, x_1^{d-1}, (x_2 + x_3)^d)$.

Theorems 3.1 and 3.2 contain positive results with estimates on $\operatorname{rk} \mathcal{J}H$, but for power linear H only. Theorem 3.4 below however brings two results with estimates on $\operatorname{rk} \mathcal{J}H$ without the requirement that H is power linear. Furthermore, the homogeneous counterexamples (9) and (10) later in this article show that the estimates in Theorem 3.4 cannot be improved, even if we have the extra condition that H is homogeneous.

THEOREM 3.4. *Assume that* $H \in K[x]^n$ *has degree* d, $H(0) = 0$ *and* $\mathcal{J}H$ *is nilpotent. If* $\operatorname{rk} \mathcal{J}H = 1$ *or* $\operatorname{rk} \mathcal{J}H = 2 = d$, *then* H *is of the form* $(*)$.

Furthermore, if we replace $(*)$ *by its equivalent in* (i) *of Theorem* 2.3, *then the condition* $H(0) = 0$ *is no longer necessary.*

Proof. We show the equivalent property in (i) of Theorem 2.3 instead of $(*)$. The case $\operatorname{rk} \mathcal{J}H \le 1$ follows from the corresponding strong nilpotence result in (2) ⇒ (3) of [B2, Th. 4.2], and from the equivalence of strong nilpotence and the property in (i) of Theorem 2.3, which is proved in [EH].

So assume that $\operatorname{rk} \mathcal{J}H = 2 = d$ and suppose without loss of generality that $H(0) = 0$. The additional claim that the diagonal is zero follows from the nilpotency of $\mathcal{J}H$, so we do not need to worry about that any more. By

Lemma 3.5 below, there exists a $T \in \mathrm{GL}_n(K)$ such that for $\tilde{H} := T^{-1}H(Tx)$, we have one of the following cases, which we treat separately.

• $\tilde{H} \in K[x_1, x_2]^n$. Then by Theorem 3.2, $(\tilde{H}_1, \tilde{H}_2)$ has the property in (i) of Theorem 2.3. Hence we can choose T such that $\mathcal{J}_{x_1,x_2}(\tilde{H}_1, \tilde{H}_2)$ is lower triangular. It follows that $\mathcal{J}\tilde{H}$ is lower triangular as well, which is the property in (i) of Theorem 2.3.

• $\tilde{H}_3 = \cdots = \tilde{H}_n = 0$. Then by [E, Th. 7.2.25], we have

$$(\tilde{H}_1, \tilde{H}_2) = (bg(ax_1 - bx_2) + d, ag(ax_1 - bx_2) + c),$$

where $a, b, c, d \in K[x_3, \ldots, x_n]$ and g is a univariate polynomial over $K[x_3, \ldots, x_n]$. Hence $a\tilde{H}_1 - b\tilde{H}_2 \in K[x_3, x_4, \ldots, x_n]$. Using that $\deg(\tilde{H}_1, \tilde{H}_2) = 2$, we see that either g is constant, or both a and b are constant.

In both cases, there exists a nontrivial K-linear combination of $\tilde{H}_1$ and $\tilde{H}_2$ which is in $K[x_3, \ldots, x_n]$. By choosing T appropriately, we can get $\tilde{H}_2 \in K[x_3, \ldots, x_n]$, in which case $\mathcal{J}\tilde{H}$ is upper triangular. By a subsequent conjugation of $\tilde{H}$ with the map $(x_n, x_{n-1}, \ldots, x_1)$, we get the desired lower triangular form of the Jacobian, which gives the property in (i) of Theorem 2.3.

• $\tilde{H}_2 = \tilde{H}_3^2 \neq 0$ and $\tilde{H}_4 = \cdots = \tilde{H}_n = 0$. If $\tilde{H}_3 \in K[x_4, \ldots, x_n]$, then $\mathcal{J}\tilde{H}$ is upper triangular, and a subsequent conjugation of $\tilde{H}$ with the map $(x_n, x_{n-1}, \ldots, x_1)$ gives the desired result. So assume that $\tilde{H}_3 \notin K[x_4, \ldots, x_n]$. Applying polynomial extension of scalars to the case $n = 3 = d + 1$ of Theorem 3.2, it follows that there exists a $\tilde{T} \in \mathrm{GL}_3(K(x_4, \ldots, x_n))$ such that $\mathcal{J}_{\tilde{x}}(\tilde{T}^{-1}(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3)|_{\tilde{x}=\tilde{T}\tilde{x}})$ is lower triangular with zeroes on the diagonal, where $\tilde{x} = x_1, x_2, x_3$.

By clearing denominators in the first row of $\tilde{T}^{-1}$, we see that there is a nonzero $\lambda \in K[x_4, , \ldots, x_n]^3$ such that $\lambda_1\tilde{H}_1 + \lambda_2\tilde{H}_2 + \lambda_3\tilde{H}_3 \in K[x_4, \ldots, x_n]$. Since $\tilde{H}_2$ and $\tilde{H}_3$ have different positive degrees with respect to $\tilde{x}$, it follows that $\lambda_1 \neq 0$ and $\tilde{H}_1 \in K[\tilde{H}_3, x_4, \ldots, x_n]$.

Now take $S \in \mathrm{GL}_n(K)$ such that the ith row of S^{-1} equals e_i^{t} for all $i \geq 4$ and the third row of S^{-1} equals $\mathcal{J}\tilde{H}_3$. Then only the first three components of $S^{-1}\tilde{H}(Sx)$ are nonzero, and we have $\tilde{H}_3(Sx) = x_3$ and $(Sx)_i = x_i$ for all $i \geq 4$. Consequently, the first three components of $S^{-1}\tilde{H}(Sx)$ are in $K[x_3, \ldots, x_n]$. Hence the Jacobian of $S^{-1}\tilde{H}(Sx)$ is upper triangular, and a subsequent conjugation of $\tilde{H}$ with the map $(x_n, x_{n-1}, \ldots, x_1)$ gives the desired lower triangular form of the Jacobian. This gives the property in (i) of Theorem 2.3. ∎

Lemma 3.5. *Assume that $H \in K[x]^n$ has degree 2, $H(0) = 0$ and $\operatorname{rk} \mathcal{J}H \leq 2$. Then there exists a $T \in \mathrm{GL}_n(K)$ such that $\tilde{H} := T^{-1}H(Tx)$ has one of the three forms that are specified in the proof of Theorem* 3.4.

Proof. We can choose T such that $\tilde{H}_1, \ldots, \tilde{H}_r$ have linearly independent quadratic parts over K, $\tilde{H}_{r+1}, \ldots, \tilde{H}_s$ are linear forms which are independent over K, and $\tilde{H}_{s+1} = \tilde{H}_{s+2} = \cdots = 0$. If $s \le 2$, then $\tilde{H} = T^{-1}H(Tx)$ has the second form in the proof of Theorem 3.4, so assume that $s \ge 3$. We distinguish three cases.

• $r \le 1$. Then $\tilde{H}_2$ and $\tilde{H}_3$ are linear forms which are independent over K. Hence we can take $S \in \mathrm{GL}_n(K)$ such that the first two rows of S^{-1} are $\mathcal{J}\tilde{H}_2$ and $\mathcal{J}\tilde{H}_3$. By the chain rule, $\mathcal{J}(\tilde{H}_2(Sx)) = e_1^{\mathrm{t}}$ and $\mathcal{J}(\tilde{H}_3(Sx)) = e_2^{\mathrm{t}}$, so $\tilde{H}_2(Sx) = x_1$ and $\tilde{H}_3(Sx) = x_2$. Hence $\tilde{H}(Sx) \in K[x_1, x_2]^n$ and $S^{-1}\tilde{H}(Sx) = (TS)^{-1}H((TS)x)$ has the first form in the proof of Theorem 3.4.

• $r \ge 3$. Since $\operatorname{rk} \mathcal{J}\tilde{H} = 2$, the rows of $\mathcal{J}(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3)$ are linearly dependent over $K(x)$ and hence also over $K[x]$. By looking at leading homogeneous parts, we see that $\operatorname{rk} \mathcal{J}(\bar{H}_1, \bar{H}_2, \bar{H}_3) \le 2$, where $\bar{H}_i$ is the leading and quadratic homogeneous part of $\tilde{H}_i$ for each $i \le 3$. By [B1, Th. 4.3.1], there exist linear forms p, q such that $\bar{H}_1, \bar{H}_2, \bar{H}_3$ are linearly dependent over K on p^2, pq and q^2. Furthermore, p and q are independent over K, and p^2, pq and q^2 are in turn linearly dependent over K on $\bar{H}_1, \bar{H}_2, \bar{H}_3$. Thus there exists an $L \in \mathrm{GL}_3(K)$ such that $L\big(\bar{H}_1, \bar{H}_2, \bar{H}_3\big) = (p^2, pq, q^2)$.

Take $S \in \mathrm{GL}_n(K)$ such that the first two rows of S^{-1} are $\mathcal{J}p$ and $\mathcal{J}q$, in that order. Then $L\big(\bar{H}_1(Sx), \bar{H}_2(Sx), \bar{H}_3(Sx)\big) = (x_1^2, x_1x_2, x_2^2)$. The 2-minor determinants of $\mathcal{J}_{x_1,x_2}(x_1^2, x_1x_2, x_2^2)$ are $2x_2^2$, $4x_1x_2$ and $2x_1^2$, which are also linearly independent over K. It follows that

$$\det \mathcal{J}_{x_1,x_2,x_i}\big(L(\tilde{H}_1(Sx), \tilde{H}_2(Sx), \tilde{H}_3(Sx))\big) \ne 0$$

if $i \ge 3$, and the last column of the Jacobian matrix on the left hand side, which can only be constant, is nonzero. Hence $L(\tilde{H}_1(Sx), \tilde{H}_2(Sx), \tilde{H}_3(Sx)) \in K[x_1, x_2]^3$. Since the first two rows of its Jacobian are linearly independent over K, and L is invertible, $\tilde{H}(Sx) \in K[x_1, x_2]^n$ as well. So $S^{-1}\tilde{H}(Sx) = (TS)^{-1}H((TS)x)$ has the first form in the proof of Theorem 3.4.

• $r = 2$. If $s \ge 4$, then we can proceed as in the case $r \le 1$, but with $\tilde{H}_3$ and $\tilde{H}_4$ instead of $\tilde{H}_2$ and $\tilde{H}_3$. So assume that $s = 3$.

Since multiplication of the third row of $\mathcal{J}\tilde{H}$ by $2\tilde{H}_3$ does not change the rank of $\mathcal{J}\tilde{H}$, we have $\operatorname{rk} \mathcal{J}(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3^2) \le 2$. Let $\bar{H}_i$ be the leading homogeneous part of $\tilde{H}_i$ for each $i \le 3$. If $\bar{H}_3^2$ is linearly independent over K of $\bar{H}_1$ and $\bar{H}_2$, then we can proceed as in the case $r \ge 3$ to obtain $\tilde{H}_i(Sx) \in K[x_1, x_2]$ for each $i \ne 3$ and $\tilde{H}_3(Sx)^2 \in K[x_1, x_2]$ for some $S \in \mathrm{GL}_n(K)$. So $S^{-1}\tilde{H}(Sx) = (TS)^{-1}H((TS)x)$ has the first form in the proof of Theorem 3.4 in that case.

Now assume that $\bar{H}_3^2$ is linearly dependent over K on $\bar{H}_1$ and $\bar{H}_2$. Then we can choose T such that $\bar{H}_2 = \bar{H}_3^2$. If the linear part of $\tilde{H}_2$ is dependent over K on $\tilde{H}_3$, then we can choose T such that even $\tilde{H}_2 = \tilde{H}_3^2$. Since $s = 3$, we see that $\tilde{H} = T^{-1}H(Tx)$ has the third form in the proof of Theorem 3.4 in that case.

Finally, assume that the linear part of $\tilde{H}_2$ is independent over K of $\tilde{H}_3$. Then $\tilde{H}_2 - \tilde{H}_3^2$ and $\tilde{H}_3$ are linear forms which are independent over K. Since $\mathcal{J}(\tilde{H}_2 - \tilde{H}_3^2) = \mathcal{J}\tilde{H}_2 - 2\tilde{H}_3\mathcal{J}\tilde{H}_3$, we can replace $\tilde{H}_2$ by $\tilde{H}_2 - \tilde{H}_3^2$ without affecting the Jacobian rank of $\tilde{H}$, and proceed as in the case $r \le 1$ to infer that e_1^{t} and e_2^{t} are in the row space of $\mathcal{J}\tilde{H}(Sx)$ for some $S \in \mathrm{GL}_n(K)$. Hence $\tilde{H}(Sx) \in K[x_1, x_2]^n$ and $S^{-1}\tilde{H}(Sx) = (TS)^{-1}H((TS)x)$ has the first form in the proof of Theorem 3.4. ∎

4. Lemmas. The lemmas in this section are required for the proofs that the counterexamples in the next section are indeed counterexamples.

Lemma 4.1. *Let $d \ge 1$ and $a_1, \ldots, a_{2d+2} \in K^n$ be pairwise linearly independent. Suppose that for all $j \ge \min\{3, d^2\}$ and all k with $3 \le k \le d+2$, the set $\{a_j, a_k, a_{k+d}\}$ consists of two or three vectors which are linearly independent* (*depending on whether $j \in \{k, k+d\}$ or not*). *If*

$$\sum_{i=1}^{2d+2} \lambda_i (a_i^{\mathrm{t}} x)^d = 0 \tag{5}$$

for some $\lambda_i \in K$, not all zero, then $\lambda_1\lambda_2 \ne 0$.

Proof. Assume that (5) holds. Since a_1 and a_2 are linearly independent, we may assume without loss of generality that $\lambda_3 \ne 0$. If $d = 1$, then $\lambda_1\lambda_2 = 0$ implies that either a_1 or a_2 is linearly dependent on a_3 and a_4, which is a contradiction. Hence the following cases remain:

• $d = 2$. Since a_4, a_5 and a_6 are linearly independent and $d = 2$, we may assume without loss of generality that a_1, a_3, a_6 are linearly independent vectors. Consequently, there exists a $b_1 \in K^n$ such that $b_1^{\mathrm{t}} a_1 = b_1^{\mathrm{t}} a_6 = 0 \ne b_1^{\mathrm{t}} a_3$. Applying $b_1^{\mathrm{t}}\partial$ to (5) gives

$$\sum_{i=2}^{5} \mu_i (a_i^{\mathrm{t}} x)^1 = 0$$

where $\mu_i = 2\lambda_i b_1^{\mathrm{t}} a_i$ for all i. Since a_3, a_4, a_5 are linearly independent and $\mu_3 \ne 0$, we have $\mu_2 \ne 0$ as well. Hence $\lambda_2 \ne 0$. In a similar manner, $\lambda_1 \ne 0$ follows.

• $d > 2$. Since a_3, a_{d+2} and a_{2d+2} are linearly independent, there exists a $b_2 \in K^n$ such that $b_2^{\mathrm{t}} a_{d+2} = b_2^{\mathrm{t}} a_{2d+2} = 0 \ne b_2^{\mathrm{t}} a_3$. Applying $b_2^{\mathrm{t}}\partial$ to (5) gives

$$\sum_{i=1}^{d+1} \mu_i (a_i^{\mathrm{t}} x)^{d-1} + \sum_{i=d+3}^{2d+1} \mu_i (a_i^{\mathrm{t}} x)^{d-1} = 0$$

where $\mu_i = d\lambda_i b_2^{\mathrm{t}} a_i$ for all i. Since $\mu_3 \ne 0$, it follows by induction on d that $\mu_1\mu_2 \ne 0$. Hence $\lambda_1\lambda_2 \ne 0$. ∎

LEMMA 4.2. *We have*

$$\sum_{i=0}^{d}(-1)^i\binom{d}{i}(x_1+ix_3)^d=\sum_{i=0}^{d}(-1)^i\binom{d}{i}(x_2+ix_3)^d \tag{6}$$

and if $d \ge 2$ and $\zeta_d \in K$ is a primitive dth root of unity, then

$$\sum_{i=0}^{d-1}\zeta_d^i(\zeta_d^i x_1+x_2+x_3)^d+\sum_{i=0}^{d-1}\zeta_d^i(\zeta_d^i x_1+x_2-x_3)^d=2d^2x_1^{d-1}x_2. \tag{7}$$

Proof. We first prove (6). Assume that (6) holds when we replace d by $d-1$. By substituting $x_2 = x_1 + x_3$ on both sides, we obtain

$$\begin{aligned}\sum_{i=0}^{d-1}(-1)^i\binom{d-1}{i}(x_1+ix_3)^{d-1} &= \sum_{i=0}^{d-1}(-1)^i\binom{d-1}{i}(x_1+(i+1)x_3)^{d-1}\\ &= -\sum_{i=1}^{d}(-1)^i\binom{d-1}{i-1}(x_1+ix_3)^{d-1}\end{aligned}$$

As $\binom{d}{i}=\binom{d-1}{i-1}+\binom{d-1}{i}$, both sides combine to

$$0=\sum_{i=0}^{d}(-1)^i\binom{d}{i}(x_1+ix_3)^{d-1}=\frac{1}{d}\partial_1\sum_{i=0}^{d}(-1)^i\binom{d}{i}(x_1+ix_3)^d.$$

Hence the left hand side of (6) is in $K[x_3]$. By a symmetry argument, (6) follows by induction on d, because the case $d = 0$ is trivial.

Assume that $d \ge 2$ and that $\zeta_d \in K$ is a primitive dth root of unity. By substituting $x_2 = x_2 \pm x_3$ in

$$\sum_{i=0}^{d-1}\zeta_d^i(\zeta_d^i x_1+x_2)^d=d^2x_1^{d-1}x_2 \tag{8}$$

we get (7). So in order to prove (7), it suffices to show (8). This can be done as follows:

$$\begin{aligned}\sum_{i=0}^{d-1}\zeta_d^i(\zeta_d^i x_1+x_2)^d &= \sum_{i=0}^{d-1}\zeta_d^i\sum_{j=0}^{d}\binom{d}{j}(\zeta_d^i x_1)^j x_2^{d-j}\\ &= \sum_{j=0}^{d}\binom{d}{j}x_1^j x_2^{d-j}\sum_{i=0}^{d-1}\zeta_d^{i(j+1)}\\ &= \binom{d}{d-1}x_1^{d-1}x_2^1\sum_{i=0}^{d-1}1=d^2x_1^{d-1}x_2. \ \blacksquare\end{aligned}$$

Notice that (7) is not true for $d = 1$, because we are using the fact that $\sum_{i=0}^{d-1}\zeta_d^{(j+1)}=0$ for $j = d$, which does not hold for $d = 1$.

Lemma 4.3. *To write $x_1^{d-1}x_2$ as a linear combination of x_1^d and other dth powers of linear forms, at least d such powers are necessary besides x_1^d.*

Proof. The case $d = 1$ is easy, so let $d \geq 2$ and suppose that $x_1^{d-1}x_2$ can be written as a linear combination of $x_1^d, (a_3^{\mathrm{t}}x)^d, \ldots, (a_{d+1}^{\mathrm{t}}x)^d$. Assume without loss of generality that $n \geq 2d+2$ and that the vectors $e_1, a_3, \ldots, a_{d+1}$ are pairwise linearly independent. By applying ∂_2 to this linear combination, we obtain

$$x_1^{d'} = \lambda_3(a_3^{\mathrm{t}}x)^{d'} + \lambda_4(a_4^{\mathrm{t}}x)^{d'} + \cdots + \lambda_{d'+2}(a_{d'+2}^{\mathrm{t}}x)^{d'}$$

where $d' = d - 1$. Take $a_1 = e_1$ and take a_2 linearly independent of $a_1, a_3, \ldots, a_{d'+2}$. Next, take a_i linearly independent of $a_1, \ldots, a_{i-1}$ for all i with $d' + 3 \leq i \leq 2d' + 2$. Then Lemma 4.1 with d replaced by d' gives a contradiction. ■

5. Counterexamples. We start by giving counterexamples $x + H$ to $(\mathrm{JC}^-) \Rightarrow (\mathrm{JC})$ and $(\mathrm{JC}^+) \Rightarrow (*)$, such that H is homogeneous of degree $d \geq 3$ and $d \geq 2$ respectively. Using known techniques, these counterexamples can be improved to counterexamples of the form $x + (Ax)^{*d}$.

Theorem 5.1. *If $n = 4$ and $d \geq 3$, then*

$$H = x_1^{d-3}\big(0, 0, x_2(x_1x_3 - x_2x_4), x_1(x_1x_3 - x_2x_4)\big) \tag{9}$$

is a homogeneous counterexample of degree d to $(\mathrm{JC}^-) \Rightarrow (\mathrm{JC})$.

If $n = 5$ and $d \geq 2$, then

$$H = (0, 0, x_2^{d-1}x_4, x_1^{d-1}x_3 - x_2^{d-1}x_5, x_1^{d-1}x_4) \tag{10}$$

is a homogeneous counterexample of degree d to $(\mathrm{JC}^+) \Rightarrow (*)$.

Furthermore, there exist a power linear counterexample to $(\mathrm{JC}^-) \Rightarrow (\mathrm{JC})$ for each $d \geq 3$, and a power linear counterexample to $(\mathrm{JC}^+) \Rightarrow (*)$ for each $d \geq 2$.

Proof. Assume first that $n = 4$ and H is as in (9). Since the components of H are composed of the invariants $x_1, x_2, x_1x_3 - x_2x_4$ of $x + H$, we see that $x + H$ is a quasi-translation, i.e. $x - H$ is the inverse of $x + H$. One can compute that the trailing principal 2-minor of $(d-1)I_4 + (d-2)\cdot(\mathcal{J}H)|_{x=(1,0,0,0)} + (\mathcal{J}H)|_{x=(1,c,0,0)}$ equals

$$\begin{pmatrix} d-1+c & -c^2 \\ d-1 & d-1-c \end{pmatrix}$$

and that its determinant equals $c^2(d-2) + (d-1)^2$. So if we take $c = \frac{d-1}{\sqrt{d-2}}\mathrm{i}$, then

$$\det\big((d-1)I_4 + (d-2)(\mathcal{J}H)|_{x=(1,0,0,0)} + (\mathcal{J}H)|_{x=(1,c,0,0)}\big) = 0,$$

which contradicts $(\mathrm{JC}^-) \Rightarrow (\mathrm{JC})$.

Assume next that $n = 5$ and H is as in (10). Then one can compute that

$$\mathcal{J}H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & b & 0 \\ * & * & a & 0 & -b \\ * & * & 0 & a & 0 \end{pmatrix}$$

for certain polynomials a, b. The form on the right hand side does not change by substitution and adding copies of $\mathcal{J}H$ with different substitutions, so $\sum_{i=1}^{n} (\mathcal{J}H)|_{x=v_i}$ is nilpotent for all $v_i \in K^n$. This gives (JC$^+$).

On the other hand, $(\mathcal{J}H)|_{x_1=0} \cdot (\mathcal{J}H)|_{x_2=0}$ is a lower triangular matrix with diagonal $(0, 0, x_1^{d-1}x_2^{d-1}, -x_1^{d-1}x_2^{d-1}, 0)$, so $(\mathcal{J}H)|_{x_1=0} \cdot (\mathcal{J}H)|_{x_2=0}$ is not nilpotent. By [EH], we see that H is a counterexample to (JC$^+$) $\Rightarrow$ $(*)$.

To obtain power linear counterexamples, we can use the concept of GZ-pairing of [GZ]. For that purpose, let H be any of the above two maps. By [GZ, Th. 1.3], there exist $N > n$ and $A \in \mathrm{Mat}_N(K)$ such that $x + H$ and $X + (AX)^{*d}$ are GZ-paired through matrices $B \in \mathrm{Mat}_{n,N}(K)$ and $C \in \mathrm{Mat}_{N,n}(K)$, where $X = (x_1, \ldots, x_N)$.

Take $M \in \mathrm{Mat}_{N,N-n}(K)$ whose columns form a basis of $\ker B$ and define $\bar{T} = (C \mid M)$. Then one can show that $\bar{T}$ is as in the proof of [Che, Th. 2], with $F = X + (AX)^{*d}$ and $F_1 = (x + H, \ldots)$. Now one can use similar techniques as in the proof of Theorem 3.1 to deduce that $(AX)^{*d}$ is a counterexample as also is H, or use the following invariance results for GZ-pairing. The GZ-invariance of (JC$^-$) follows from [GZ, Th. 1.3(9)] and that of $(*)$ from [LDS, Th. 3(2)]. The GZ-invariance of (JC) and (JC$^+$) can be proved with the techniques of [GZ, proof of Th. 2.4]. ∎

Example 5.2. Let $x = (x_1, \ldots, x_5)$ and $X = (x_1, \ldots, x_{13})$. Take H as in (10) and

$$G = \big(0, 0, (x_4 - x_1)^3, (x_4 + x_1)^3, x_4^3, (x_4 - x_2)^3, (x_4 + x_2)^3,$$
$$(x_3 - x_1)^3, (x_3 + x_1)^3, x_3^3, (x_5 - x_2)^3, (x_5 + x_2)^3, x_5^3\big).$$

Then $\ker \mathcal{J}_x G$ is trivial and $6H = BG$, where

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & -1 & -1 & 2 \\ 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has full rank. Hence there exists a matrix C such that $BC = I_5$. Consequently, $x + H$ and $X + \frac{1}{6}G(BX)$ are GZ-paired through B and C.

In the next theorem, we give threedimensional counterexamples $F = x + H$ to $(*) \Rightarrow (**)$ and $(**) \Rightarrow (***)$, such that H is homogeneous of degree $d \geq 3$. The techniques in the proof of the previous theorem to get counterexamples of the form $F = x + (Ax)^{*d}$ do not work, so we improve our counterexamples to that form by hand.

Theorem 5.3. *Assume that $d \geq 2$, and either*

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ \vdots \\ H_{d+2} \\ H_{d+3} \\ H_{d+4} \\ \vdots \\ H_{2d+2} \end{pmatrix} := \begin{pmatrix} 0 \\ \nu x_1^d \\ x_1^d - x_2^d \\ (x_1 + 2x_3)^d \\ \vdots \\ (x_1 + dx_3)^d \\ (x_2 + x_3)^d \\ (x_2 + 2x_3)^d \\ \vdots \\ (x_2 + dx_3)^d \end{pmatrix} \tag{11}$$

or

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ \vdots \\ H_{d+2} \\ H_{d+3} \\ H_{d+4} \\ \vdots \\ H_{2d+2} \end{pmatrix} := \begin{pmatrix} 0 \\ \nu x_1^d \\ x_1^{d-1} x_2 \\ (\zeta_d x_1 + x_2 + x_3)^d \\ \vdots \\ (\zeta_d^{d-1} x_1 + x_2 + x_3)^d \\ (x_1 + x_2 - x_3)^d \\ (\zeta_d x_1 + x_2 - x_3)^d \\ \vdots \\ (\zeta_d^{d-1} x_1 + x_2 - x_3)^d \end{pmatrix} \tag{12}$$

for some $\nu \in K$, where ζ_d is a primitive root of unity of K in the case of (12). *Then $2d + 2 \geq 6$,*

$$d(x_1 + x_3)^d = H_3 + \sum_{i=2}^{d} (-1)^i \binom{d}{i} H_{i+2} - \sum_{i=1}^{d} (-1)^i \binom{d}{i} H_{i+d+2} \tag{13}$$

in the case of (11) *and*

$$(14) \qquad (x_1 + x_2 + x_3)^d = 2d^2 H_3 - \zeta_d H_4 - \zeta_d^2 H_5 - \cdots - \zeta_d^{2d-1} H_{2d+2}$$

in the case of (12), *and there exists a* $T \in \mathrm{GL}_{2d+2}(K)$ *such that* $T(H(T^{-1}x))$ *is power linear if* $n = 2d + 2$.

If $3 \le n \le 2d+2$, *then* $H = (H_1, \ldots, H_n)$ *is of the form* $(*)$ *and we have the following:*

- (i) *If* H *is of the form* $(**)$, *then* c_1 *and* c_2 *are linearly independent linear combinations of* e_1 *and* e_2.
- (ii) H *is of the form* $(**)$ *if and only if either* H *is as in* (11), *or* H *is as in* (12) *with* $H_2 = 0 = d - 2$.
- (iii) H *is of the form* $(***)$ *if and only if* H *is as in* (11) *with* $H_2 \neq 0$.

Proof. Since $\mathcal{J}H$ is lower triangular with zeroes on the diagonal, it follows from (i) of Theorem 2.3 that H is of the form $(*)$. By (6) and (7) in Lemma 4.2, we get (13) and (14) respectively. So H is a linear triangularization of a power linear map if $n = 2d + 2$.

In the case of (11), set

$$a_{2+i}^{\mathrm{t}} x := x_1 + i x_3, \qquad a_{d+2+i}^{\mathrm{t}} x := x_2 + i x_3,$$

for $i = 1, \ldots, d$. In the case of (12), set

$$a_{2+i}^{\mathrm{t}} x := \zeta_d^{i-1} x_1 + x_2 + x_3, \qquad a_{d+2+i}^{\mathrm{t}} x := \zeta_d^{i-1} x_1 + x_2 - x_3,$$

for $i = 1, \ldots, d$. Then $H_i = (a_i^{\mathrm{t}} x)^d$ for all $i \ge 4$ and the left hand side of (13) or (14) respectively is a multiple of $(a_3^{\mathrm{t}} x)^d$. Hence the linear span S of $H_3, \ldots, H_n$ is contained in that of $(a_3^{\mathrm{t}} x)^d, \ldots, (a_{2d+2}^{\mathrm{t}} x)^d$.

(i) We have

Claim. *If* $\mu^{\mathrm{t}} H$ *and* $\mu_2 H_2$ *are both linearly dependent over* K *on the same power of a linear form in* x_1 *and* x_2 *for some* $\mu \in K^n$, *then* μ *is a linear combination of* e_1 *and* e_2.

To prove the claim, assume that $\mu^{\mathrm{t}} H$ and $\mu_2 H_2$ are as above. Then there exists a nontrivial linear combination a_2 of e_1 and e_2 such that both $\mu^{\mathrm{t}} H$ and $\mu_2 H_2$ are linearly dependent on $(a_2^{\mathrm{t}} x)^d$. On account of $H_1 = 0$, we have

$$0 x_4^d + (\mu_2 H_2 - \mu^{\mathrm{t}} H) + \mu_3 H_3 + \mu_4 H_4 + \cdots + \mu_n H_n = 0$$

Take $a_1 = e_4$. By (13) and (14) respectively, there exists a $\lambda \in K^{2d+2}$ with $\lambda_1 = 0$ such that

$$\lambda_1 (a_1^{\mathrm{t}} x)^d + \lambda_2 (a_2^{\mathrm{t}} x)^d + \lambda_3 (a_3^{\mathrm{t}} x)^d + \cdots + \lambda_{2d+2} (a_{2d+2}^{\mathrm{t}} x)^d = 0.$$

Furthermore, there exists an injective linear map which maps $(\mu_3, \ldots, \mu_n)$ to $(\lambda_3, \ldots, \lambda_{2d+2})$. By Lemma 4.1 and $\lambda_1 = 0$, we have $\lambda = 0$. Thus $\mu_3 = \cdots = \mu_n = 0$. So μ is a linear combination of e_1 and e_2 and the claim has been proved.

Suppose that H is of the form $(**)$. Since $c_j^{\mathrm{t}} b_i = 0$ for all $i \geq j \geq 1$, we have

$$c_1^{\mathrm{t}} H = c_1^{\mathrm{t}} \sum_{i=1}^{n-1} (c_i^{\mathrm{t}} x)^d b_i = \sum_{i=1}^{n-1} c_1^{\mathrm{t}} b_i (c_i^{\mathrm{t}} x)^d = 0, \tag{15}$$

thus c_1 is a linear combination of e_1 and e_2 on account of the above claim. Using (15) again, we see that c_1 is linearly dependent on e_1 if $H_2 \neq 0$. Hence $(c_1^{\mathrm{t}} x)^d$ and H_2 are linearly dependent on the same power of a linear form in x_1 and x_2. By using $c_j^{\mathrm{t}} b_i = 0$ for all $i \geq j \geq 1$ again, we obtain

$$c_2^{\mathrm{t}} H = c_2^{\mathrm{t}} \sum_{i=1}^{n-1} (c_i^{\mathrm{t}} x)^d b_i = \sum_{i=1}^{n-1} c_2^{\mathrm{t}} b_i (c_i^{\mathrm{t}} x)^d = c_2^{\mathrm{t}} b_1 (c_1^{\mathrm{t}} x)^d. \tag{16}$$

It follows from the above claim that c_2 is a linear combination of e_1 and e_2 as well.

Suppose that c_1 and c_2 are linearly dependent. Then there exist a nontrivial linear combination a_2 of e_1 and e_2 such that both c_1 and c_2 are linearly dependent on a_2. Using the claim with $\mu_1 = \mu_2 = 0$, and $\mu^{\mathrm{t}} H = 0$ and $\mu^{\mathrm{t}} H = (a_2^{\mathrm{t}} x)^d$ respectively, we obtain $\dim S = n - 2$ and $(a_2^{\mathrm{t}} x)^d \notin S$.

The space S^* generated by $(c_1^{\mathrm{t}} x)^d, \ldots, (c_{n-1}^{\mathrm{t}} x)^d$, which contains S, is generated by $n - 2$ powers of linear forms, namely $(a_2^{\mathrm{t}} x)^d, (c_3^{\mathrm{t}} x)^d, \ldots, (c_{n-1}^{\mathrm{t}} x)^d$. Hence $S^* \supseteq S$ and $\dim S^* \leq n - 2 = \dim S$. It follows that $S = S^*$. Since $(a_2^{\mathrm{t}} x)^d \notin S = S^*$, we have $c_1 = c_2 = 0$ and $\dim S^* < n - 2$. This contradicts $S = S^*$ and $\dim S = n - 2$, so c_1 and c_2 are linearly independent.

(ii) If H is as in (11), then we can take

$$\begin{aligned} c_1 &= e_1, & b_1 &= \nu e_2 + e_3, \\ c_2 &= e_2, & b_2 &= -e_3, \\ c_i &= a_{i+1}, & b_i &= e_{i+1}, \end{aligned}$$

for all $i > 2$, which shows that H is of the form $(**)$. If H is as in (12) with $H_2 = 0 = d - 2$, then we can take

$$\begin{aligned} c_1 &= e_1 + e_2, & b_1 &= \tfrac{1}{4} e_3, \\ c_2 &= e_1 - e_2, & b_2 &= -\tfrac{1}{4} e_3, \\ c_i &= a_{i+1}, & b_i &= e_{i+1}, \end{aligned}$$

for all $i > 2$, which shows that H is again of the form $(**)$.

Conversely, suppose that H is as in (12) and of the form $(**)$. By Lemma 4.3, at least d powers of linear forms are necessary to write H_2 and H_3 as their linear combinations if $H_2 = 0$, and at least $d + 1$ such powers otherwise. Now assume that $H_2 \neq 0$ or $d \geq 3$. Then there are at least three powers of linear forms necessary to write H_2 and H_3 as their linear combinations. Hence there exists a linear combination h of H_2 and H_3 which is not a linear combination of $(c_1^{\mathrm{t}}x)^d$ and $(c_2^{\mathrm{t}}x)^d$.

Since $\dim S^* \le n-1 < n$, there exists a nonzero $\mu \in K^n$ such that

$$\mu_1(c_1^{\mathrm{t}}x)^d + \mu_2(c_2^{\mathrm{t}}x)^d + \mu_3 h + \mu_4 H_4 + \mu_5 H_5 + \cdots + \mu_n H_n = 0.$$

By applying ∂_3 to both sides, we get

$$\begin{aligned}(17)\qquad d\mu_4(a_4^{\mathrm{t}}x)^{d'} + \cdots + d\mu_{d+2}(a_{d+2}^{\mathrm{t}}x)^{d'} - d\mu_{d+3}(a_{d+3}^{\mathrm{t}}x)^{d'}\\ - d\mu_{d+4}(a_{d+4}^{\mathrm{t}}x)^{d'} - \cdots - d\mu_{2d+2}(a_{2d+2}^{\mathrm{t}}x)^{d'} = 0\end{aligned}$$

where $d' = d-1$ and $\mu_{n+1} = \mu_{n+2} = \cdots = \mu_{2d+2} = 0$. Take $a_1' = e_4$ and $a_2' = a_{d+3}$. Additionally set $a_{i+1}' = a_{i+2}$ and $a_{i+d}' = a_{i+d+2}$ for all i with $2 \le i \le d$. By Lemma 4.1 with d and a replaced by d' and a' respectively, we get $\mu_4 = \cdots = \mu_{2d+2} = 0$. Hence h is a linear combination of $(c_1^{\mathrm{t}}x)^d$ and $(c_2^{\mathrm{t}}x)^d$, a contradiction, so H is not of the form $(**)$.

(iii) Assume first that $H_2 \ne 0$. If H is as in (11), then we can take the c_j's and the b_i's as in (ii), and we have $(***)$. If H is as in (12), then by (ii), H is not of the form $(**)$ and hence neither of the form $(***)$.

Assume next that $H_2 = 0$ and that H is of the form $(***)$. From $H_1 = H_2 = 0$ and the fact that c_2 is linearly dependent on e_1 and e_2, we have $c_2^{\mathrm{t}}H = 0$. Consequently, $c_2^{\mathrm{t}}b_1 = 0$ on account of (16). By definition of $(***)$, we have $c_1^{\mathrm{t}}b_i = c_2^{\mathrm{t}}b_i = 0$ for all i. Since c_1 and c_2 are linearly independent, we have a contradiction with the independence of the b_i's. ∎

We can make nonhomogeneous variants of (9) and (10) as follows. In (9), we can replace x_2 by 1, remove H_2, and replace x_{i+1} by x_i for all $i \ge 1$. In (10), we can replace x_2^{d-1} by x_1^{d-2}, remove H_2, and replace x_{i+1} by x_i for all $i \ge 1$. In this manner, we get rid of the second coordinate, so that the dimension and the Jacobian rank each decrease by one, in return for abandoning homogeneity, just as with most of Theorem 3.2 with respect to Theorem 3.1.

The maps $H = (0, x_1^d - x_1^{d-1})$ and $H = (0, 0, x_1^d - x_1^{d-1})$ are additional nonhomogeneous counterexamples to $(*) \Rightarrow (**)$ and $(**) \Rightarrow (***)$ respectively. By comparing the counterexamples with the positive results of Theorems 3.1, 3.2 and 3.4, we get the following four questions.

The first two questions are whether (JC) implies (JC$^+$) in general and whether (JC$^+$) implies $(*)$ in dimension three if $\mathcal{J}H$ is nilpotent (if F satisfies (JC$^+$), then by [GBDS, Th. 3.9], $\mathcal{J}H$ gets nilpotent in addition if we compose F with some linear map). In case H is homogeneous, the questions are whether (JC) implies (JC$^+$) in general and whether (JC$^+$) implies $(*)$ in dimension four, which are the last two questions. By Theorems 3.1 and 3.2, the last and the second question have an affirmative answer when the degree is at most three.

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